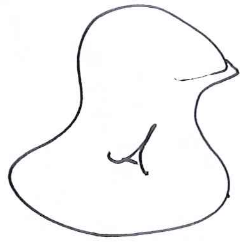
Shape optimization (Application)

Question: Given a curve



in $\mathbb{R}^2$

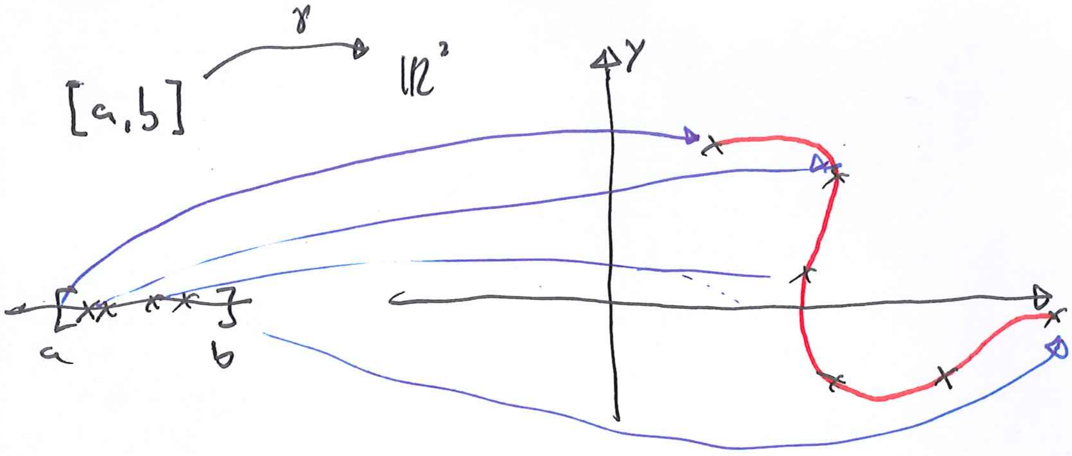
What is the largest area $\mathcal{A}$ that can be enclosed by a curve Γ of length l?

This is not a mathematical question! Not before we can define "curve", "area", "enclose" and "length of a curve" etc.

We want to answer the question by means of Fourier analysis, so we need to be able to define the concepts in terms of analysis.

Definition: We say that a set, Γ, in $\mathbb{R}^2$ is a curve if there exists a function $\gamma(t) : [a,b] \to \mathbb{R}^2$ s.t. $\gamma(t) \in C^1[a,b]$.

The curve is regular if $|\gamma'(t)| > 0$.



If we want an enclosed area we need to define closed curves, we will also need some other definitions.

Definition: We say that a curve Γ, parametrized by $\gamma \in C([a,b])$

i) is closed if $\gamma(a) = \gamma(b)$

ii) is simple if $\gamma(x) \neq \gamma(y)$ for all $x \neq y$, unless $(x,y) = (a,b)$ [or $(x,y) = (b,a)$]

(This means that Γ does not intersect itself)

iii) has length $\int_a^b |\gamma'(t)|\,dt$.

iv) parametrized by arclength if $|\gamma'(t)| = 1$ for all $t \in [a,b]$

v) If γ is closed then we say that the area enclosed by the curve Γ is

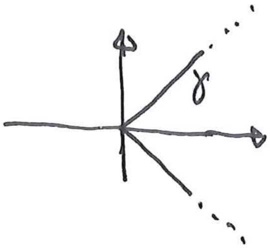
$$A = \frac{1}{2}\left|\int_\Gamma x\,dy - y\,dx\right| = \frac{1}{2}\left|\int_a^b \big(x(t)\,y'(t) - y(t)\,x'(t)\big)\,dt\right|$$

Remarks:

1) This lecture is about mathematical creation. We have a geometric problem in everyday language and we define terms that ties the everyday language problem to mathematical concepts that allows us to use powerful mathematical tools to find a solution.

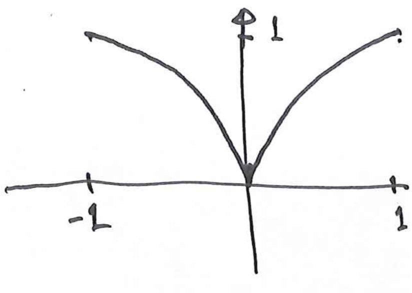
2)

i) When we define a curve to be the graph of a C^1-function γ we impose a strong restriction. Why wouldn't $\gamma(t) = (|t|, t)$ be a curve?

