Lecture 3.

Last week we proved that if

1) $S_N(f) = 0$ for all N then $f(x_0) = 0$

for every point where f is continuous.

2) 1) $\Rightarrow$ $S_N(f) = S_N(g)$ for two continuous functions

then $f = g$ (since $S_N(f-g) = 0$ for all N)

3) ~~If $S_N(f) \to g$ uniformly~~

If $\sum |\hat{f}(n)|$ converges then

$S_N(f) \to f$ uniformly

Today we will continue to analyse this set of ideas.

The first question we ask is when can we say that $\sum_{n=0}^{\infty} |\hat{f}(n)|$ converges? Given an f the only way, it would appear, to do so would be to calculate infinitely many $\hat{f}(n)$ which isn't practical in reality.

We will state a corollary, prove it, and then try to analyse it. It is a little weird and the proof seems to be a trick so at first we just accept it and then discuss it.

Corollary 7.4. Suppose that

i) $f \in C^2$ on the circle.

Then

1) $\hat{f}(n) = O\left(\frac{1}{n^2}\right)$

2) $S_N(f) \to f$ uniformly.

Remark: This gives our first answer to the question: ~~When~~ Does $S_N(f) \to f$?
Yes, if $f \in C^2$ on the circle

Proof: By definition of $\hat{f}(n)$; $|n| \geq 1$

$$2\pi \hat{f}(n) = \int_0^{2\pi} f(x) e^{-inx}\, dx = \int_0^{2\pi} f(x) \left[\frac{1}{-in} \frac{d e^{-inx}}{dx} \right] dx$$

$$= \begin{Bmatrix} \text{integration} \\ \text{by parts} \end{Bmatrix} = \underbrace{\left[f(x) \cdot \frac{-e^{-inx}}{in} \right]_0^{2\pi}}_{\substack{=0 \text{ since} \\ f(0) = f(2\pi)}} + \frac{1}{in} \int_0^{2\pi} f'(x) e^{-inx}\, dx$$

$$= \begin{Bmatrix} \text{integration} \\ \text{by parts} \end{Bmatrix} = \frac{1}{in} \underbrace{\left[f'(x) \frac{-e^{-inx}}{in} \right]_0^{2\pi}}_{=0} + \frac{1}{(in)^2} \int_0^{2\pi} f''(x) e^{-inx}\, dx$$

So

$$\left.\begin{aligned} |n^2|\,|\hat{f}(n)| &\leq \left|\int_0^{2\pi} f''(x)\, e^{-inx}\, dx\right| \leq \\ &\leq \int_0^{2\pi} |f''(x)|\,\underbrace{|e^{-inx}|}_{\leq 1}\, dx \leq 2\pi \max_{x\in[0,2\pi]} |f''(x)|. \end{aligned}\right\} \quad (1)$$

But f'' is a continuous function on the closed bounded intervall $[0,2\pi]$ so $\max_{[0,2\pi]} |f''(x)|$ exsists, say that it equals M. Thus (1) implies that

$$|\hat{f}(n)| \leq \frac{2\pi M}{n^2}, \quad \text{for} \quad n \neq 0.$$

So $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{n\neq 0} |\hat{f}(n)| \leq |\hat{f}(0)| + \frac{2\pi M}{\cancel{\pi}} \sum_{n=0} \frac{1}{n^2} < \infty.$

■

Remark: To me the proof is sort of a trick. We make an "arbitrary" assumption $f \in C^2$ and then it happens to be an integration by parts formula that spits out the result.

So let us try to understand why $f \in C^2$, that is f being smooth, implies that $\hat{f}(n)$ should be small when n is large.

Corollary 2.4

f is smooth (has 2 cont derivatives) $\Longrightarrow$ $\hat{f}(n)$ is small for large n $\left(|\hat{f}(n)| < \frac{C}{n^2} \right)$

Can we make sense of this?

What would happen if $f \in C^1$?

What is the absolute "best" smoothness assumption on f?

