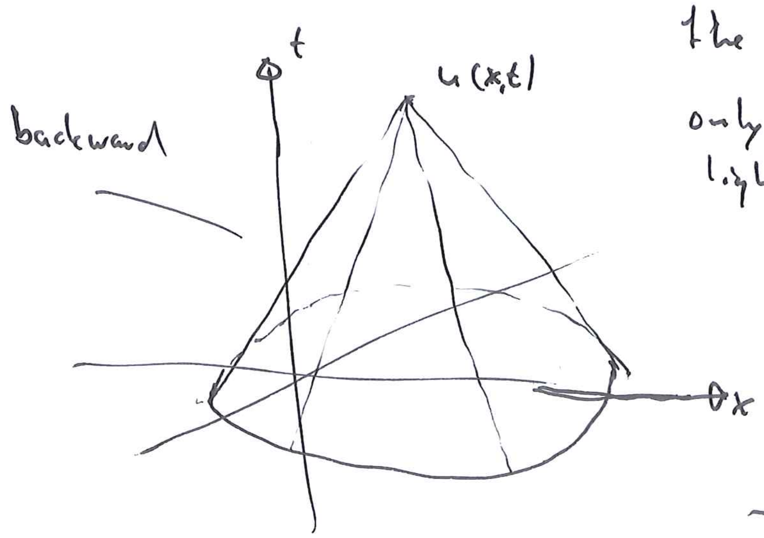
Last weak we saw that (in the notation of the book) if

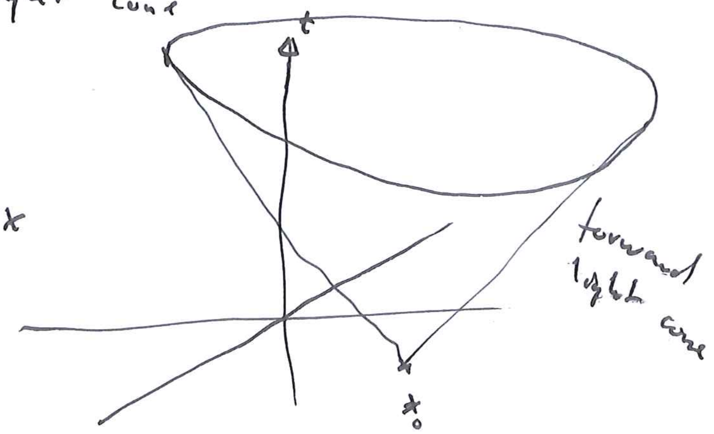
$$M_t(f)(x) = \frac{1}{4\pi} \int_{\underbrace{S^2}_{\text{sphere in } \mathbb{R}^3}} f(x-t\gamma)\, \underbrace{d\sigma(\gamma)}_{\text{spherical measure}}$$

then

$$\left.\begin{aligned} & u(x,t) = \frac{\partial}{\partial t}\left(t M_t(f)(x)\right) + t M_t(g)(x) \\ & \text{solves} \\ & \frac{\partial^2 u}{\partial t^2} = \Delta u \qquad \text{in} \quad \mathbb{R}^3 \times \mathbb{R} \\ & u(x,0) = f(x) \in S(\mathbb{R}^3) \\ & \frac{\partial u(x,0)}{\partial t} = g(x) \in S(\mathbb{R}^3) \end{aligned}\right\} \text{In } \mathbb{R}^3$$

