Fourier Transform

Remember that we defined the Fourier-Transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

Then

Prop 1.2 (p. 136) If $f \in S(\mathbb{R})$ then

i) $f(x+h) \longrightarrow \hat{f}(\xi) e^{2\pi i h \xi}$ (Relation translation Fourier transform)

⋮

v) $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi} \hat{f}(\xi)$

$\left(f'(x) \longrightarrow 2\pi i \xi \hat{f}(\xi) \right)$ by Fourier inversion. (Relation different Fourier)

We have also studied convolutions so what is the relation between convolutions and the Fourier transform? We know that

$$\widehat{f * g}(n) = \hat{f}(n) \hat{g}(n)$$ for Fourier series

so it shouldn't be a surprize that

Proposition 1.11 iii) If $f, g \in S(\mathbb{R})$ then

$$\widehat{f*g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi).$$

Proof: The proof is simple. Since $f, g \in S(\mathbb{R})$ we will have no problems changing order in any integrals.

$$\widehat{f*g}(\xi) = \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \right)}_{f*g(x)} e^{-i\pi x \xi}\, dx =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\, g(x-y)\, e^{-i\pi x \xi}\, dy\, dx. \qquad (1)$$

Choose different dummy variables since we want to write this as a double integral

And

$$\hat{f}(\xi)\,\hat{g}(\xi) = \underbrace{\left(\int_{-\infty}^{\infty} e^{-i\pi y \xi} f(y)\, dy \right)}_{\text{constant w.r.t. } x} \underbrace{\left(\int_{-\infty}^{\infty} e^{-i\pi x \xi} g(x)\, dx \right)}_{\text{constant w.r.t. } y} =$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(y)\, g(x)\, e^{-i\pi (y+x)\xi}\, dx\, dy \right] = \left\{ \begin{array}{l} x \to \tilde{x} - y \\ y + x = \tilde{x} \\ x \end{array} \right\} =$$

want this to be $x-y$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\, g(\tilde{x} - y)\, e^{-i\pi \tilde{x} \xi}\, \underset{dy\, dx}{\cancel{d\tilde{x}\, dy}} = (1)$$

Before we get to the punch line I would want to prove another little THM

Thm 1.12 (Plancherel)

If $f \in S(\mathbb{R})$ then $\|\hat{f}\|^2 = \|f\|^2$.

Proof (just a discussion):

We want to connect

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} f(y)\, \overline{f(y)} \, dy$$

to $$\|\hat{f}\|^2 = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)} \, d\xi \qquad (2)$$

The (rather weird observation is that)

$$\int_{-\infty}^{\infty} f(y)\, \overline{f(y)}\, dy = \int_{-\infty}^{\infty} f(y)\, \overline{f(-(0-y))}\, dy = \underbrace{\left(f(y) * \overline{f(-y)}\right)}_{h}(0)$$

So $h(0) = \|f\|^2$ (3) if $h(x) = \left(f(y) * \overline{f(-y)}\right)(x)$

but then (since we know ~~the~~ how to handle convolutions)

$$\hat{h}(\xi) = \hat{f}(\xi)\, \widehat{\overline{f}}(-\xi) = \hat{f}(\xi)\, \overline{\hat{f}(\xi)} \qquad \text{compare to } (2)$$

So

$$\int_{-\infty}^{\infty} \hat{f}(\xi)\,\overline{\hat{f}(\xi)}\,d\xi = \|\hat{f}\|^2 = \int_{-\infty}^{\infty} \hat{h}(\xi)\,d\xi = \int_{-\infty}^{\infty} \hat{h}(\xi)\,\underbrace{e^{2\pi i\cdot 0\cdot \xi}}_{=1}\,d\xi =$$

$$= \left\{\begin{array}{l} x=0 \\ \text{in the} \\ \text{Fourier inversion} \end{array}\right\} = h(0) = \left\{ 3 \right\} = \|f\|^2.$$

$\blacksquare$

Remark: This proof seems to be a bunch of formula manipulation.
I do not know if there is a great idea behind it. It should be true in view of the analogy

$$\int |f|^2 = \sum |\hat{f}(n)|^2 \quad \text{for Fourier series.}$$

My guess is that there was a more difficult proof at one point.
~~(maybe similar to the proof of~~
Then someone stumbled upon this simpler - but less intuitive - proof.

Can you find an alternative approach.