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f being ~~contin~~ a constant is the best smoothness assumption. then

$$\int_0^{2\pi} f\cos(nx)\,dx = \int_0^{2\pi} f\sin(nx)\,dx = 0 \qquad n \neq 0.$$

So if f is very close to a constant $f(0)$ then

$$\int_0^{2\pi} f(x)\cos(nx)\,dx \approx \int_0^{2\pi} f(0)\cos(nx)\,dx = 0.$$

So if f is continuous. ~~Then there exist~~

Then f is uniformly continuous (since $[0,2\pi]$ is compact)

So

$$\int_0^{2\pi} f(x)\cos(nx)\,dx = \sum_{k=0}^{n-1} \int_{\frac{(2\pi)k}{n}}^{\frac{2\pi(k+1)}{n}} f(x)\cos(nx)\,dx =$$

$$= \sum_{k=0}^{n-1} \left[\underbrace{\int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} f\left(\frac{2\pi k+\pi}{n}\right)\cos(nx)\,dx}_{=0} + \int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} \underbrace{\left(f(x) - f\left(\frac{2\pi k+\pi}{n}\right)\right)}_{\text{must be small if } n \text{ is large.}}\cos(nx)\,dx \right.$$

So in particular if, for a given $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ s.t.

$$|x-y| < \delta_\varepsilon \longrightarrow |f(x)-f(y)| < \varepsilon$$

and $n > \frac{1}{\delta_\varepsilon}$ then

$$\left| f(x) - f\left(\frac{2\pi k + \pi}{n}\right) \right| \leq \varepsilon \qquad \text{for all} \quad x \in \left[\frac{2\pi k}{n}, \frac{2\pi(k+1)}{n}\right]$$

So

$$\left| \int_0^{2\pi} f(x) \cos(nx)\, dx \right| = \left|\sum_{k=0}^{n-1} \int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} \underbrace{\left(f(x) - f\left(\frac{2\pi k + \pi}{n}\right)\right)}_{<\varepsilon \text{ if } n > \frac{1}{\delta_\varepsilon}} \underbrace{\cos(nx)}_{<1}\, dx\right| \leq$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} \varepsilon\, dx \leq \varepsilon .$$

So $f \in C([0, 2\pi]) \Rightarrow |\hat{f}(n)| = o(1)$.

Play with math Let $f \in C$ periodic on $\mathbb{R}$

$$S_N(f)(x) = \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \, dy \right) e^{inx} =$$

$$= \left\{ \begin{array}{l} \text{we may} \\ \text{move the} \\ e^{inx} \text{ term} \\ \text{under the integral} \\ \text{sign} \end{array} \right\} = \sum_{n=-N}^{N} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(y) \underbrace{\cancel{e^{-iny}\, e^{inx}}}_{e^{in(x-y)}} \, dy \right) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left[\sum_{n=-N}^{N} e^{in(x-y)} \right]}_{D_N(x-y)} dy$$

Definition: Convolutions, we ~~say that~~ define the operator $*$ between two functions (periodic) f and g

$$f * g(x) = \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy.$$

Thus

$$S_N(f)(x) = \frac{1}{2\pi} f * D_N(x)$$

So we get a new way to express the N:th partial sum.

Notice that

$$(e^{ix}-1)D_N(x) = (e^{ix}-1)\sum_{n=-N}^{N} e^{inx} = \sum_{n=-N+1}^{N+1} e^{inx} - \sum_{n=-N}^{N} e^{inx} =$$

$$= e^{i(N+1)x} - e^{-iNx}.$$

$$\Rightarrow \quad D_N(x) = \frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} \cdot \frac{e^{-\frac{ix}{2}}}{e^{-\frac{ix}{2}}} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}}$$