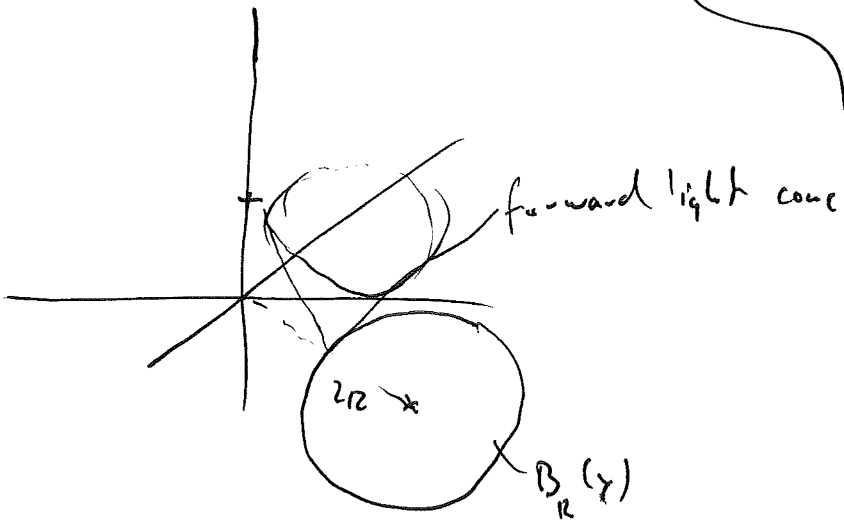
We also noticed that this implies that u only depend on its backward light cone, and that the value of $f(x_0)$, $g(x_0)$ only influence the forward light cone





x is in $\mathbb{R}^3$ (even though' drawn in $\mathbb{R}^2$)

Notice that if we change the values of $f(x), g(x)$ in some ball $B_R(y)$ $|y| = 2R$ then this change will not affect the solution at $x=0$ before the time $t=R$. We call this "finite speed of propagation".



We can use the finite speed of propagation to create solutions to the wave equation in $\mathbb{R}^2 \times \mathbb{R}$ as follows. Consider the problem

$$\frac{\partial^2 u(x_1, x_2, t)}{\partial t^2} = \Delta u(x_1, x_2, t)$$

$$u(x_1, x_2, 0) = f(x_1, x_2) \in S(\mathbb{R}^2)$$

$$\frac{\partial u(x_1, x_2, 0)}{\partial t} = g(x_1, x_2) \in S(\mathbb{R}^2)$$

If we want to calculate $u(x_1^0, x_2^0, t^0)$

then we can consider u as a function in $\mathbb{R}^3 \times \mathbb{R}$ $\tilde{u}$

and calculate $\tilde{u}(x_1, x_2, 0, t)$ instead, where

$$(*)\begin{cases} \dfrac{\partial^2 \tilde{u}}{\partial t^2} = \Delta \tilde{u} & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \tilde{u} = \tilde{f}(x_1, x_2, x_3) \notin S(\mathbb{R}^3) & \\ \dfrac{\partial \tilde{u}}{\partial t} = \underbrace{\tilde{g}(x_1, x_2, x_3)}_{\text{independent of } x_3} \notin S(\mathbb{R}^3) & \end{cases}$$

$$\underbrace{\tilde{f}(x_1, x_2, x_3)}_{\text{independent of } x_3} = f(x_1, x_2)$$

We have a formula for the solutions to $(*)$, if $\tilde{f}$ and $\tilde{g}$ where in $S(\mathbb{R}^3)$. To get around this we consider $\tilde{f}, \tilde{g}$ to be defined

$$\tilde{f}(x_1, x_2, x_3) = f(x_1, x_2)\, e(x_3)$$

$$\tilde{g}(x_1, x_2, x_3) = g(x_1, x_2)\, e(x_3)$$

where
$$e(x_3) = \begin{cases} 1 & \cancel{|x_1|+|x_2|},\ |x_3| \leq 100\left(|x_1^0| + |x_2^0| + |t^0|\right) \\ 0 & |x_3| \geq 200\left(|x_1^0| + |x_2^0| + |t^0|\right) \end{cases}$$

and $e \in C^\infty(\mathbb{R})$.

