Theorem: Suppose that

i) $u(x,t)$ is continuous in $\mathbb{R} \times [0,\infty)$ ~~with its first two derivatives~~

ii) u satisfies the heat equation in $t>0$ (here we also assume that $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ are continuous for $t>0$)

iii) $u(x,0)=0$

iv) There are constants C_0, C_t, C_x s.t.

$$|u(x,t)| \le \frac{C_0}{1+|x|}, \quad \left|\frac{\partial u}{\partial t}\right| \le \frac{C_t}{1+|x|} \quad \text{and} \quad \left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial^2 u}{\partial x^2}\right| \le \frac{C_x}{1+|x|}$$

Then $u=0$.

Proof: Define the energy

$$E(t) = \int_{\mathbb{R}} |u(x,t)|^2 \, dx \ge 0 \quad \left(\text{welldefined since } \overset{\text{iv)}}{|u|^2 \le \frac{C_0^2}{1+|x|^2}}\right)$$

Then $E(0) \overset{\text{iii)}}{=} 0$. We aim to show that $\frac{\partial E}{\partial t} \le 0$ then it follows that

$$0 \le E(t) = \underbrace{E(0)}_{=0} + \underbrace{\int_0^t E'(s)\,dx}_{\le 0} \le 0 \implies E(t)=0$$

and thus $\left.\begin{array}{l} \int_{\mathbb{R}} |u(x,t)|^2 dx = 0 \\ u \text{ continuous } ^{\text{i)}} \end{array}\right\} \implies u(x,t)=0.$

Since we assume that $|u| \leq \frac{C_0}{1+|x|}$, $\left|\frac{\partial u}{\partial t}\right| \leq \frac{C_1}{1+|x|}$ iv)

we may differentiate under the integral sign and conclude

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\int_{\mathbb{R}} u(x,t)\,\bar{u}(x,t)\,dx = \int_{\mathbb{R}}\left[\frac{\partial u}{\partial t}\bar{u} + u\frac{\partial \bar{u}}{\partial t}\right]dx =$$

$$= \left\{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ ii)}\right\} = \int_{\mathbb{R}}\left[\frac{\partial^2 u}{\partial x^2}\bar{u} + u\frac{\partial^2 \bar{u}}{\partial x^2}\right]dx = \left\{\begin{matrix}\text{int}\\ \text{by}\\ \text{parts}\end{matrix}\right\} =$$

$$= -\int_{\mathbb{R}} 2\frac{\partial u}{\partial x}\frac{\partial \bar{u}}{\partial x}\,dx = -2\int_{\mathbb{R}}\left|\frac{\partial u}{\partial x}\right|^2 dx \leq 0.$$

$\boxtimes$

Remark: This is a very simple technique, but also very versatile.

I) ~~It is~~ The same calculation (without the assumption $u(x,0)=0$) leads to

$$\int_{\mathbb{R}} |u(x,T)|^2 dx + 2\int_0^T\int_{\mathbb{R}}\left|\frac{\partial u}{\partial x}\right|^2 dx \leq \int_{\mathbb{R}} |u(x,0)|^2 dx$$

and it works for a wide class of complicated PDE.

II) Obviously, this is important since if we have two solutions of the Heat eq u, v s.t. $u(x,0) = v(x,0)$ then $u-v=0$ and thus $u=v$.
So we have uniqueness for solutions vanishing at infinity.

Heisenberg's uncertainty principle.

We know that if f is a function of moderate decrease and f' is of moderate decrease then

