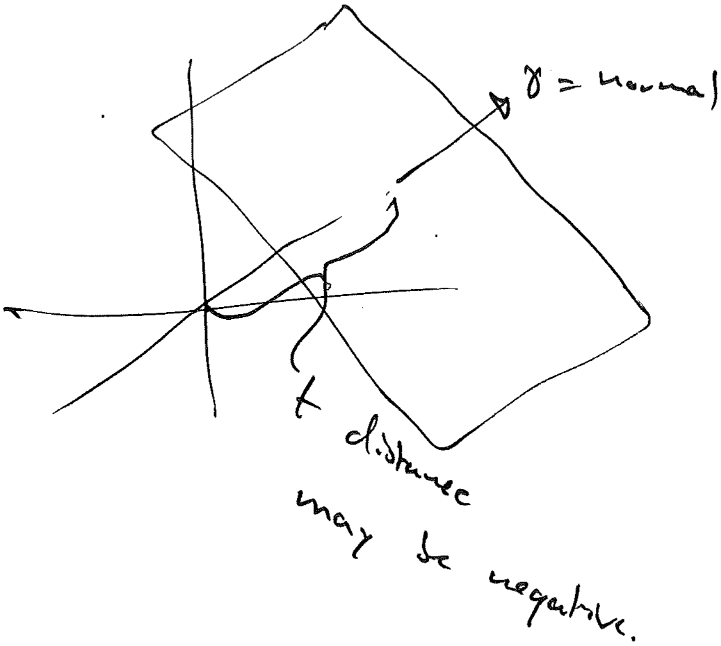
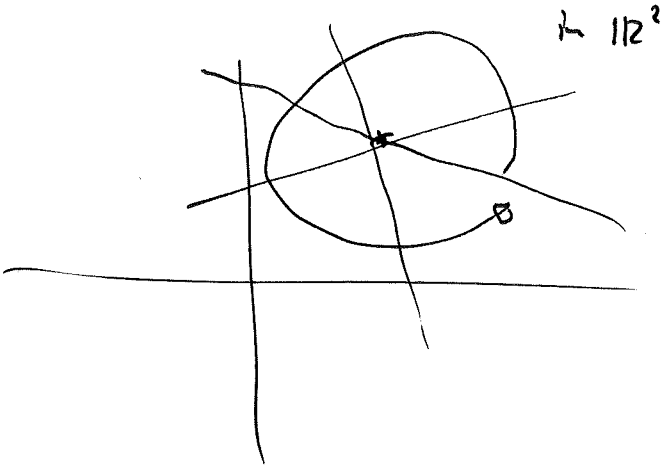
Last week we defined the Radon-transform:
Let $f \in S(\mathbb{R}^3)$

$R(f)(t,\gamma) = \int_{P_{t,\gamma}} f$ = integration over the plane $P_{t,\gamma}$



We also ~~show~~ defined the dual Radon transform

$R^*(F(t,\gamma))(x) = \int_{S^2} F(x\cdot\gamma,\gamma)\, d\sigma(\gamma)$ = integral over all planes containing x.



We also indicated that

$$\mathcal{R}^{*}(\mathcal{R}(f))(x) = 2\pi \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^2}\, dy = -8\pi^2 \oint f * N(x)$$

where $N(x) = \dfrac{1}{4\pi |x|}$.

Finally we showed that

$$\underbrace{\hat{\mathcal{R}}(f)(t,\gamma)}_{\text{Fourier transform in } t} = \hat{f}(t\gamma).$$

So $\mathcal{R}^*(\mathcal{R}(f))$ is a convolution operator.

We are now ready to prove the main thm of Radon transforms

Thm: If $f \in \mathcal{S}(\mathbb{R}^3)$, then

$$\Delta\left(\mathcal{R}^* \mathcal{R}(f)\right) = -8\pi^2 f.$$

Corollary: Let $f \in \mathcal{S}(\mathbb{R}^3)$ and define

$$u(x) = f * N(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} f(y)\frac{1}{|x-y|}\,dy$$

then $u(x)$ is a solution to

$$\Delta u(x) = f(x)$$

Proof of the theorem:

Since $\hat{\mathcal{R}}(f)(t,\gamma) = \hat{f}(t\gamma)$ we have by fourier inversion

$$\mathcal{R}(f)(t,\gamma) = \int_{-\infty}^{\infty} \hat{f}(s\gamma)\, e^{2\pi i t s}\, ds \qquad \text{so}$$

$$\mathcal{R}^*(\mathcal{R}(f))(x) = \int_{S^2}\int_{-\infty}^{\infty} \hat{f}(s\gamma)\, e^{2\pi i\, x\cdot\gamma\, s}\, ds\, d\sigma(\gamma)$$