$$= \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}.$$

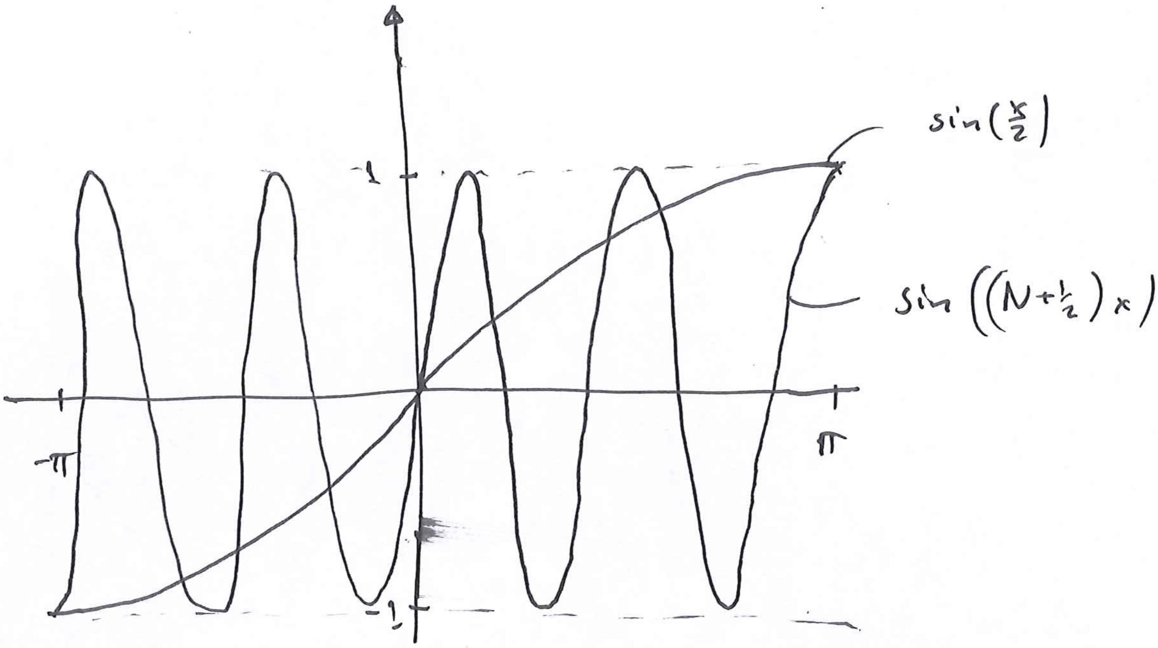
Thus

$$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \underbrace{\frac{\sin\left((N+\frac{1}{2})(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)}}_{D_N(x-y)} dy.$$

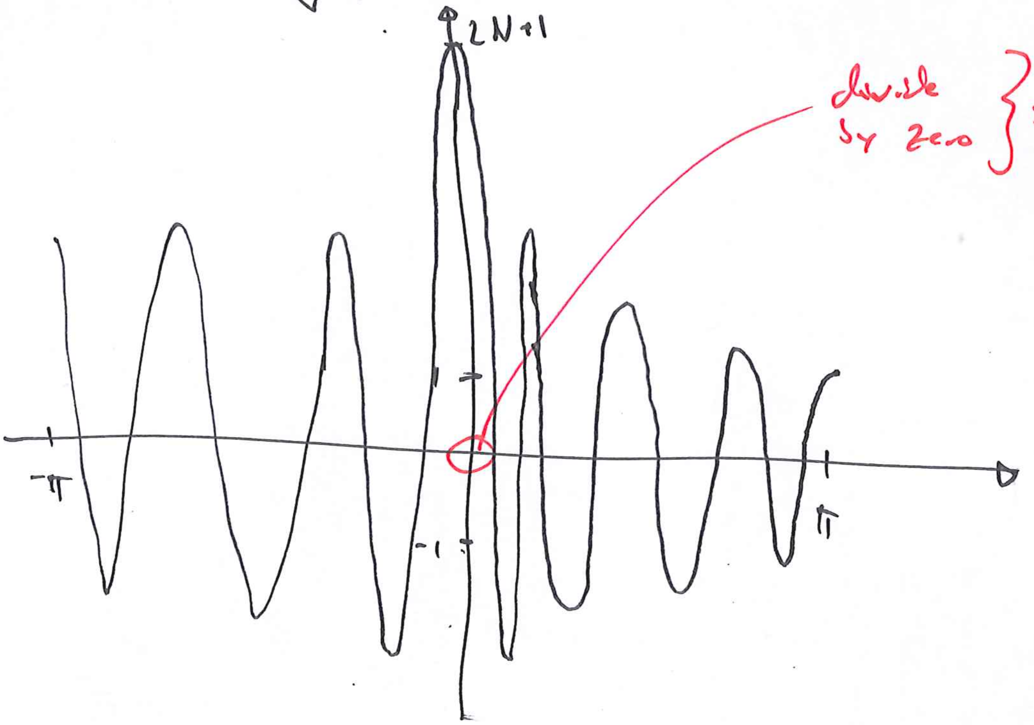
So if we can understand $D_N(x-y)$ then we can understand the rather more complicated object

$S_N(f)(x)$.

So how does this mysterious Dirichlet kernel $D_N(x)$ behave



$\Downarrow$ divide



$$\left.\begin{matrix}\text{divide}\\ \text{by zero}\end{matrix}\right\} \Rightarrow \lim_{x\to 0} \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} =$$