1) consider f with spt. in $[-M, M]$

Then Parseval's identity states

$$\underbrace{L\sum_{n=-\infty}^{\infty}\left(a_n\left(\tfrac{n}{L}\right)\right)^2}_{\text{Riemann approximation}} = \int_{-L/2}^{L/2} |f(x)|^2 \, dx$$

$$= L\sum_n \left(\frac{1}{L}\int_{-L/2}^{L/2} e^{-\frac{2\pi i n x}{L}} f(x)\, dx\right)^2 \simeq \int_{-\infty}^{\infty}\left(\int_{-L/2}^{L/2} e^{-2\pi i \xi x} f(x)\, dx\right)^2 d\xi$$

More natural.

In particular if we remember that

Parseval's identity

$$f \sim \sum_{n=-\infty}^{\infty} a_n \underbrace{e^{inx}}_{\text{basis in a vector space}} \Rightarrow \|f\|^2 = \sum_{n=-\infty}^{\infty} a_n^2$$

which is an (infinite dimensional) linear algebra result.

Applications to PDE

Two things makes Fourier transforms important for PDE theory

I) $F(f'(x)) = 2\pi i \xi \hat{f}(\xi)$

That is it changes differentiation to multiplication

II) Since, $\widehat{f * g} = \hat{f}\,\hat{g}$ we have that if $\hat{u} = \hat{f}\,\hat{g}$ then $u = f * g$.

Consider for instance the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{and} \quad u(x,0) = f(x) \in S(\mathbb{R})$$

take a Fourier transform in x (Heuristically)

$$F_x\left(\frac{\partial u}{\partial t}\right) = F_x\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$\Downarrow$ using I twice

Not really justified, but it works

$$\left\{\frac{\partial \hat{u}(\xi,t)}{\partial t} = -4\pi^2 \xi^2 \hat{u}(\xi,t)\right.$$ which is an ODE

So for fixed ξ we get the solution

$$\hat{u}(\xi,t) = e^{-4\pi^2\xi^2 t}\,\hat{u}(\xi,0) = e^{-4\pi^2\xi^2 t}\,\hat{f}(\xi)$$

Make an inverse Fourier transform to deduce, using II,

$$u(x,t) = \underbrace{\mathcal{F}^{-1}\left(e^{-4\pi^2\xi^2 t}\right)}_{\text{This we can calculate}} * \underbrace{f(x)}_{\text{this we know.}}$$

&

So we get a formula for the solution.

As a matter of fact we have already calculated $\mathcal{F}^{-1}\left(e^{-4\pi^2\xi^2 t}\right)$ in corollary 1.5 p. 139

$$\mathcal{F}^{-1}\left(e^{-4\pi^2\xi^2 t}\right) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}} = \mathcal{H}_t(x)$$

So, at least Heuristically,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy \qquad (1)$$

which is an absolutely wonderfull formula.

Compare, if $x^2+ax+b=0 \Rightarrow x = -\frac{a}{2} \pm \sqrt{\frac{a^2-4b}{4}}$

The solution formula is similar, but (1) is for a much more abstract and difficult problem (and hence a little more complicated).

The above is not a proof. We are just throwing around some ideas to see what is reasonable to expect. Now when we know we can formulate ~~an~~ Theorem and try to prove that.

Thm: Given $f \in S(\mathbb{R})$, let

$$u(x,t) = (f * H_t)(x) \quad \text{for} \quad t > 0. \qquad (1)$$

Then

i) $u \in C^2$ for $x \in \mathbb{R}$, $t > 0$,

ii) u solves the heat equation

iii) $\lim\limits_{t \to 0^+} u(x,t) \to f(x)$ uniformly

iv) $\int\limits_{-\infty}^{\infty} |u(x,t) - f(x)|^2 \, dx \to 0$ as $t \to 0$.

Remark: Observe that this theorem ~~gives~~ states that u defined as in (1) solves $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ (from i) and ii))

with the initial data

$$u(x,0) = f \qquad \text{(from iii) and iv))}.$$

Proof: The proof is straightforward, since

$$u(x,t) = \underbrace{f * \mathcal{H}_t}_{\substack{\text{Both in } S(\mathbb{R}) \\ \text{so } u(x,t) \in S(\mathbb{R}) \\ \text{for each } t>0}} = \int_{-\infty}^{\infty} \underbrace{\frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x-y|^2}{4t}}}_{\substack{\text{decays fast} \\ \text{with all it's} \\ \text{derivatives}}} \underbrace{f(y)}_{\substack{\text{decays} \\ \text{fast}}} \, dy$$

i)

we may differentiate under the integral sign and conclude that

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \cancel{\int_{-\infty}^{\infty} \left[\frac{|x-y|^2}{t^2} e^{-\frac{|x-y|^2}{4t}} - \frac{1}{2t} + \frac{|x-y|^2}{4t^2} \right.}$$

$$= \int_{-\infty}^{\infty} \left[-\frac{1}{2t} + \frac{|x-y|^2}{4t^2} \right] \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}} f(y)\, dy$$

equal

and

$$\frac{\partial u(x,t)}{\partial t} = \int_{-\infty}^{\infty} \left[-\frac{1}{2t} + \frac{|x-y|^2}{4t^2} \right] \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}} f(y)\, dy$$

proves ii).

