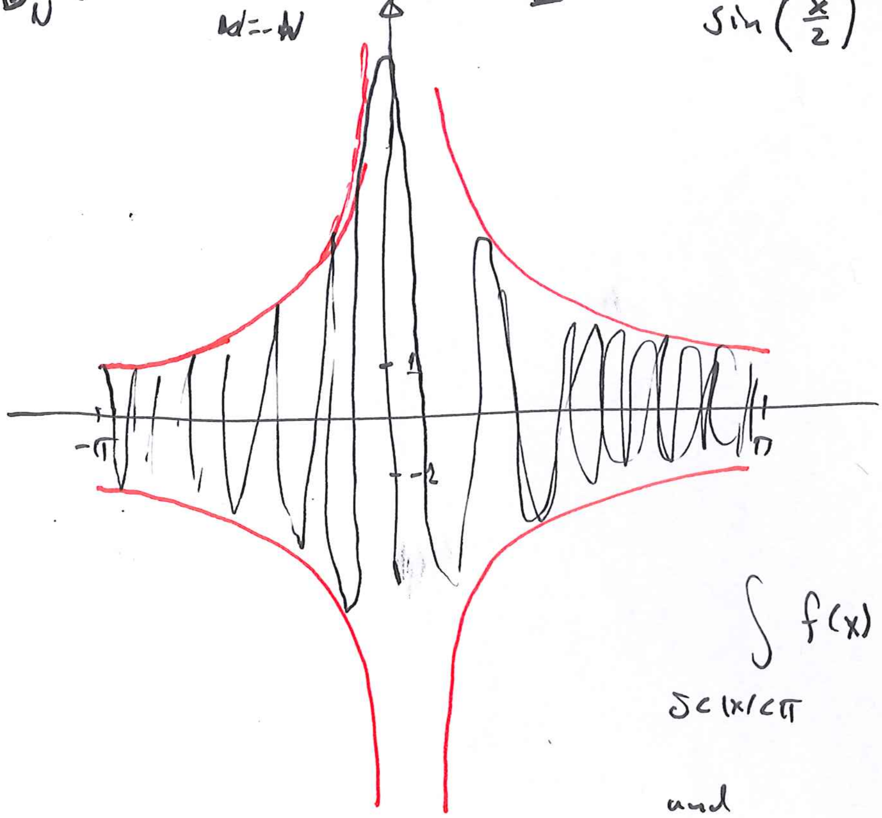
Lecture 4.

Last week we looked at the Dirichlet kernel

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$



Somehow we hoped that the oscilations for $|x|>\delta$ should imply

$$\int_{\delta<|x|<\pi} f(x)\, D_N(-x)\, dx \longrightarrow 0$$

and

$$\int_{|x|<\delta} f(x)\, D_N(-x)\, dx \approx f(0).$$

so $\lim_{N\to\infty} \int_{-\pi}^{\pi} f(x)\, D_N(-x)\, dx \Rightarrow \cancel{0}$. $\approx \cancel{f}$

But this is not true, so we can not hope to prove it (it was probably a very time consuming and expensive experience bought by hundreds of hours of vain work.)

But we have some good intuition that we can use to prove an important result. If we forget the Dirichlet kernel and consider an abstract kernel $K_N(x)$ instead.

We want $\lim_{N\to\infty} f*K_N(x) = \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_N(x-y)\, dy = f(x).$ (1)

We have the intuition that such a $\{K_N\}_{N=1}^{\infty}$ should satisfy Definition: We say that $\{K_N\}$ is a family of good kernels if:

