Lecture 5.

We now prove that there exists a function $f(x)$ that is continuous on the circle and $S_N(f)(0)$ diverges.

How to proceed. Assume that we know the following

Lemma 1. Let $f_N(x) = \sum_{1 \le |n| \le N} \frac{e^{inx}}{n}$, $\tilde{f}_N = \sum_{-N \le n \le -1} \frac{e^{inx}}{n}$.

then

i) $|\tilde{f}_N(0)| \geq c \log(N)$

ii) $|f_N(x)|$ is uniformly bdd in x & N.

We will not prove this ~~later, if we have time~~.

Notice that the Lemma is intimately related to the order of summation. By convention

$$|f_N(0)| = \left|\sum_{1 \le |n| \le N} \frac{e^{inx}}{n}\right| = \left| e^{ix} - e^{-ix} + \frac{e^{2ix}}{2} - \frac{e^{-2ix}}{2} + \dots + \frac{e^{iNx}}{N} - \frac{e^{-iNx}}{N} \right| \le C$$

but

$$\Bigg| \underbrace{-\frac{e^{-iNx}}{N} - \frac{e^{-i(N-1)x}}{N-1} + \dots - \frac{e^{-ix}}{1}}_{\geq c \log(N) \text{ for } x=0} + e^{ix} + e^{i2x} + \dots + e^{iNx} \Bigg| \le C$$

That is if we sum $e^{iNx} f_N(x)\Big|_{x=0}$ then

the sum goes to $\approx c\ln(N)$ and then down to C.

So define

$$e^{2iNx} f_N(x) = P_N(x) = \sum_{\substack{n=N \\ n\neq 2N}}^{3N} \frac{e^{inx}}{n-2N} = \text{trig poly}$$

So P_N is continuous for any N and $|P_N(x)| \leq C$

So if N_k, $\alpha_k > 0$ are sequences s.t. $\sum_{k=1}^{\infty} \alpha_k < \infty$ (1)

then $\sum_{k=1}^{K} \alpha_k P_{N_k}(x)$ ~~converges~~ is a sequence of continuous functions that converges uniformly to a, necessarily continuous function, $f(x)$.

Moreover, if $N_{k+1} > 3N_k$ then (2)

$$S_{2N_l}(f)(0) = \underbrace{\sum_{k=1}^{l-1} \alpha_k P_{N_k}(0)}_{\leq C\sum \alpha_k \leq C} + \alpha_l \underbrace{e^{i2N_l\cdot 0} \sum_{n=-N_l}^{N_l} \frac{e^{in0}}{-n}}_{\geq c\ln(N_l)} \geq c\alpha_l \ln(N_l) - C'$$

So $\limsup_{l\to\infty} S_{2N_l}(f)(0) \geq \lim_{l\to\infty} \left(c\alpha_l \ln(N_l) - c\right) \to \infty$ if $\alpha_l \ln(N_l) \to \infty$ (3)

So we need to choose $\alpha_k > 0$ s.t. $\sum \alpha_k < \infty$ and N_k s.t. $\alpha_l \ln(N_l) \to \infty$. Then

$$\lim_{l\to\infty} S_{2N_l}(f)(0) = \infty.$$

This is easily done, say $\alpha_k = \frac{1}{k^2}$,

and $N_k = e^{k^3}$ $\left(\text{So that } \alpha_k \ln(N_k) = \frac{1}{k^2} \ln(e^{k^3}) = k \right)$.

and $N_{k+1} = e^{(k+1)^3} \geq e^{k^3+3k} \geq e^3 e^{k^3} > 3e^{k^3} \geq 3N_k$.

Proposition: There exists a function f on the circle s.t. f is continuous and

$$\lim_{N\to\infty} S_N(f)(0) \quad \text{diverges}$$

The only thing left to prove is Lemma 1.