To show iii) we just notice that by Thm 1.6 p. 139 $\mathcal{H}_t(x) = K_{4\pi t}(x)$ is a family of good kernels so the uniform convergence follows by the result for good kernels (same argument at least

To prove iv) we use Plancherel's formula

$$\int_{-\infty}^{\infty} |u(x,t) - f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\underbrace{\hat{u}(\xi,t)}_{=\hat{f}\hat{H}_t} - \hat{f}(\xi)|^2 \, d\xi \doteq$$

$$= \int_{-\infty}^{\infty} \underbrace{|\hat{f}(\xi)|^2}_{\leq \frac{C}{1+|\xi|^2}} \underbrace{\left| e^{-4\pi^2 t \xi} - 1 \right|^2}_{} d\xi \leq$$

$\leq \frac{C}{1+|\xi|^2}$ So the contribution of this $< \varepsilon$ for $|\xi| > N$

This term ~~is~~ is < 4, and $\to 0$ uniformly on compact sets

$$\leq \underbrace{\int_{|\xi| > |N|} \frac{4C}{1+|\xi|^2} \, d\xi}_{\leq \frac{8C}{N}} + \underbrace{\int_{|\xi| \leq N} C \left| e^{-4\pi t \xi} - 1 \right|^2 d\xi}_{< \varepsilon} \leq \frac{8C}{N} + \varepsilon < 2\varepsilon$$

$< \varepsilon$ if $0 < t < \delta_\varepsilon$

since $e^{-4\pi t \xi} \to 1$ uniformly in $|\xi| \leq N$ as $t \to 0$

if we choose N large enough. ∎

This is a fantastic result! However, do we know that we get the "right" solution? We need to investigate if there are other solutions as well.

Thm 2.3. Suppose that $u(x,t)$ satisfies

i) u is continuous on the closure of $\mathbb{R} \times \{t \geq 0\}$

ii) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for $t > 0$

iii) $u(x,0) = 0$

iv) There exists constants C_0, C_t, C_x s.t

$$|u(x,t)| \leq \frac{C_0}{1+|x|}, \quad \left|\frac{\partial u}{\partial t}(x,t)\right| \leq \frac{C_t}{1+|x|}, \quad \left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial^2 u}{\partial x^2}\right| \leq \frac{C_x}{1+|x|}$$

then $u(x,t) = 0$.

Proof: Consider $E(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dx$ which is well defined since $|u(x,t)|$ is integrable.

Also $0 \leq E(t) \leq C$ and $E(0) = 0$.

Moreover

$$E'(t) = \int_{-\infty}^{\infty} \Big[\underbrace{u_t(x,t)}_{=\frac{\partial^2 u}{\partial x^2}} \bar{u}(x,t) + u(x,t) \underbrace{\bar{u}_t(x,t)}_{=\frac{\partial^2 \bar{u}}{\partial x^2}}\Big] dx =$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial^2 u}{\partial x^2} \bar{u} + u \frac{\partial^2 \bar{u}}{\partial x^2}\right] dx = \left\{\begin{matrix}\text{integration} \\ \text{by} \\ \text{parts}\end{matrix}\right\} = -\int_{-\infty}^{\infty} 2 \frac{\partial u}{\partial x} \frac{\partial \bar{u}}{\partial x} dx = -\int_{-\infty}^{\infty} 2\left|\frac{\partial u}{\partial x}\right|^2 dx \leq 0$$

So E is non-increasing and thus

$$0 = E(0) \geq \int_{-\infty}^{\infty} |u(x,t)|^2 dx \geq 0$$

since u continuous

so $\int_{-\infty}^{\infty} |u|^2 dx = 0 \implies u = 0.$

Remarks: This theorem is very important.
In particular, the technique.

i) If u & v are solutions to the heat eq s.t $u(x,0) = v(x,0)$ and u, v satisfies the conditions in Thm 2.3 (except iii)) then $u - v = 0$ so $u = v$.
We get uniqueness of the solutions.

ii) Notice that we actually show that

$E'(t) \leq 0$ even if $u(x,0) \neq 0$.

So theorem 2.3 shows that ~~the~~

$$\| u(x,0) \|^2 \geq \| u(x,t) \|^2 + \int_0^t \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x} \right|^2 dx \geq \| u(x,t) \|^2$$

Which implies many things about the solution.