Then $\tilde{f}, \tilde{g} \in S(\mathbb{R}^3)$ and that we change the values for $\tilde{f}, \tilde{g}$ when $|x_3| > 100\left(|x_1^0| + |x_2^0| + |t^0|\right)$ doesn't affect the solution $\tilde{u}(x_1^0, x_2^0, 0, t^0)$ (whatever that means)

So we may write

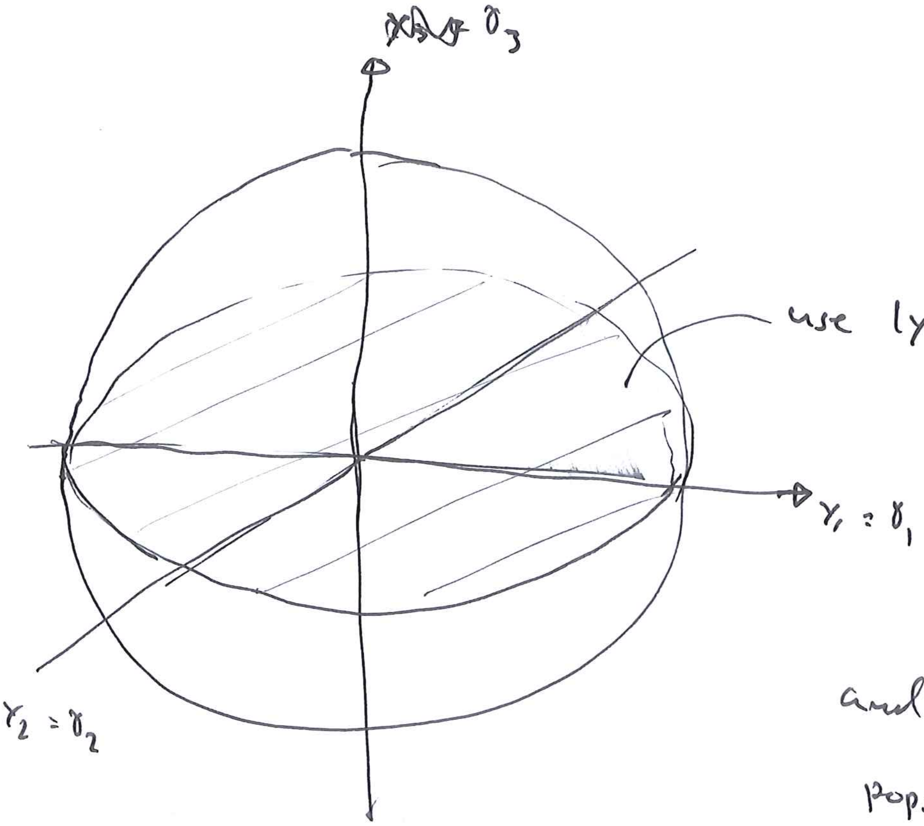
$$\tilde{u}(x_1^\circ, x_2^\circ, 0, t^\circ) = \frac{\partial}{\partial t}\left(t_0 M_{t_0}(\tilde{f})(x^\circ)\right) + t_0 M_{t_0}(\tilde{g})(x^\circ) =$$

$$= \frac{\partial}{\partial t}\left(\frac{t_0}{4\pi} \int_{S^2} f(\underbrace{\bar{x}^\circ - \gamma' t_0}_{\bar{x} = (x_1, x_2)}) \overbrace{\varrho(x_3^\circ - \gamma_3 t_0)}^{=1} \, d\sigma(\gamma) \right) + \frac{t_0}{4\pi} \int_{S^2} g(\bar{x}^\circ - \gamma' t_0) \overbrace{\varrho(x_3^\circ - \gamma_3 t_0)}^{=1} \, d\sigma(\gamma)$$

independent of x_3 — independent of x_3

$$= \left\{ \begin{array}{l} \text{Polar} \\ \text{coordinates} \\ \text{\& calculation} \end{array} \right\} = \underbrace{\frac{\partial}{\partial t}\left(\frac{t_0}{2\pi} \int_{|y| \le 1} f(x^\circ - t_0 y)(1 - |y|^2)^{-\frac{1}{2}} dy \right) + \frac{t_0}{2\pi} \int_{|y| < 1} g(x^\circ - t_0 y)(1 - |y|^2)^{-\frac{1}{2}} dy}_{1}$$