Proof of lemma 1:

i) $|\tilde{f}_N(0)| \geq c\ln(N)$: $\quad |\tilde{f}_N(0)| = \left| \sum_{n=-N}^{-1} \frac{e^{-in\cdot 0}}{n} \right| = \sum_{n=-N}^{-1} \frac{1}{n} \approx c\ln(N)$

~~ii) Notice that~~

$$\cancel{f_N(x) = \sum_{n=1}^{N} \frac{e^{inx} - e^{-inx}}{n}, \quad \text{so} \quad |f_N(x)| \leq 2\sum_{n=1}^{N} \frac{1}{n}}$$

~~Now~~ $\cancel{\sum_{n=1}^{N} \frac{1}{n}}$ ~~is Abel summable~~ : $\cancel{\sum_{n=1}^{\infty} \frac{r^n}{n} < \infty}$

~~for~~ $\cancel{0<r<1}$

Start, differential equations

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2}$$

Naive idea

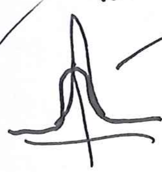
$y(x,t) = X(x)\,T(t)$

$\Rightarrow \; y(x,t) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx} e^{int}$

Does it converge.

Convergence properties

Idea



distinguishes between continuous functions

$\sum |a_n| < \infty$, or $f \in C^2$

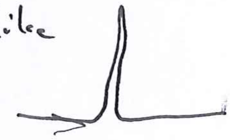
$S_N(x) \to f(x)$

$\cancel{f \in C^1} \overset{!}{=} |f(x) - f(y)| < C|x-y|$

$\Rightarrow S_N(f)(y) \to f(y).$

Dirichlet kernel

not like



Good kernels

"More or less rearrangements of D_N" converges

Not pointwise convergence

A new perspective.

Notice that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} e^{imx}\,dx = \begin{cases} 1 & n=-m \\ 0 & \text{else.}\end{cases}$

So if $p(x) = \sum_{n=-N}^{N} a_n e^{inx}$, $q(x) = \sum_{n=-N}^{N} b_n e^{inx}$ $\begin{cases}(a_{-N}, a_{-N+1}, \dots, a_0, a_1) \\ (b_{-N}, \dots, b_0, b_1, \dots, b_N)\end{cases}$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)\overline{q(x)}\,dx \underbrace{=}_{\overline{e^{inx}} = e^{-inx}} \sum_{n=-N}^{N} a_n \overline{b_n}. = \left((a_{-N}, a_{(-N+1)}, \dots, a_0, \dots, a_N)\cdot(b_{-N}, \dots, b_N) = \sum a_n \overline{b_n}\right)$$

So the trigonometric polynomials behaves as vectors in Linear algebra with the inner product $\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)\overline{q(x)}\,dx}_{\text{Makes sense for } p,q \text{ integrable}} = (p,q)$.

Definition: We will denote by $\mathcal{R}$ the space of integrable functions on the circle.

If $f,g \in \mathcal{R}$ then we define the inner product $(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

We define the norm

$$\|f\| = |(f,f)|^{1/2}.$$

Remark: Here we take what we know from linear algebra and add what we know from Fourier analysis, that is that integrable functions should - in some sense - be approximated by trigonometric polynomials.

Remark: 1) We have the right to call $\mathcal{R}$ a space since if $f, g \in \mathcal{R}$ then $f \pm g \in \mathcal{R}$, and $\lambda f \in \mathcal{R}$ for any $\lambda \in \mathbb{R}$. (Compare to vectors v & w in linear algebra)

2) This connects to Fourier analysis since if we can write $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ then e^{inx} can be viewed as an orthogonal basis of $\mathcal{R}$.

The question ~~means~~ is what would $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$ mean for $f \in \mathcal{R}$.

In lin alg. $v = w$ if $|v - w| = 0$, that is $|(v-w, v-w)|^{\frac{1}{2}} = 0$. This leads to the question. Will

$$\|f - S_N(f)\| = |(f - S_N(f), f - S_N(f))|^{\frac{1}{2}} \to 0 \quad \text{as } N \to \infty \, ?$$

Definition: We define a norm on $\mathcal{R}$ according to

$\|f\| = |(f, f)|^{\frac{1}{2}}$ (Note $|v| = |(v, v)|^{\frac{1}{2}}$ in lin alg)

Definition: We say that f and g are equal $f = g$ as elements of $\mathcal{R}$ if

$\|f - g\| = 0$.

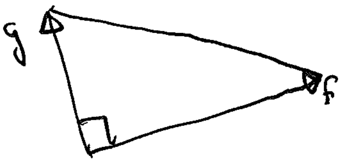
Example: $f = \begin{cases} 1 & x = 0 \\ 0 & \text{else} \end{cases}$, then $f = 0$ as an element of $\mathcal{R}$.