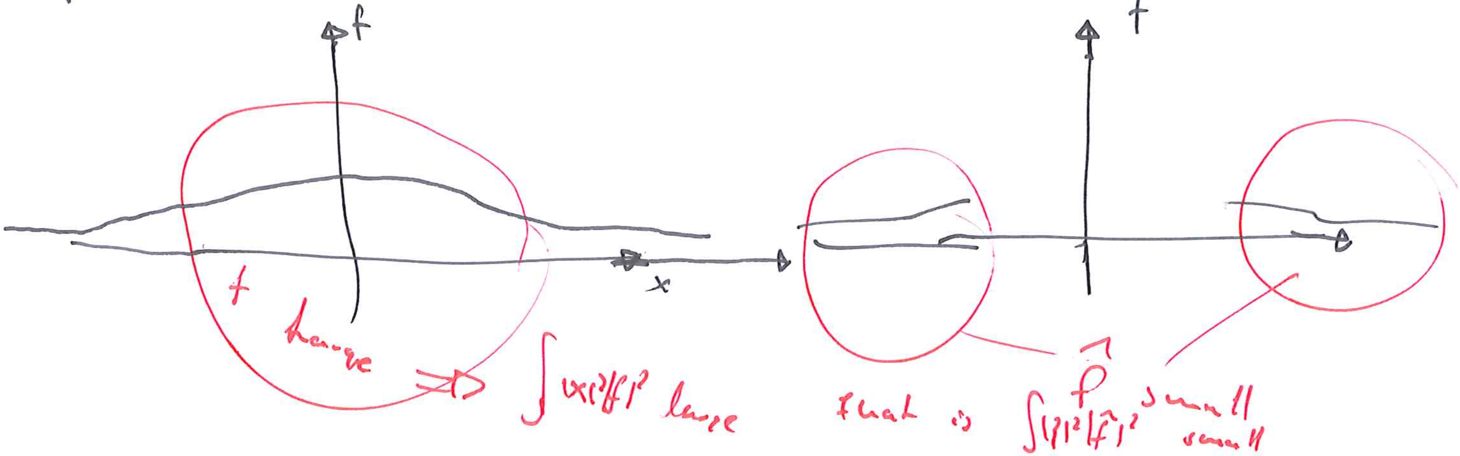
$$C \geq \int_{\mathbb{R}} |f'|^2 = \int_{\mathbb{R}} |\widehat{f'}|^2 \, d\xi = \underbrace{4\pi \int_{\mathbb{R}} |\xi|^2 |\hat{f}|^2 \, d\xi}_{\text{can only be bounded if } |\hat{f}| \text{ is small for very large } \xi}$$

Similarly if $f \in C^k$ ~~then~~ and all calculations work then

$$\int_{\mathbb{R}} |\xi|^{2k} |\hat{f}|^2 \, d\xi < \infty.$$

This is the old relation $f \in C^k \Rightarrow |\hat{f}(\xi)| \leq \frac{C}{|\xi|^k}$ again. Can we formulate this differently ~~precisely~~?

There are many ways. For instance if the "derivatives of f are small" then "f is almost constant" on intervals



So there is some relation $\int_{\mathbb{R}} |\xi|^2 |\hat{f}|^2 d\xi$ small $\Rightarrow$ $\int_{\mathbb{R}} |x|^2 |f|^2 dx$ large. Can we quantify this?

Thm: (Heissenberg's unc. prin.) Suppose $\psi \in \mathcal{S}(\mathbb{R})$ and $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$ then

$$\left(\int_{\mathbb{R}} x^2 |\psi|^2 dx \right) \Big(\overbrace{\int_{\mathbb{R}} |\xi|^2 |\hat{f}|^2 d\xi}^{\text{Involves a derivative}} \Big) \geq \frac{1}{16\pi^2}$$

with equality iff $\psi = A e^{-Bx^2}$.

observe that this is more general than $f^{(k)} \subseteq C \Rightarrow |\hat{f}| \leq \frac{C}{|\xi|^k}$

Proof: We assume

$$1 = \int_{\mathbb{R}} |\psi(x)|^2 dx = \left\{ \begin{array}{l} \text{let us create} \\ \text{a derivative} \end{array} \right\} = \int_{\mathbb{R}} \frac{dx}{dx} |\psi(x)|^2 dx =$$

$$= - \int_{\mathbb{R}} x \psi(x) \bar{\psi}'(x) + x \psi'(x) \bar{\psi}(x) dx \leq 2 \int_{\mathbb{R}} |x| |\psi| |\psi'| dx$$

$$\leq \left\{ \begin{array}{l} \text{Cauchy} \\ \text{Schwartz} \end{array} \right\} \leq 2 \left(\int_{\mathbb{R}} |x|^2 |\psi|^2 dx \right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{R}} |\psi'|^2 dx \right)^{\frac{1}{2}}}_{= \left\{ \begin{array}{l} \text{Plancherel's} \\ \text{formula} \end{array} \right\} = \left(4\pi \int |\xi|^2 |\psi'|^2 d\xi \right)^{\frac{1}{2}}} = 4\pi (\quad)^{\frac{1}{2}} (\quad)^{\frac{1}{2}}$$

$\blacksquare$

Fourier series in $\mathbb{R}^n$

Notice that if $f(x) \in \mathcal{S}(\mathbb{R}^n)$, that is f infinitely differentiable ($f \in C^\infty$) and for each $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$ we have

$$\sup_{x \in \mathbb{R}^n} |x|^k \left| \frac{\partial^{|\alpha|} f(x)}{\partial x^\alpha} \right| < \infty$$

where

$$\frac{\partial^{|\alpha|} f(x)}{\partial x^\alpha} \equiv \frac{\partial^{|\alpha|} f \quad (x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. then the following is well defined

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx =$$

$$\int_{\mathbb{R}} e^{-2\pi i x_1 \xi_1} \left[\int_{\mathbb{R}} \left[\dots e^{-2\pi i x_{n-1} \xi_{n-1}} \left[\underbrace{\int_{\mathbb{R}} e^{-2\pi i x_n \xi_n} f(x', x_n) dx_n}_{\text{function in } (x_1, x_2, \dots, x_{n-1}, \xi_n)} \right] \right] \right]$$

function in $(x_1, x_2, \dots, x_{n-2}, \xi_{n-1}, \xi_n)$

function in $(x_1, \xi_2, \xi_3, \dots, \xi_n)$

Corollary. (To prop 2.1) The fourier transform maps $S(\mathbb{R}^n)$ to itself.