Remember

$$\mathcal{R}^*(F) = \int_{S^2} F(\underbrace{x\cdot\gamma}_{t \text{ slot}}, \gamma)\, d\sigma(\gamma)$$

Thus

$$\Delta\left(\mathcal{R}^*(\mathcal{R}(f))(x)\right) = \int_{S^2}\int_{-\infty}^{\infty} -4\pi^2 s^2 \hat{f}(s\gamma)\, e^{2\pi i x\cdot\gamma s}\, ds\, d\sigma(\gamma) =$$

$$= -8\pi^2 \underbrace{\int_{S^2}\int_{0}^{\infty}} \hat{f}(s\gamma)\, e^{2\pi i x\cdot\gamma s}\, \underbrace{s^2\, ds\, d\sigma(\gamma)} = \left\{ \begin{array}{c} s\gamma = \xi \\ \text{in cartesian} \end{array} \right\}$$

integration in polar coordinates

$$= -8\pi^2 \int_{\mathbb{R}^3} \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi = \left\{ \begin{array}{c} \text{Fourier} \\ \text{inversion} \end{array} \right\} = -8\pi^2 f(x).$$

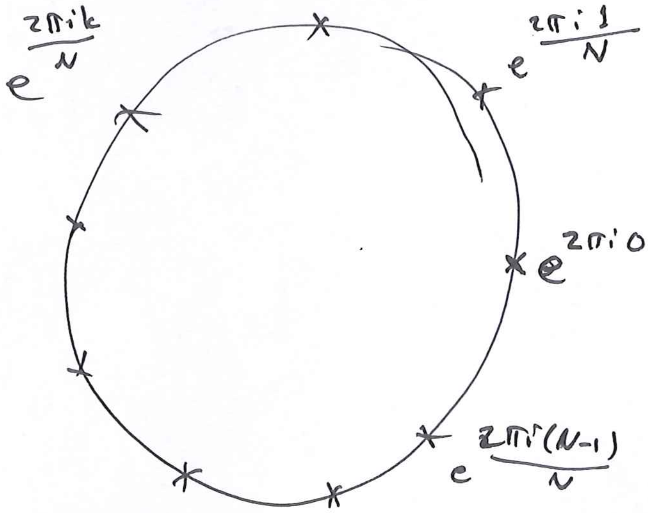
$\blacksquare$

Finite Fourier analysis.

Now we are going to extend the ideas of fourier analysis to finite sets.

~~The Fourier series~~ We remarked that the Fourier series can be seen as a way to view the space of square differentiable functions ($f(x)$ s.t $\int_{-\pi}^{\pi}|f(x)|^2dx<\infty$) as an infinite dimensional vector space with basis $e^{2\pi i n x}$ $n\in\mathbb{Z}$ and inner product $(f,g)=\int_{-\pi}^{\pi} f\bar{g}\,dx$.

We can naturally try to consider a similar basis on a finite dimensional vector space consisting of N points $\mathbb{Z}_N=\{0,1,\dots,N-1\}$. We may view this as discrete points on the unit circle.



The space of (complex valued) functions on $\mathbb{Z}_N$ is an N-dimensional vector space, with basis

$$\underbrace{\hat{e}_l(k)=\begin{cases}1 & l=k\\ 0 & \text{else}\end{cases}}_{\text{Observe that the basis consists of functions.}}$$

So we may write

$$f(\cdot)=\sum_{l=0}^{N-1} f(l)\,\hat{e}_l(\cdot).$$

We get the inner product $(f,g)=\sum_{k=0}^{N-1} f(k)\bar{g}(k)$

We could also choose another basis

$$e_l(k) = e^{\frac{2\pi i l k}{N}} \qquad \text{for} \quad l = 0, \dots, N-1.$$

It is easy to verify

Lemma 1.1 The family $\{e_0, e_1, \dots, e_{N-1}\}$ is orthogonal

$$(e_l, e_m) = \begin{cases} N & m = l \\ 0 & m \neq l \end{cases}$$

Proof: We calculate

$$(e_m, e_l) = \sum_{k=0}^{N-1} e^{\frac{2\pi i m k}{N}} \overline{e^{\frac{2\pi i l k}{N}}} = \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(m-l)k} =$$

$$= \begin{cases} \sum_{k=0}^{N-1} 1 = N & \text{if } (m-l) = 0 \\ \dfrac{1 - e^{\frac{2\pi i (m-l)}{N} \cdot N}}{1 - e^{\frac{2\pi i (m-l)}{N}}} = 0 & \text{by the formula for ~~tr~~ geometric series if } m \neq l. \end{cases}$$