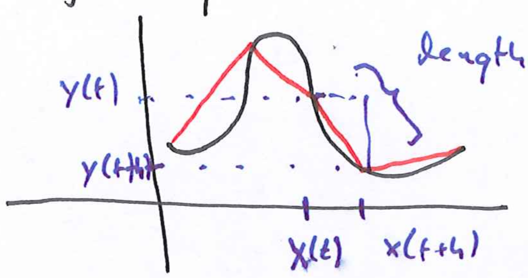


ii) We use regular curve if $|\gamma'| > 0$ since if $|\gamma'| = 0$ then the curve might ~~B~~ have cusps.

Say $\gamma(t) = (t^3, t^2)$ for $t \in [-1, 1]$



iii) The definition of length is based on the Pythagorean Theorem



length $h\sqrt{\frac{(x(t+h)-x(t))^2}{h^2}+\frac{(y(t+h)-y(t))^2}{h^2}}$

b) ~~R~~ The definition of the area is way to complicated to be a definition since it doesn't relate to anything we intuitively would call area. (Assignment 3, however, shows that the definition is reasonable.)

It also hides complicated topological notions such as "inside" and "outside" of a closed curve.

Take 5 minutes to discuss the above definition.

We can now formulate the question mathematically. Rather we will formulate the answer

Theorem 1.1 Suppose that Γ is a simple closed curve in $\mathbb{R}^2$ of length l, let $\mathcal{A}$ denote the area of the region enclosed by the curve. Then

$$\mathcal{A} \leq \frac{l^2}{4\pi}.$$

with equality if and only if Γ is a circle.

Proof: Step 1. (Dimension analysis.)

There is no loss of generality in assuming that $l = 2\pi$.

Let Γ be a curve of length $2\pi\delta$. Then the curve $\frac{1}{\delta}\Gamma$ parametrized by $\frac{1}{\delta}\gamma(t)$ has length

$$l_\delta = \int_a^b \left|\frac{1}{\delta}\gamma'(t)\right| dt = \frac{1}{\delta}\underbrace{\int_a^b |\gamma'(t)|\,dt}_{\text{length } 2\pi\delta} = 2\pi,$$

and encloses an area

$$\mathcal{A}_\delta = \underbrace{\int_a^b \left(\frac{1}{\delta^2}x(t)y'(t) - \frac{1}{\delta^2}y(t)x'(t)\right)dt}_{\text{area within } \frac{1}{\delta}\Gamma} = \frac{1}{\delta^2}\underbrace{\int_a^b x(t)y'(t) - y(t)x'(t)\,dt}_{\substack{\text{Area within } \Gamma \\ = \mathcal{A}}} = \frac{1}{\delta^2}\mathcal{A}$$

So $\mathcal{A} \le \frac{l^2}{4\pi} \iff \mathcal{A}_\delta \le \frac{l_\delta^2}{4\pi}$.

This means that if there isn't any contradiction to $\mathcal{A} \le \frac{l^2}{4\pi}$ for a curve of length 2π then there isn't any contradiction for any curve of length $2\pi\delta$ for any $\delta > 0$.

Step 2. We may assume that Γ is parametrized by arclength.

Proof step 2 (See ~~assignment~~ Exercise 1 p. 120).

Step 3. We may thus assume that Γ is parametrized by $\gamma : [0, 2\pi] \to \mathbb{R}^2$ s.t. $|\gamma(t)| = 1$. Since γ is closed we may extend γ to be a 2π-periodic function.

Under these conditions the Thm is true.