Proof: We need to show that

$$\sup_{\xi\in\mathbb{R}^n} |\xi|^k \left|\frac{\partial^{|\alpha|}\hat{f}(\xi)}{\partial\xi^\alpha}\right| < \infty.$$

But

$$\left|\xi_i^k \frac{\partial^{|\alpha|}\hat{f}(\xi)}{\partial\xi^\alpha}\right| = \left|\widehat{\underbrace{\frac{(2\pi i)^{|\alpha|}}{(2\pi i)^k} \frac{\partial^k}{\partial x_i^k}\left(x^\alpha f(x)\right)}_{\text{of moderate decrease}}}\right|$$

so $\left|\int \frac{\partial}{\partial x_i^k}\left(x^\alpha f(x)\right) \underbrace{e^{-2\pi i x\cdot\xi}}_{|\cdot|\le 1} dx\right| \le C$

for all ξ

Furthermore we may freely change the order of integration and integrate in x_i first etc.

Since, the Fourier transform in $\mathbb{R}^n$ is built up on the 1-d counterpart most results for the 1-d case are applicable to the $\mathbb{R}^n$-case.

In particular:

Prop 2.1 Let $f \in S(\mathbb{R}^n)$ then

i) $\widehat{f(x+h)} = e^{2\pi i \xi \cdot h} \hat{f}(\xi)$

ii)

⋮

v) For any multiindex $\alpha \in \mathbb{N}^n$

$$\widehat{(-2\pi i x)^\alpha f(x)} = \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} \hat{f}(\xi).$$

vi) ~~Whenever~~ Let R be a rotation of $\mathbb{R}^n$ (say a matrix s.t. $Re_i = \tilde{e}_i$ where $\tilde{e}_i$ is an orthonormal basis of $\mathbb{R}^n$)

then
$$\widehat{f(Rx)} = \hat{f}(R\xi).$$

Proof: i) - v) are the same as before

To prove vi) we notice that

$$\widehat{f(Rx)} = \int_{\mathbb{R}^n} f(Rx)\, e^{-2\pi i x\cdot\xi}\, dx = \left\{\begin{array}{l}\text{change of variables}\\ y = Rx\end{array}\right\} =$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i (R^{-1}y)\cdot(\xi)}\, \cancel{dy}\, \underbrace{\det(R)}_{=1}\, dy =$$

$$= \left\{\begin{array}{l}\text{Now } R^{-1}y\cdot\xi = \left[R(R^{-1}y)\right]\cdot\left[R\xi\right] = y\cdot R\xi\\ \text{since the scalar product is independent of}\\ \text{rotations}\end{array}\right\} =$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i\, y\cdot(R\xi)}\, dy = \hat{f}(R\xi).$$

$\blacksquare$

Thm 2.4. Suppose $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi. \qquad \text{(Fourier inversion)}$$

And

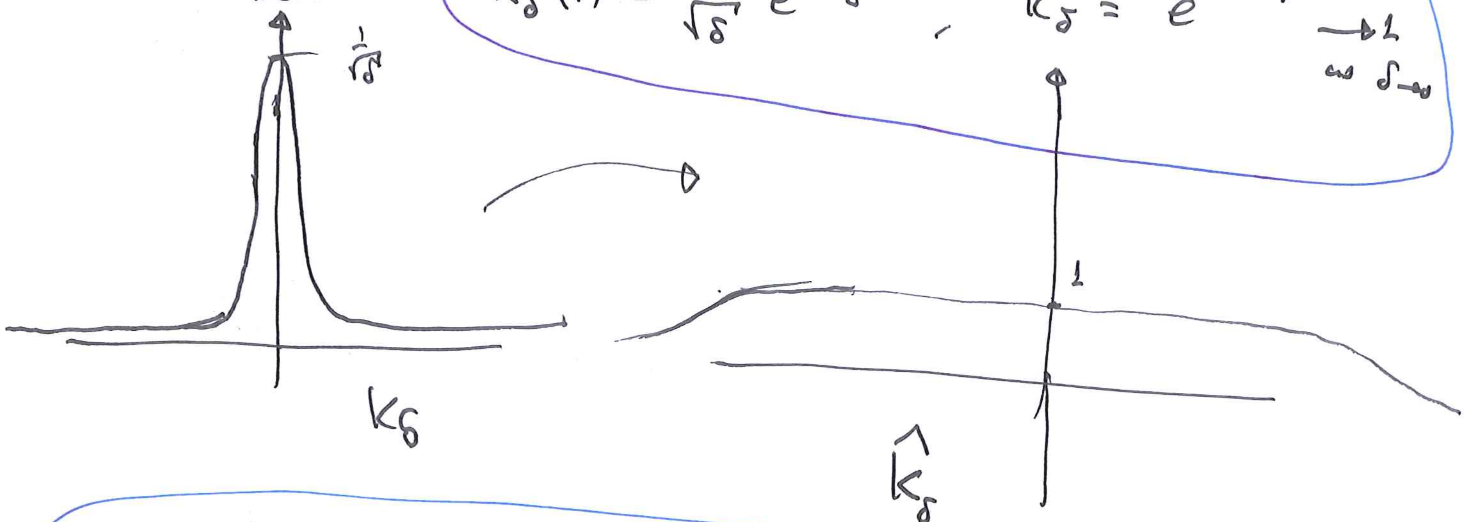
$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} |f(x)|^2 \, dx \qquad \text{(Plancherel's Thm)}$$

