The lecture that comes after the last (5?, 6?)

Remember that we proved the "Best approximation lemma."

Lemma If f is integrable on the circle with fourier coefficients a_n, then

$$\|f - S_N(f)\| \leq \| f - \sum_{|n|\leq N} c_n e_n \|$$

for every c_n, with equality only if $a_n = c_n$

We also mentioned the following

Thm: Let f be integrable on the circle then $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_N(f)|^2 dx \right)^{1/2} \to 0$ as $N \to \infty$

Proof: We prove this using the following

i) If h is continuous on the circle then we may approximate h uniformly by a trigonometric polynomial. That is for every $\varepsilon > 0$ there exsist a trigonometric polynomial $P_M(x)$ s.t. the degree of P_M is M and

$$|h(x) - P_M(x)| < \varepsilon \qquad \text{for every } x$$

Reminder of the proof: We know that

$h * F_N \cancel{(x)} \longrightarrow h(x)$ uniformly, where F_N is the Fejér kernel. Also $h * F_N(x)$ is a trig poly.

So if N is large enough then $|\underbrace{h * F_N(x)}_{\text{trig poly} = P_N(x)} - h(x)| < \varepsilon$.

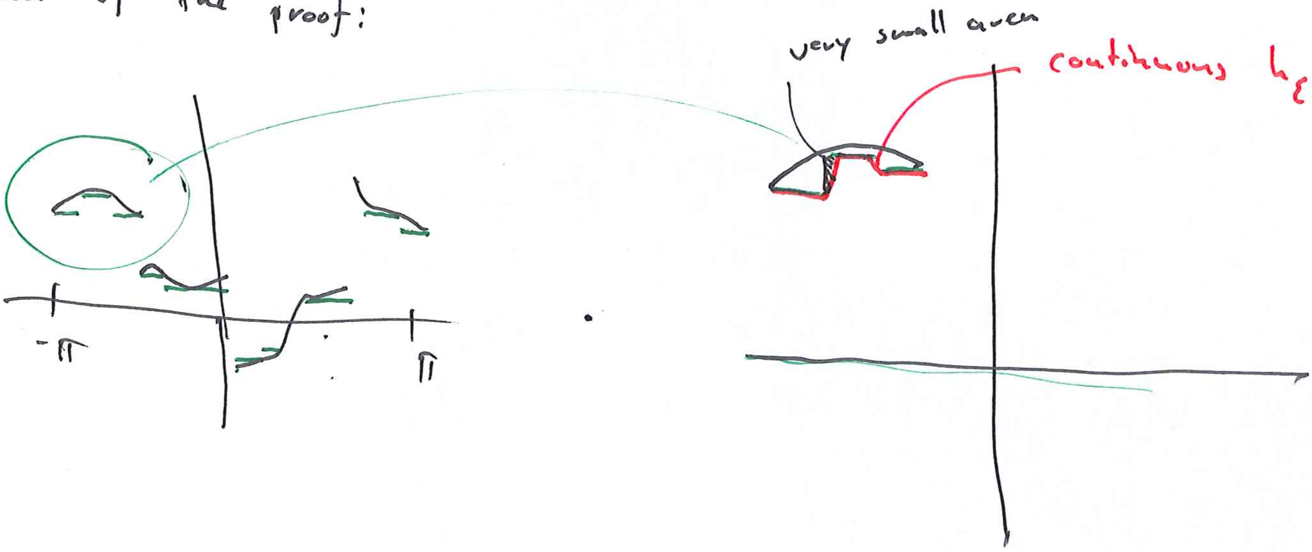
ii) If $f(x)$ is integrable then there exist, for every $\varepsilon > 0$, a continuous function h_ε s.t

$$\sup_x |h_\varepsilon| \leq \sup |f(x)|$$

and

$$\int_{-\pi}^{\pi} |f - h_\varepsilon| \, dx < \varepsilon .$$

Idea of the proof:



So let f be as in the theorem and $1>\varepsilon>0$ be given. We need to show that there exists an $N_\varepsilon > 0$ s.t.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_N|^2\,dx < \varepsilon \qquad \text{for} \qquad N > N_\varepsilon .$$