→ We integrate over S^2



use $|y| < 1$, $y = (y_1, y_2)$ to parametrize the upper and lower half spheres and the integral pops out.

$$\boxed{\text{Thm. The function } (1) \text{ solves the wave equation in } \mathbb{R}^2 \times \mathbb{R}}$$

Remarks. Notice that whereas in $\mathbb{R}^3 \times \mathbb{R}$

$u(x,t)$ only depend on the boundary of the light cone

in $\mathbb{R}^2 \times \mathbb{R}$ ~~the~~ $u(x,t)$ will depend on the solid

light cone.

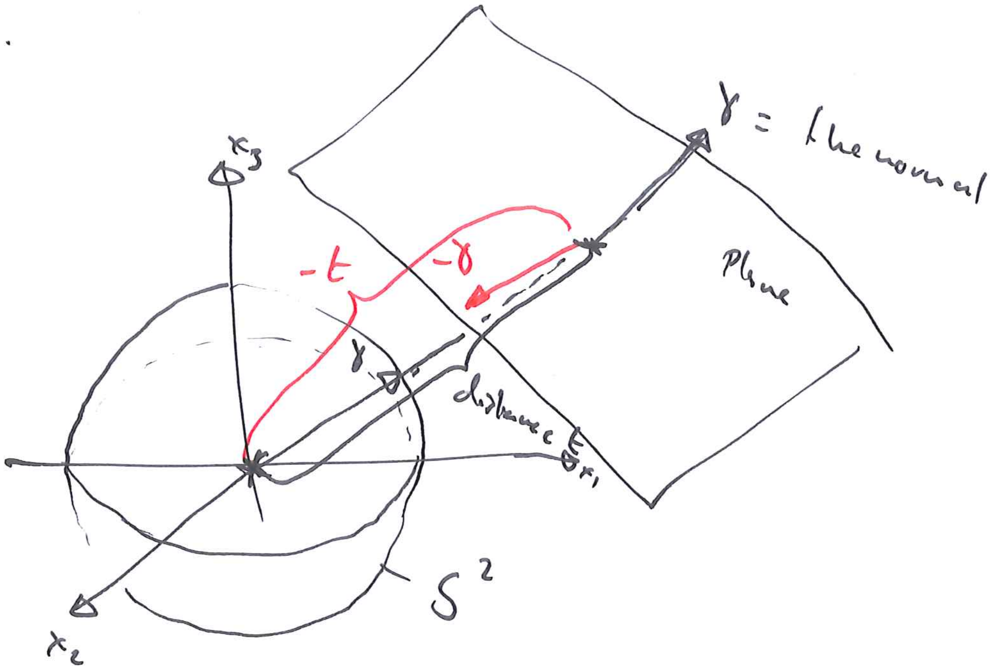
The Radon Transform.

Assume that we have some material in $\mathbb{R}^3$ with unknown density $f(x) \in S(\mathbb{R}^3)$. Assume furthermore that we have some device to measure $\overbrace{\int_{\text{plane}} f(x)\,dx}^{\text{known}}$, where the integral is taken over any plane. Can we reconstruct $f(x)$ from the data?

Think of $f(x)$ being some body $K \subset \mathbb{R}^3$ and $\int_{\text{plane}} f(x)\,dx$ as some kind of x-ray scanning the body in the plane

We need to formulate this mathematically, which in this case is basically to introduce notation.

We will parametrize the planes in $\mathbb{R}^3$ according to $\mathbb{R} \times S^2$.



So (t, γ) = the plane with normal $\gamma \in S^2$ and distance t (negative distance in the $-\gamma$ direction) from the origin.