$\boxtimes$

Thus

$$e_l^* = \frac{1}{\sqrt{N}} e_l \quad \text{forms an orthonormal basis}$$

for $\mathbb{C}^N$ (Functions on $\mathbb{Z}_N$)

Since e_ℓ^* is an orthonormal basis we may write for any $f \in \mathbb{C}^N$

$$f(k) = \sum_{n=0}^{N-1} \underbrace{(f, e_n^*)}_{a_n} e_n^*(k) = \sum_{n=0}^{N-1} \underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1} f(k) e^{-\frac{2\pi i k n}{N}}\right)}_{a_n} e^{\frac{2\pi i k n}{N}}$$

since $f(k)$ and $\sum_{n=0}^{N-1} a_n e^{\frac{2\pi i k n}{N}}$ are the same function it clearly follows that

$$(f(k), f(k)) = \left(\sum a_n e^{\frac{2\pi i k n}{N}}, \sum a_n e^{\frac{2\pi i k n}{N}} \right)$$

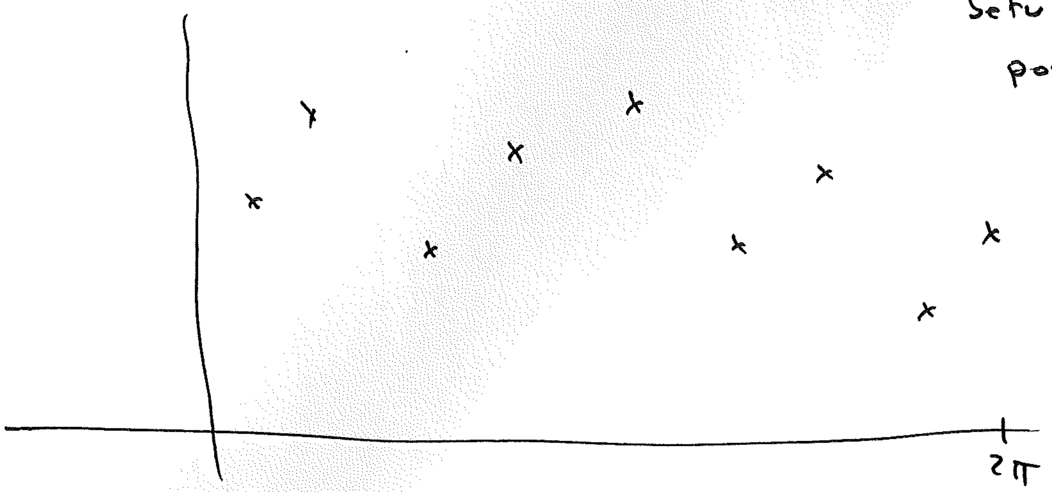
$$\Downarrow$$

$$\frac{1}{N} \sum_{k=1}^{N-1} |f(k)|^2 = \sum_{k=1}^{N-1} |a_n|^2$$ (Parseval's formula)

Remark. It is not absolutely clear to me why this is important!

At this point we have only made a very specific change of ~~variables~~ basis on the vector space $\mathbb{C}^N$.

Could it be that we are given values of a continuous function $f(x)$, $x \in [0, 2\pi)$ at certain points $x_k = \frac{2\pi k}{N}$ and want to interpolate between these points?



If we make an assumption that $f(x) = \sum_{k=0}^{N-1} a_k e^{\frac{2\pi i k x}{N}}$.

then this approach is natural!

Or are we thinking of an applied situation where $f(x)$ is a signal and we want it in a form that makes it easier to manipulate for certain purposes, such as cutting of high frequencies or measure the intencity of a certain frequency?

I am not sure. Mathematically, however, we have only made a change of basis in a finite dimensional vector space.

The fast Fourier transform.