But by the best approximation lemma we only need to show that there exists one trigonometric polynomial P_M s.t.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - P_M|^2\,dx < \varepsilon$$

since then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_N(f)|^2\,dx \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-P_M|^2\,dx < \varepsilon \qquad \text{for all } N > M.$$

To that end we fix a continuous function h possible by ii) s.t.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - h|\,dx < \varepsilon_1 < (\text{to be chosen later}) < \left(\frac{\sup|f|}{8}\right)^{-1}\varepsilon .$$

By i) there exists a P_M, trig poly of degree M, s.t.

$$|h - P_M| < \varepsilon_2 < (\text{to be chosen later}) < \inf\left(\left(\frac{\sup|f|}{8}\right)^{-1}\varepsilon, \frac{2\varepsilon}{4}\right)$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - P_M|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f - h + h - P_M|^2 dx \leq$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-h|^2 dx + \frac{2}{2\pi}\int_{-\pi}^{\pi} |f-h||h-P_M| dx + \int_{-\pi}^{\pi} |h-P_M|^2 dx \leq$$

$$\leq \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-h| dx}_{<\varepsilon_1} \cdot \sup|f-h| + \underbrace{2\sup|f-h| \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|h-P_M|}_{<\varepsilon_2} dx}_{<\sup|f-h|\varepsilon_2} + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|h-P_M|^2}_{<\varepsilon_2^2} dx}_{\varepsilon_2^2}$$

$$\leq 2\sup|f|(\varepsilon_1+\varepsilon_2) + \varepsilon_2^2 < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

$\blacksquare$

Corollary: Let f be integrable on the circle $f \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

Proof

$$\left| \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 dx \right| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_N(f)|^2 dx \right| =$$

$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx - \sum_{n=-N}^{N} a_n^2 \right| \to 0 \quad \text{as } N \to \infty.$$

But this is the definition of $\sum_{n=-\infty}^{\infty} a_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 dx$. ∎

Corollary (Riemann-Lebesgue lemma): If f is integrable on the circle then $\underbrace{\hat{f}(n)}_{a_n} \to 0$ as $|n| \to \infty$.

Proof: Since $\sum_{-\infty}^{\infty} a_n^2$ converges $a_n^2 \to 0$ as $|n| \to \infty$

so $a_n \to 0$ as $|n| \to \infty$.

∎

Theorem 2.1 Let f be integrable on the circle. Assume furthermore that f is differentiable at x_0. Then $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$.

Proof (Idea): we need to show that

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, D_N(x_0 - y)\, dy - f(x_0) \right| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(y) - f(x_0))\, D_N(x_0 - y)\, dy \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{2\frac{f(y) - f(x_0)}{y - x_0 \cancel{\sin\left(\frac{y - x_0}{2}\right)}}}\ \sin\left(\frac{2N+1}{2}(y - x_0)\right) dy \right| \to 0$$

$$F(y) \approx \frac{f(y) - f(x_0)}{y - x_0} \approx f'(x_0)$$

should be bounded

and $\int F(y) \sin(Ny)\, dy \to 0$

by Riemann Lebesgue lemma.

Proof (Formal): Define

$$F(y) = \begin{cases} \dfrac{f(x_0 - y) - f(x_0)}{y} & y \neq 0 \\ -f'(x_0) & y = 0 \end{cases}$$

Then $F(y)$ is ~~diff~~ bounded since f is differentiable at x_0 and F is clearly integrable on $[-\pi, -\delta] \cup [\delta, \pi]$ for any $\delta > 0$. We can conclude that F is integrable

Next we calculate

$$S_N(f)(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x_0-y) - f(x_0)\right) D_N(y)\,dy =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(y)\, y\, D_N(y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{F(y)\frac{y}{\sin(\frac{y}{2})}}_{\substack{\text{integrable} \\ \text{since } f(y) \text{ is} \\ \text{and } \frac{y}{\sin(\frac{y}{2})} \text{ is.}}} \sin\left(\frac{2N+1}{2}y\right) dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left[F(y)\frac{y}{\sin(\frac{y}{2})}\cos\left(\frac{y}{2}\right)\right]}_{\substack{\text{integrable} \\ G_1}} \sin\left(2N\frac{y}{2}\right) + \underbrace{\left[F(y)\frac{y}{\cancel{\sin(\frac{y}{2})}}\cancel{\sin\left(\frac{y}{2}\right)}\right]}_{\substack{\text{integrable} \\ G_2}} \cos\left(2N\frac{y}{2}\right) dy$$