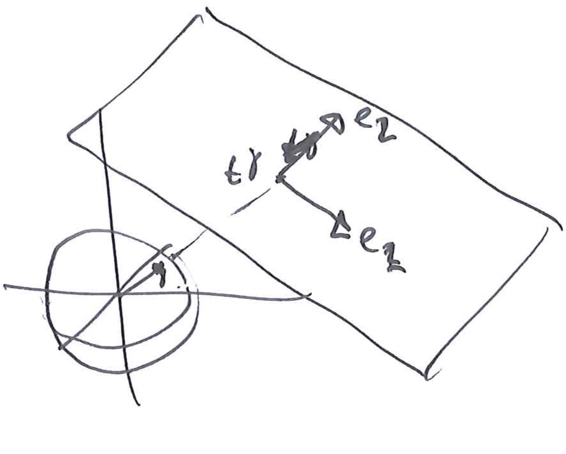
We will denote this plane $P_{t,\gamma}$.

The integral of f over $P_{t,\gamma}$ will be denoted

$$\int_{P_{t,\gamma}} f = \int_{\mathbb{R}^2} f(t\gamma + u_1 e_1 + u_2 e_2)\, du_1 du_2$$

where (γ, e_1, e_2) is an orthonormal basis for $\mathbb{R}^3$

So $\gamma + u_1 e_1 + u_2 e_2$ is a point in the plane



Notice that $\int_{P_{t,\gamma}} f$ is indep of the choice e_1 & e_2

Now we define the Radon transform.

$$\mathcal{R} : \mathcal{S}(\mathbb{R}^3) \longrightarrow \text{functions on } \mathbb{R} \times S^2$$

$$\mathcal{R}(f)(t,\gamma) = \int_{P_{t,\gamma}} f.$$

Question: 1) Given $\mathcal{R}(f)(t,\gamma)$ [Notice that this is just a function on $\mathbb{R} \times S^2$] can we reconstruct f? What assumptions on $\mathcal{R}(f)$ and f?

2) Assume that $\mathcal{R}(f)(t,\gamma) = \mathcal{R}(g)(t,\gamma)$ does it follow that $f = g$?

We will start with 2). That is:

If $\displaystyle\int_{P_{t,\gamma}} f = \int_{P_{t,\gamma}} g$ for all planes $P_{t,\gamma}$.

(we assume that $f, g \in \mathcal{S}(\mathbb{R}^3)$) will $f(x) = g(x)$ for all x?

First we will investigate the relation between the fourier transforms of $\mathcal{R}(f)(t,\gamma)$ and $\hat{f}(\kappa)$?

the FT of

So, not the least because this is a Fourier analysis course we want to find a relation between the Fourier transform of

$\mathcal{R}(f)(t,\gamma)$

function on $\mathbb{R} \times S^2$

We haven't done any fourieranalysis on S^2 so the only thing we can do is to take the Fourier transform in $t \in \mathbb{R}$ leaving $\gamma \in S^2$ fixed.

Observe that we use that $t \in \mathbb{R}$ here and thus the awkward symmetry $P_{t,\gamma} = P_{-t,-\gamma}$ so t may have negative values

$f(x) \in S(\mathbb{R}^3)$ function on $\mathbb{R}^3$

~~To do the Fou~~

To take a Fouriertransform of $\mathcal{R}(f)(t,\gamma)$

we need $\mathcal{R}(f)(\cdot,\gamma) \in S(\mathbb{R})$ (or have moderate decrease in any case).

Notice that, for any k, l we have

$$|t|^l \left|\frac{\partial^k \mathcal{R}(f)(t,\gamma)}{\partial t^k}\right| \le |t|^l \int_{P_{t,\gamma}} \left|\frac{\partial^k f}{\partial t^l}(\gamma t + u_1 e_1 + u_2 e_2)\right| du_1\, du_2 \le \left\{\begin{array}{l}\text{since}\\ f \in S(\mathbb{R}^3)\end{array}\right\} \le$$

$$\le |t|^l \int_{P_{t,\gamma}} \frac{A_{k,2(l+3)}}{(1+|t|)^l(1+|u_1|^{l+3}+|u_2|^{l+3})}\, du_1\, du_2 \le \int \frac{A_{k,2(l+3)}}{(1+|u_1|^{l+3}+|u_2|^{l+3})}\, du_1\, du_2 \le C$$

for some $A_{k,2(l+3)}$. We thus have:

~~Theorem~~ Lemma: ~~For each There is a constant, independent of γ (but dependent on the constants in the definition of $f \in S(\mathbb{R}^3)$) s.t.~~ If $f \in S(\mathbb{R}^3)$,

$$\mathcal{R}(f)(t,\gamma) \in S(\mathbb{R}) \quad \text{in } t, \text{ uniformly in } \gamma$$

(But the constants depend on the constants for $f \in S(\mathbb{R}^3)$).