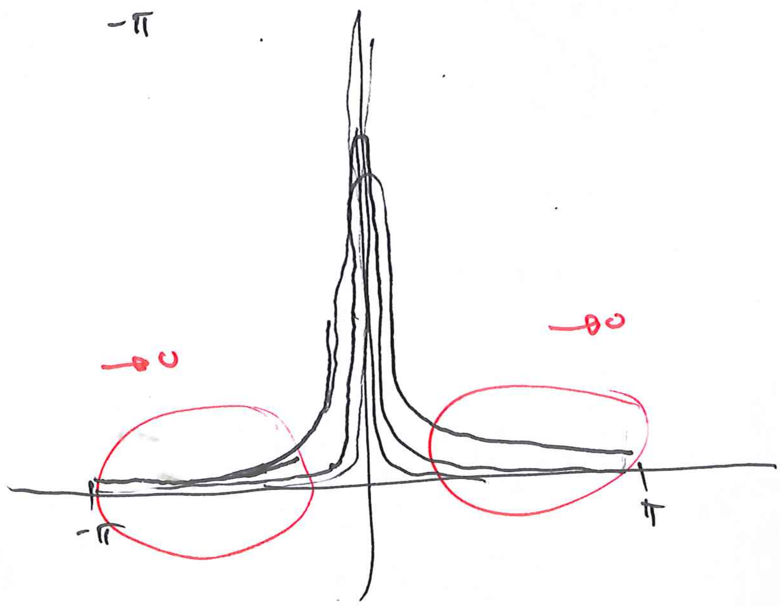
For every $\delta > 0$

i) $\int_{\delta<|x|<\pi} |K_N(x)|\, dx \to 0$ as $N \to \infty$ (concentrates at 0)

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\, dx = 1$ (so that (1) holds if f is a constant)

iii) There exists an M s.t.

$\int_{-\pi}^{\pi} |K_N(x)|\, dx \leq M$ (don't exactly know why?)



Theorem 4.1 Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels and f integrable on the unit circle. Then if f is continuous at x_0 then

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_n(x-y)\, dy = f(x).$$

If f is continuous everywhere, then the limit is uniform in x.

Proof: We need to show that for any $\varepsilon>0$ $\exists N_\varepsilon>0$ s.t.

$$n>N_\varepsilon \implies \left| K_n * f(x) - f(x)\right| < \varepsilon.$$

To that end we consider

$$\left| K_n * f(x) - f(x)\right| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_n(x-y)\, dy - f(x)\right| =$$

$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_n(x-y)\, dx - \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x-y)\, f(x)\, dy}_{\text{since } \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x-y)\,dy = 1} \right| =$$

by assumption ii) for good kernels

$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(y)-f(x)\big)\, K(x-y)\, dx \right|$$

we still have to use i) (& iii)) in the def of good kernels (as well as continuity of f). To that end we write for some $\delta > 0$ (to be chosen later)

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(y) - f(x)) \, K_n(x-y) \, dx \right| \leq \underbrace{\left| \frac{1}{2\pi} \int_{\delta < |x| < \pi} |f(y) - f(x)| \, |K_n(x-y)| \, dy \right|}_{\text{here we can use i) that is if } n \text{ is large then } \int_{\pi > |x| > \delta} |K_n| \, dy \text{ is small.}}$$

$$+ \underbrace{\left| \int_{-\delta}^{\delta} (f(y) - f(x)) \, K_n(x-y) \, dy \right|}_{\text{If } \delta \text{ is small then } |f(y) - f(x)| \text{ will be small}} = \mathrm{I} + \mathrm{II}$$

Notice that

$$\mathrm{II} \leq \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(y) - f(x)| \, |K_n(x-y)| \, dy \right| \leq \left(\sup_{|x-y| < \delta} |f(y) - f(x)| \right) \underbrace{\left(\frac{1}{2\pi} \int_{-\delta}^{\delta} |K_n(x-y)| \, dy \right)}_{\leq M \text{ by iii)}}$$

$$\leq M \sup_{|x-y| < \delta} |f(y) - f(x)|$$

$$\mathrm{I} \leq 2 \sup_{|x| < \pi} |f(x)| \left(\frac{1}{2\pi} \int_{\delta < |x| < \pi} |K_n(x-y)| \, dy \right)$$

Now fix an $\varepsilon > 0$ then since f is continuous at x there exist a $\delta_\varepsilon > 0$ s.t.

$$|x-y| < \delta \implies |f(y) - f(x)| < \frac{\varepsilon}{2M}.$$

and by i) there exists an N_ε s.t

$$\frac{1}{2\pi}\int_{\delta_\varepsilon < |x| < \pi} |K_n(x-y)|\,dy < \frac{\varepsilon}{4 \sup_x |f(x)|} \quad * \qquad \text{for all } n > N_\varepsilon.$$

In particular, for every $\varepsilon > 0$ there exist an N_ε s.t.

$$|f * K_n(x) - f(x)| \leq I + II < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\blacksquare$

* $f = 0$, if $f = 0$ for all x then the proof is trivial since $f * K_n(x) = 0$ for all x.