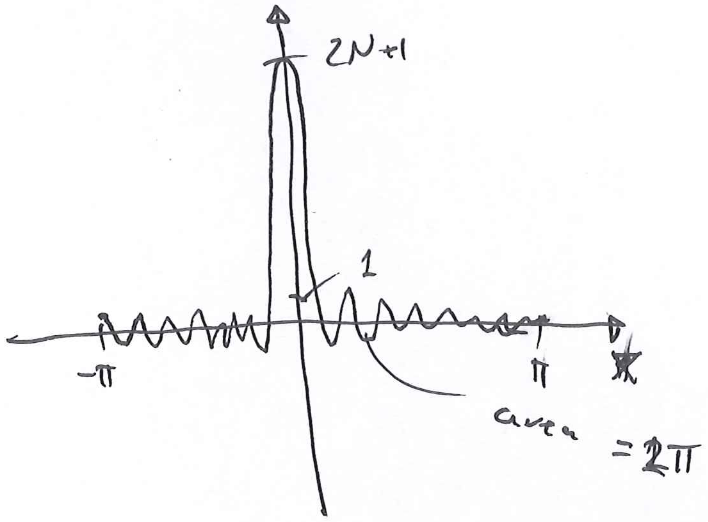
$$= \left\{\begin{matrix}\text{l'Hospital's}\\ \text{rule}\end{matrix}\right\} =$$

$$= (2N+1) \lim_{x\to 0} \frac{\cos(N+\frac{1}{2}x)}{\cos(\frac{x}{2})} =$$

$$= 2N+1$$

Also

$$\int_{-\pi}^{\pi} D_N(x)\,dx = \underbrace{\int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{inx}\,dx}_{\substack{=0\\ \text{unless } n=0}} = 2\pi$$



So somehow

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, D_N(\cancel{x}-y)\,dy \qquad x=0$$

calculates some average of $f(x)$ in 2π, but an average that peaks, or puts extra weight, at the point $x=0$.

~~Also, if~~

Can we pass to the limit $\lim_{N\to\infty} D_N(x)$? NO!

If so then we could hope $\lim_{N\to\infty} S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, D_\infty(x-y)\,dy.$

The answer is NO! At least in any reasonable interpretation,

1) $\lim_{N\to\infty} D_N(x)$ can not exist pointwise since

$$D_N(x) = 2N+1 \to \infty$$

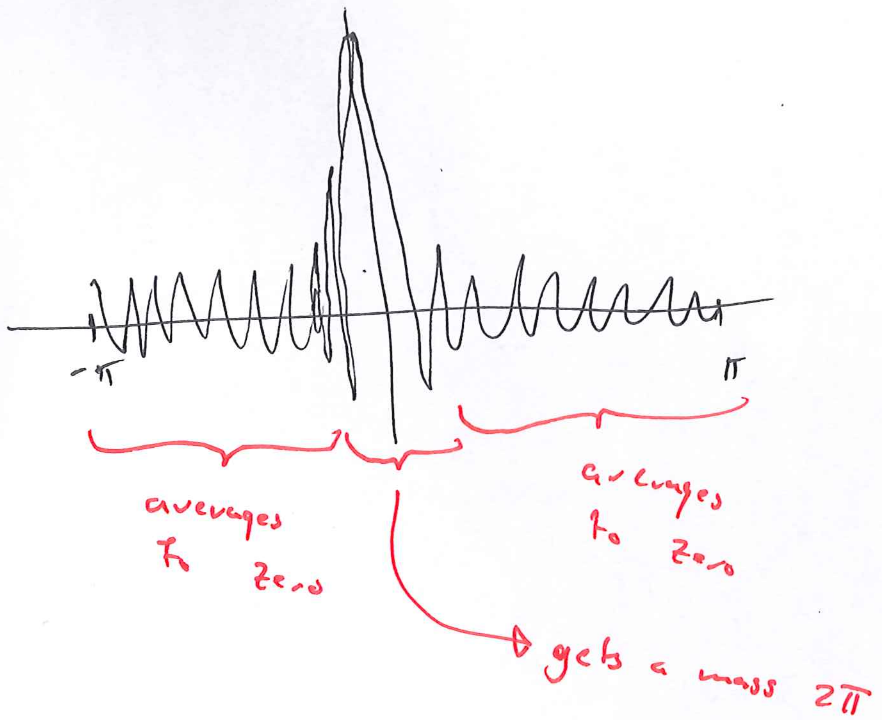
2) Also $\int_{-\pi}^{\pi} |D_N(x)|\,dx \geq c\ln(N) \to \infty$ (exercise).

~~So we are given the stark choice to abandon the~~

But as $N \to \infty$ we get more and more oscillations so intuitively

$$\int_{\pi > |x| > \delta} \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} dx \to 0 \quad \text{as } N \to \infty$$

so ~~But~~ $$\int_{|x|<\delta} \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} dx \to 2\pi .$$



So it is reasonable that the average

$$\int_{-\pi}^{\pi} f(y)\, D_N(x-y)\, dy \to f(x) \quad \text{when } N \to \infty.$$

But D_N behaves to wildly for us to prove this. We need to take a step back and consider something nicer, and that we will do next week.