Lemma: Let $f \in S(\mathbb{R}^3)$ then

$$\hat{\mathcal{R}}(f)(s,\gamma) = \hat{f}(s\gamma).$$

Proof:

$$\hat{\mathcal{R}}(f)(s,\gamma) = \int_{-\infty}^{\infty} \left(\int_{P_{t\gamma}} f\right) e^{-2\pi i s t}\, dt = \int_{-\infty}^{\infty}\int_{\mathbb{R}^2} f(t\gamma + u_1 e_1 + u_2 e_2)\, \underbrace{e^{-2\pi i s t}}_{= e^{-2\pi i\, s\gamma\cdot(t\gamma + e_1 u_1 + e_2 u_2)}}\, du_1\, du_2\, dt$$

$$= \int_{-\infty}^{\infty}\int_{\mathbb{R}^2} f(\underbrace{t\gamma + u_1 e_1 + u_2 e_2}_{\xi})\, e^{-2\pi i\, s\gamma\cdot\overbrace{(t\gamma + e_1 u_1 + e_2 u_2)}^{}}\, \underbrace{du_1\, du_2\, dt}_{d\xi} = \int_{\mathbb{R}^3} f(\xi) e^{-2\pi i\, s\gamma\cdot\xi}\, d\xi = \hat{f}(s\gamma).$$

Corollary: If $f, g \in S(\mathbb{R}^3)$ and $\mathcal{R}(f)(t,\gamma) = \mathcal{R}(g)(t,\gamma)$ for all $(t,\gamma) \in \mathbb{R} \times S^2$. Then $f \equiv g$.

Proof: Since $\mathcal{R}(f) = \mathcal{R}(g)$ we have that

$$\hat{f}(t\gamma) = \hat{\mathcal{R}}(f)(t,\gamma) = \hat{\mathcal{R}}(g)(t,\gamma) = \hat{g}(t\gamma)$$

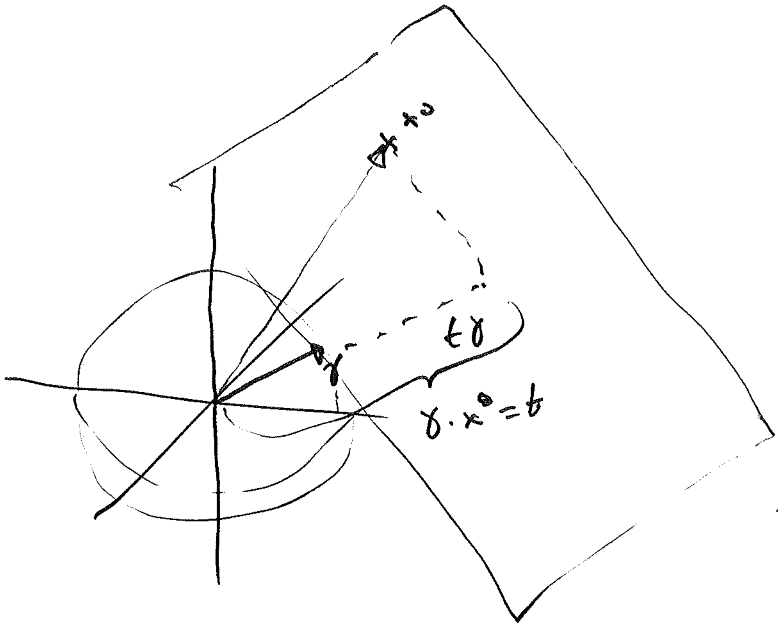
for all $(t,\gamma) \in \mathbb{R} \times S^2$. Thus $\hat{f} = \hat{g} \Rightarrow f = g$ by the fourier inversion. $\blacksquare$

Now we turn to the more difficult problem: If $f \in S(\mathbb{R}^3)$ and we know what $\mathcal{R}(f)(t,\gamma)$ is. Can we then calculate f?