The idea in $\mathbb{R}$.

Remember that we proved the inversion in $\mathbb{R}$ by the marvelous properties of the Gaussian kernels

1

$$K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x^2}{\delta}}, \qquad \hat{K}_\delta = e^{-\pi \delta \xi^2} \underset{\text{as } \delta \to 0}{\longrightarrow} 1$$



K_δ $\qquad$ $\hat{K}_\delta$

$K_\delta(x)$ is a family of good kernels. 2

$$f(0) \underset{\text{as } \delta \to 0}{\longleftarrow} \int_{\mathbb{R}} f(x) K_\delta(x) \, dx = \int_{\mathbb{R}} f(x) \underbrace{\hat{G}_\delta(x)}_{\hat{G}_\delta(x) = K_\delta(x)} dx =$$

That is $G_\delta(x) = \hat{K}_\delta(-x) = \hat{K}_\delta(x)$

3 Multiplication formula

$$= \int_{\mathbb{R}} f(x) \hat{K}_\delta(x) \, dx = \int_{\mathbb{R}} \hat{f} \, \hat{K}_\delta \, dx = \int_{\mathbb{R}} \hat{f} \underbrace{e^{-\pi\delta|x|^2}}_{\to 1} dx \to \int_{\mathbb{R}} \hat{f}(\xi) \, d\xi$$

Then we called $f(x+y) = F(y)$ so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi)\, d\xi = \Bigg\{ \underbrace{\widehat{f(x+y)} = e^{2\pi i x \xi} \hat{f}(\xi)}_{\text{translation formula}} \Bigg\} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi.$$

4

So we need to prove

1) $$K_\delta(x) = \frac{1}{\delta^{n/2}} e^{-\frac{\pi x^2}{\delta^2}} \Rightarrow \hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$$

$$\int_{\mathbb{R}^n} \frac{1}{\delta^{n/2}} e^{-\pi\frac{(x_1^2+x_2^2+\dots x_n^2)}{\delta^2}} e^{-2\pi i (x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n)}\, dx = \Big\{ K_\delta \in S(\mathbb{R}^n)$$

$$= \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x_1^2}{\delta^2}} e^{-2\pi i x_1 \xi_1} dx_1}_{= e^{-\pi\delta\xi_1^2}} \cdot \underbrace{\int_{\mathbb{R}} \quad dx_2}_{= e^{-\pi\delta\xi_2^2}} \cdots \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x_n^2}{\delta^2}} e^{-2\pi i \dots} dx_n}_{= e^{-\pi\delta\xi_n^2}}$$

$$= e^{-\pi\delta(\xi_1^2+\xi_2^2+\dots+\xi_n^2)} = e^{-\pi\delta|\xi|^2}$$

2) Need to show

$$\int_{\mathbb{R}^n} K_\delta(x)\, dx = 1 \qquad (1)$$

$$\int_{\mathbb{R}^n} |K_\delta(x)|\, dx \le 1 \qquad (2)$$ Follows from 1 since $K_\delta > 0$

$$\int_{|x|>\eta} |K_\delta(x)|\, dx \to 0 \quad \text{as } \delta \to 0 \qquad (3)$$ Simple.

3/ Multiplication formula $\int_{\mathbb{R}^n} f(x)\,\hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x)\,g(x)\,dx.$

But this was done by a change of order of integration which is valid if $f, g \in S(\mathbb{R}^n)$

4) Translation formula, mentioned in the beginning of this lecture..

So the inversion theorem (that is its proof) goes through with very small changes in $\mathbb{R}^n$.

The Plancherel's theorem is proved in a similar fashion as the 1d case.