Assume that we want to write the function

$f(k) : \mathbb{Z}_N \to \mathbb{C}$ as a finite Fourier series.

$$f(k) = \sum_{n=0}^{N-1} a_n^N(f)\, e^{\frac{2\pi i n k}{N}} \quad \text{where} \qquad (1)$$

$$a_n^N(f) = \frac{1}{N}\sum_{r=0}^{N-1} f(r)\, e^{-\frac{2\pi i r k}{N}}. \qquad (2)$$

In principle this is simple, but if N is large then this might become very time consuming.

To calculate (2) we need to make $N-1$ additions and ~~N~~ $N+1$ multiplications (assuming that we already know $e^{-\frac{2\pi i r k}{N}}$) so each coefficient requires $\underbrace{(N+1)+(N-1)}_{2N}$ operations, for each (of N) coeff. and to calculate (1) we need N multiplications and $N-1$ additions. So in all we need to make

$O(N^2) = 2N^2 + 2N - 1$ operations to determine $f(k)$.

If N is large and we need to do this many times then this might become very time consuming.

Can we do this faster?

In certain cases we can. If $N = 2^n$ then we can calculate the even and odd k separately and significantly improve the speed.

Theorem 1.3 Given $\omega_N = e^{-\frac{2\pi i}{N}}$ with $N = 2^n$, it is possible to calculate the fourier coefficients of a function on $\mathbb{Z}(N)$ with at most

$$4 \cdot 2^n n = 4N \log_2(N) = O(N \log N)$$

operations.

We need a Lemma first. If we denote the number of operations needed to calculate the fourier coefficients of a function on $\mathbb{Z}(M)$ by $\#(M)$.

Lemma 1.4. If we are given $\omega_{2N} = e^{-\frac{2\pi i}{2M}}$, then

$$\#(2M) \leq 2\#(M) + 8M.$$

Proof: First we need to know

$\omega_{2M}, \omega_{2M}^2, \ldots, \omega_{2M}^{2M} \Rightarrow 2M-1$ operations.

We now need to use the assumption that we know $\#(M)$. In order to use that we define

$$\left.\begin{array}{l} F_0(r) = F(2r) \\ F_1(r) = F(2r+1) \end{array}\right\} \text{ Functions on } \mathbb{Z}(M)$$

so we may calculate the fourier coefficients $a_k^M(F_0)$ and $a_k^M(F_1)$ in $2\#(M)$ operations.

Now

$$a_k^{2M}(F) = \frac{1}{2M}\sum_{r=0}^{2M-1} F(r)\,\omega_{2M}^{kr} = \frac{1}{2}\Bigg(\underbrace{\frac{1}{M}\sum_{l=0}^{M-1} \underbrace{F(2l)}_{=F_0(l)}\,\omega_{2M}^{k(2l)}}_{a_k^M(F_0)} + \underbrace{\frac{1}{M}\sum_{m=0}^{M-1} \underbrace{F(2m+1)}_{F_1(m)}\,\omega_{2M}^{k(2m+1)}}_{a_k^M(F_1)}\Bigg)$$

$$= \underbrace{\frac{1}{2}\left(a_k^M(F_0) + a_k^M(F_1)\right)}_{3 \text{ operations}}$$

So in total we need

$$\underbrace{\left(\text{calculate } \omega_{2M}^1, \ldots, \omega_{2M}^{2M}\right)}_{(2M-1)} + \underbrace{\left(\text{calculate } a_k^M(F_0), a_k^M(F_1)\right)}_{2\#(M)} + \underbrace{\left(3 \text{ operations for each } a_k^{2M}\right)}_{3(2M)}$$

$$(2M-1) + 2\#(M) + 3(2M) \leq 2\#(M) + 8M$$

∎

Proof of the theorem.

We will do this by induction.

When $n=1$ we can calculate the fourier coefficients

$$a_0^N(F) = \frac{1}{2}\left(F(1) + F(-1)\right) \quad \text{and} \quad a_1^N(F) = \frac{1}{2}\left(F(1) \cancel{+} (-1)F(-1)\right)$$

in 5 operations. And $5 \leq 4 \cdot 2^1 \cdot 1 = 8$.

Now suppose that the theorem is true for all ~~...~~ $n-1$, then by the lemma

$$\#(2^n) \leq 2 \underbrace{\#(2^{n-1})}_{\substack{\leq 4 2^{n-1}(n-1) \\ \text{by induction} \\ \text{hypothesis}}} + 8 \cdot 2^{n-1} \leq 4 \cdot 2^n n - \cancel{4 \cdot 2^n} + \cancel{8 \cdot 2^{n-1}}$$

$\blacksquare$