How would we proceed? Pick an $x^0 \in \mathbb{R}^3$, how can we get the maximum information about $f(x^0)$ from $\mathcal{R}(f)(t,\gamma)$.

Since $\mathcal{R}(f)$ involves an integral, we can not deduce pointwise information about f directly!

But if $x^0 \in P_{t,\gamma}$ then $\mathcal{R}(f)(t,\gamma)$ contains some information of $f(x^0)$ - or actually since f is continuous $f(x^0)$ will influence the value of $\mathcal{R}(f)(t,\gamma)$ if $x^0 \in P_{t,\gamma}$, that is if $x^0 \cdot \gamma = t$



$$\mathcal{P}_{t,\gamma} = \mathcal{P}_{x^0\cdot\gamma,\gamma}$$

So let us take all the planes that intersect x^0, that is all the planes $\mathcal{P}_{x^0\cdot\gamma,\gamma}$ for $\gamma \in S^2$ and integrate:

Definition: We ~~denote~~ define the dual Radontransform

$$\mathcal{R}^*(F)(x) = \int_{S^2} F(x\cdot\gamma, \gamma)\, d\sigma(\gamma)$$

for every function $F: \mathbb{R} \times S^2 \to \mathbb{R}$

In partikular ~~the idea~~

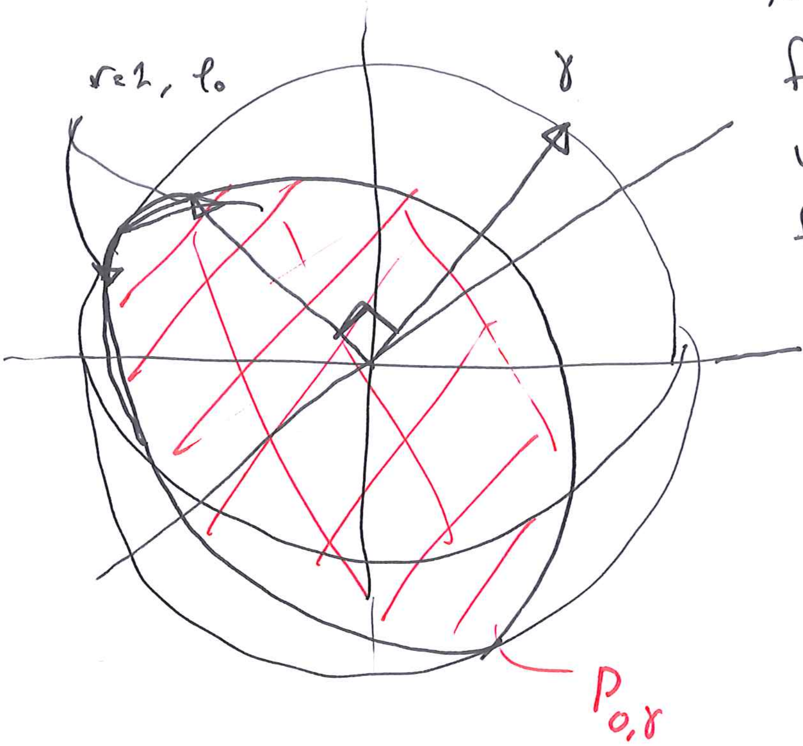
$$\mathcal{R}^*(\mathcal{R}(f))(x) = \int_{S^2} \mathcal{R}(f)(x\cdot\gamma, \gamma)\, d\sigma(\gamma)$$

Should contain information about $f(x)$.