But by the Riemann Lebesgue lemma

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G_2(y)\cos\left(2N\frac{y}{2}\right) dy \to 0 \quad \text{as} \quad N \to \infty.$$

Thus $S_N(f)(x_0) - f(x_0) \to 0$ as $N \to \infty$

$\boxtimes$

Applications

Shape optimization

Thm 1.1 Suppose that Γ is a closed curve without self intersections of length l and let A be the ~~region~~ area of the region enclosed by Γ. Then

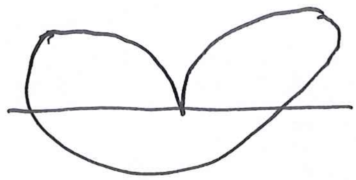
$$A \leq \frac{l^2}{4\pi}$$

with equality iff Γ is a circle.

Def: We say that $\Gamma \subset \mathbb{R}^2$ is a curve if

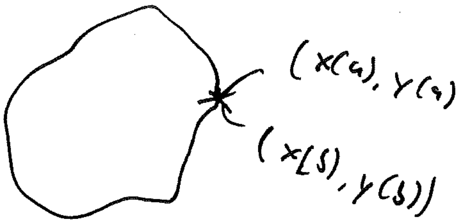
$\Gamma = \{ (x(s), y(s))$; for some functions $x(s), y(s) \in C^1[a,b]$ such that not both $x'(s) = 0$ and $y'(s) = 0$ for any $s \in [a,b] \}$

Remark: $x, y \in C^1$ is rather strong and this excludes many curves s.a.



Definition

1) Γ is closed if $(x(a), y(a)) = (x(b), y(b))$



2) Γ is without self intersections if

$s_1, s_2 \in [a, b)$ or $(a,b] \Longrightarrow (x(s_1), y(s_1)) \neq (x(s_2), y(s_2))$
$s_1 \neq s_2$

3) We define the length of Γ to be

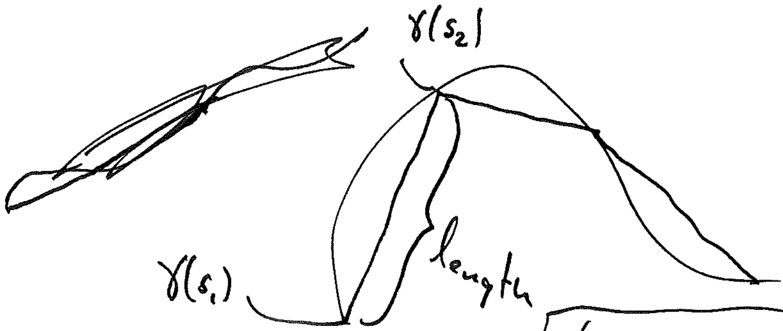
$$\int_a^b \sqrt{(x'(s))^2 + (y'(s))^2}\, ds$$

4) We define the area $\mathcal{A}$ (as in the Theorem) to be

$$\mathcal{A} \doteq \frac{1}{2} \left| \int_\Gamma x\, dy - y\, dx \right| = \frac{1}{2} \left| \int_a^b (x(s) y'(s) + y(s) x'(s))\, ds \right|$$

Remark: 1) Observe that we tie the geometric notions to the language of analysis. This is what makes this interesting, that it allows us to use analytical tools to solve geometric problems

Remark 2. Notice that 1), 2) in the definition are very intuitive and obvious. 3) Is sort of the pythagorean Theorem



$$\sqrt{(x(s_2)-x(s_1))^2+(y(s_2)-y(s_1))^2}$$

$$\approx \sqrt{\left(\frac{x(s_2)-x(s_1)}{s_2-s_1}\right)^2+\left(\frac{y(s_2)-y(s_1)}{s_2-s_1}\right)^2}\,(s_2-s_1)$$

$$\approx \sqrt{(x'(s_1))^2+(y'(s_1))^2}\;ds$$

if s_2-s_1 are small

But 4) is to complicated to be a definition. It is also restrictive, why would we need a C^1-parametrisation of the boundary to calculate the area?

Try to convince yourself that the definition is bad, to complicated and that we only make it out of convenience.