Situation:

1) $\left.\begin{array}{l} K_N \text{ good kernel} \\ f \text{ continuous} \end{array}\right\} \Rightarrow \quad f * K_n(x) \to f(x) \quad \text{as } ~~N~~ \to \infty$

2) $\left.\begin{array}{l} D_N \text{ not good kernel} \\ \exists f \text{ continuous} \end{array}\right\} \overset{\text{Next few weeks}}{\Longrightarrow} \quad f * D_N(x) \text{ diverges.}$

So $f * D_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \sum_{n=-N}^{N} e^{-in(x-y)}\, dy$

behaves badly as $N \to \infty$. That is $\sum_{n=-N}^{N} e^{-in(x-y)}$ behaves badly. But that is probably because the sum isn't absolutely convergent. Compare

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots = \ln(2)$$

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} = \frac{\ln(2)}{2}$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} \dots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots\right) = \frac{\ln(2)}{2}.$$

So changing the order of summation can change series that aren't absolutely convergent.

And quite fantastically there are two spectacular ways to sum the Dirichlet kernel that gives rise to "Good Kernels."

Definition: We define the N:th Fejér kernel

$$F_N(x) = \frac{D_0(x) + D_1(x) + \dots + D_{N-1}(x)}{N} = \frac{1}{N} \frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$$

and the Poisson kernel

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{-inx}$$ (Absolutely convergent)

We will focus on F_N - but P_r is just as important.

Notice that F_N is somehow just a rearrangement of D_N.

Lemma 1 F_N forms a family of "good kernels".

Proof: We need to prove three things

i) For all $\delta > 0$: $\displaystyle\int_{\delta < |x| < \pi} F_N(x)\,dx \to 0$ as $N \to \infty$.

Notice that when $\delta \leq |x| \leq \pi$ then $\left|\sin\left(\frac{x}{2}\right)\right| > \gamma_\delta > 0$ for some γ_δ (since $x=0$ is the only zero of $\sin\left(\frac{x}{2}\right)$ when $|x| \leq \pi$, and $[\pm\delta, \pm\pi]$ are compact.)

Also $\left|\sin\left(\frac{Nx}{2}\right)\right| \leq 1$.

Therefore

$$\left| \int_{\delta<|x|<\pi} F_N(x)\,dx \right| \leq \int_{\delta<|x|<\pi} \frac{1}{N} \frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}\,dx \leq \frac{1}{N} \int_{\delta<|x|<\pi} \frac{1}{y^2}\,dx =$$

$$= \underbrace{\frac{2(\pi-\delta)}{y^2}}_{\text{constant}} \cdot \underbrace{\frac{1}{N}}_{\to 0 \text{ as } N\to\infty} \to 0.$$

This proves 1. & Sandwich Thm

ii) We need to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_N(x)|\,dx = 1.$$

But

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = \sum_{n=0}^{N-1} \frac{1}{2\pi N} \underbrace{\int_{-\pi}^{\pi} D_n(x)\,dx}_{=2\pi} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1$$

iii) We need to show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_N(x)|\,dx \leq C$

but $F_N \geq 0$ so $|F_N| = F_N$ and thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_N(x)|\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1.$$

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Corollary: Let f be integrable on the circle and continuous at x_0. Then

$$\lim_{N \to \infty} f * F_N(x_0) = f(x_0)$$

$$\left(\lim_{N \to \infty} f * \left(\frac{1}{N} \cdot \sum_{n=0}^{N-1} D_N \right) x_0 = f(x_0) \right)$$

Proof: $\{F_N\}$ is a family of good kernels.

Corollary: If $f(x)$ is continuous on the unit circle then

$$f * F_N(x) \to f(x) \quad \text{uniformly as } N \to \infty \quad (1)$$

Proof: $\{F_N\}$ is a family of good kernels.

Corollary: Let f be continuous on the circle. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial P_N s.t

$$|f(x) - P_N(x)| < \varepsilon \quad \text{for all } x.$$

Proof: Notice that $P_N = f * F_N(x)$ is a trigonometric polynomial. So the corollary follows from (1) if N is large enough.

$\boxtimes$

Remark: This states again that the class of trigonometric functions is rich enough to distinguish continuous functions.

Before we knew that if

$$S_N(f) = S_N(g) \quad \text{for all } N$$

and $f, g \in$ Continuous then $f = g$.

But what we now prove is somewhat stronger. That we can get arbitrarily close to any continuous functions by means of a trig poly.

Next time, show that $D_N(f)(x)$ may diverge, even though f is continuous and summarize the pointwise convergence properties.