Lemma: Let H be a vector space with inner product (f,g) and norm $\|f\| = |(f,f)|^{\frac{1}{2}}$

then

i) Pythagorean thm: If f,g are orthogonal $(f,g)=0$

then
$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$



ii) Cauchy-Schwarz inequality,

$$|(f,g)| \leq \|f\| \, \|g\|$$

iii) triangle inequality.
$$\|f+g\| \leq \|f\| + \|g\|$$

Proof: Simple.

The important thing is how much we generalize the notions from normal Euclidean geometry to infinite dimensional vector spaces.

Next we notice, in analogy with lin alg, that

$f - \frac{(f,g)g}{\|g\|^2}$ is orthogonal to g.) or write $g = S_N(f)$
) This leads to

Lemma 1.2: Let f be integrable on the circle with Fourier coefficients a_n. Then

$$\|f - S_N(f)\| \leq \|f - \sum_{|n|\leq N} c_n e^{inx}\|$$

for any complex numbers c_n. Equality only if $c_n = a_n$ for all n.

Proof: Let $b_n = a_n - c_n$, then

$$\|f - \sum c_n e^{inx}\|^2 = \|f - S_N + \underbrace{\sum b_n e^{inx}}_{=g}\|^2 = \left\{ \begin{matrix} f - S_N \perp g \\ \text{and} \\ \text{Pyth. Theo} \end{matrix} \right\} = \|f - S_N(f)\|^2 + \|g\|^2$$

$\geq \|f - S_N(f)\|$ with equality only when $\|g\|^2 = \sum |a_n - c_n|^2 = 0$.

That $f - S_N(f) \perp g$ follows if $f - S_N(f) \perp e^{inx}$ for all $|n| \leq N$. But

$$(f - S_N(f), e^{inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x) - \sum_{k=-N}^{N} a_n e^{ikx}\right) e^{inx}\, dx =$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\, dx}_{a_{-n}} + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(-\sum_{k=-N}^{N} a_n e^{ikx}\right) e^{inx}\, dx}_{-a_{-n}} = 0.$$

Thm: Suppose f is integrable on the circle. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 \, dx \to 0 \quad \text{as} \quad N \to \infty$$

Proof: Idea:

We know that if g is continuous on the circle then there exist a trigonometric polynomial $P_M(x)$ s.t.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - P_M(x)|^2 dx < \varepsilon^2 < \varepsilon$$

And if f is integrable on the circle, then there exists a $g(x)$ that is continuous s.t.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g(x)|^2 \leq \underbrace{\frac{\sup(f(x) - g(x))}{2\pi}}_{\leq C} \int_{-\pi}^{\pi} |f(x) - g(x)| \, dx \leq \varepsilon.$$

So we may approximate f by continuous g and a continuous g by $P_M(x)$, a trig poly.

Now let f be any integrable function and choose $g(x)$ continuous on the circle s.t.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f-g|\,dx < \varepsilon_1 < \frac{\sup|f|}{8}\varepsilon$$

and a $P_M(x)$, trig poly, s.t.

$|g - P_M| < \varepsilon_2 < \inf\left(\frac{\sup|f|}{8}\varepsilon, \frac{\varepsilon}{2}\right)$ on the circle. Then for any $0<\varepsilon<1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f-P_M|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f-g+g-P_M|^2\,dx =$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f-g|^2 + 2|f-g||g-P_M| + |g-P_M|^2\,dx \leq$$

$$\leq \frac{1}{2\pi}\sup|f-g|\underbrace{\int_{-\pi}^{\pi}|f-g|}_{2\pi\varepsilon_1} + \frac{1}{\pi}\sup|f-g|\underbrace{\int_{-\pi}^{\pi}|g-P_M|\,dx}_{2\pi\varepsilon_2} + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}|g-P_M|^2dx}_{\varepsilon_2^2}.$$

$$= 2\sup|f|(\varepsilon_1+\varepsilon_2) + \varepsilon_2^2 \leq \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2} = \varepsilon.$$

Thus, by Lemma 1.2 for $N>M$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-S_N(f)(x)|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f-P_M(x)|^2\,dx < \varepsilon.$$

So for any $\varepsilon>0$ $\exists M_\varepsilon$ s.t. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f-S_N(f)|^2 < \varepsilon$ for all $N > M_\varepsilon$. ☒