Notice that, by definition,

$$\mathcal{R}^*(\mathcal{R}(f))(x) = \left\{\begin{matrix}\text{lets} \\ \text{translate} \\ \text{so} \\ 0 = x\end{matrix}\right\} = \int_0^{2\pi}\int_0^{\pi}\left[\int_0^{2\pi}\int_0^{\infty} f(\cancel{\cdot})\, r\, dr\, d\varphi\right]\cos\theta\, d\theta\, d\gamma \qquad (*)$$

So for every $\gamma \in S^2$ we can fix (θ, γ).



Now if we instead fix φ and r and vary (θ, γ) then, for each $\varphi \in [0, 2\pi)$ we will cover the sphere of radius r once.

So (~~this~~ is a change of order of integration)

$$(*) = \int_0^{2\pi}\underbrace{\left[\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{f}{r}\right) r^2 \cos\theta\, dr\, d\theta\, d\gamma\right]}_{= \int_{\mathbb{R}^3} \frac{f(y)}{|y|}\, dy} d\varphi = 2\pi \int_{\mathbb{R}^3} \frac{f(y)}{|y|}\, dy =$$

$$= -8\pi^2 f * N(0) \quad \text{where} \quad N(x) = \frac{-1}{4\pi|x|}.$$

[N is called the Newtonian kernel]

So $\mathcal{R}^*(\mathcal{R}(f))(x) = -8\pi^2 f * N(x). = 2\pi \int_{\mathbb{R}^3} f(y) \frac{1}{|x-y|}\, dy$

So $\mathcal{R}^*(\mathcal{R}(f))$ is a convolution operator.
We are now ready to prove the main thm of Radon transforms

Thm: If $f \in \mathcal{S}(\mathbb{R}^3)$, then

$$\Delta\left(\mathcal{R}^* \mathcal{R}(f)\right) = -8\pi^2 f.$$

Corollary: Let $f \in \mathcal{S}(\mathbb{R}^3)$ and define

$$u(x) = f * N(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} f(y)\frac{1}{|x-y|}\,dy$$

Then $u(x)$ is a solution to

$$\Delta u(x) = f(x)$$

Proof of the theorem:

Since $\hat{\mathcal{R}}(f)(t,\gamma) = \hat{f}(t\gamma)$ we have by fourier inversion

$$\mathcal{R}(f)(t,\gamma) = \int_{-\infty}^{\infty} \hat{f}(s\gamma)\, e^{2\pi i t s}\, ds \qquad \text{so}$$

$$\mathcal{R}^*(\mathcal{R}(f))(x) = \int_{S^2}\int_{-\infty}^{\infty} \hat{f}(s\gamma)\, e^{2\pi i \underbrace{x\cdot\gamma}\, s}\, ds\, d\sigma(\gamma)$$

Remember

$$\mathcal{R}^*(F) = \int_{S^2} F(\overbrace{\underbrace{x\cdot\gamma}_{t\text{ slot}}}, \gamma)\, d\sigma(\gamma)$$

We may now calculate

$$\Delta \mathcal{R}^{\#}(\mathcal{R}(f)) = \int_{S^2}\int_{-\infty}^{\infty} \hat{f}(s\gamma)\,(-4\pi^2 s^2)\, e^{2\pi i x\cdot\gamma s}\, ds\, d\sigma(\gamma) = \left\{ \cancel{\text{since}} \right\}$$

$$= 2\int_{S^2}\int_{0}^{\infty} (-4\pi^2)\, \hat{f}(s\gamma)\, s^2 e^{2\pi i x\cdot\gamma s}\, ds\, d\sigma(\gamma) = \left\{ \begin{matrix}\text{cartesian}\\ \text{coordinates}\end{matrix} \right\} =$$

$$= -8\pi^2 \int_{\mathbb{R}^3} \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi = \left\{ \begin{matrix}\text{Fourier}\\ \text{inversion}\end{matrix} \right\} = -8\pi^2 f(x).$$

$\blacksquare$