Proof of step 3. Let $\gamma(t) = (x(t), y(t))$ where

$$x(t) \approx \sum_{n=-\infty}^{\infty} a_n e^{int} \qquad y(t) \approx \sum_{n=-\infty}^{\infty} b_n e^{int}$$

$$x'(t) \sim \sum in a_n e^{int} \qquad y'(t) \approx \sum in b_n e^{int}.$$

By Parseval's identity we have

$$\underbrace{\frac{1}{2\pi} \int_0^{2\pi} \left((x'(t))^2 + (y'(t))^2 \right) dt = 1}_{\text{Use parametrization by arclength, not the def. of length wich has a square root.}} \Rightarrow \sum_{n=-\infty}^{\infty} |n|^2 (|a_n|^2 + |b_n|^2) = 1.$$

Applying parseval's identity on

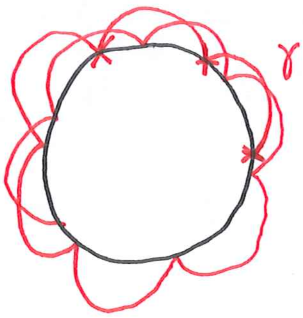
$$\mathcal{A} = \frac{1}{2} \int_0^{2\pi} \left(x(t) y'(t) - y(t) x'(t) \right) dt = \pi \left| \sum_{n=-\infty}^{\infty} n \underbrace{(a_n \bar{b}_n - b_n \bar{a}_n)}_{\substack{\leq 2|a_n||b_n| \leq \\ \leq |a_n|^2 + |b_n|^2}} \right| \leq$$

$$\leq \pi \left| \sum_{n=-\infty}^{\infty} |n| \left(|a_n|^2 + |b_n|^2 \right) \right| \leq \pi \left| \sum_{n=-\infty}^{\infty} |n|^2 \left(|a_n|^2 + |b_n|^2 \right) \right| \leq \pi = \frac{(2\pi)^2}{4\pi} =$$

$$= \frac{\ell^2}{4\pi}.$$

Application (Weyl's equidistribution thm)

Question: Consider the circle (with circumference 1)



and assume that you jump γ units along the circle again and again.

If $\gamma = \frac{p}{q}$ then you will get back to your starting point after q jumps.

But if γ is irrational, say $\sqrt{2}-1$, then you will never get back.

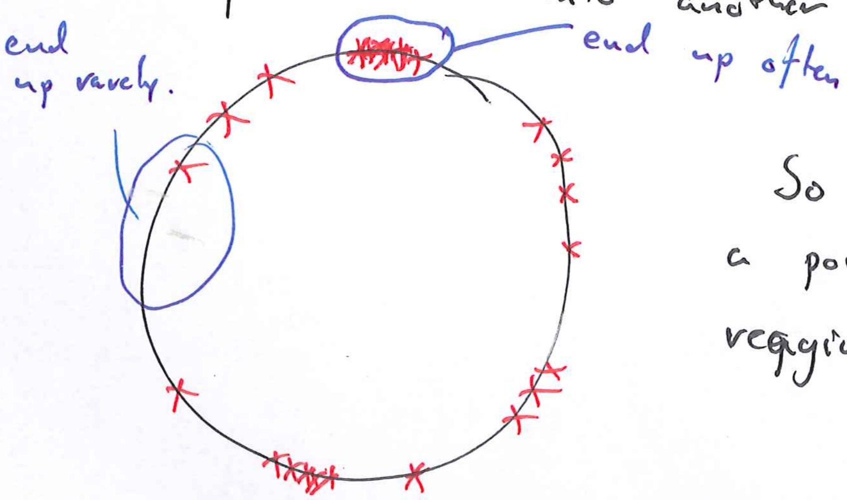
This since if you jumped around the circle exactly p times in q jumps then

$$q(\sqrt{2}-1) = p \quad \Rightarrow \quad \sqrt{2} = \frac{p}{q} + 1 = \frac{p+q}{q}$$

but $\sqrt{2}$ is irrational!

Let us ask a more sophisticated question:

Q: Are you more likely to jump into some reagion than into another reagion?



So instead of considering a point you consider a reagion of the circle.

Mathematical formulation:

Definition: We say that a sequence of numbers $\xi_1, \xi_2, \xi_3, \dots$ in $[0,1)$ (The circle with circumfrance 1) is equidistributed if for every interval $(a,b) \subset [0,1)$ we have

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N;\ \xi_n \in (a,b)\}}{N} = b-a.$$

~~Are~~ Q3: Is the sequence $\langle n\gamma \rangle$ (= decimal part) equidistributed for irrational γ?

Theorem: Yes! (That is $\{\langle n\gamma\rangle\}_{n=1}^{\infty}$ is equidistributed for irrational γ).

How do we prove this?

The relation to Fourier-Analysis is that we are considering periodic objects, But we need to formulate this in terms of functions.

That is we need to <u>create</u> a framework for analyzing the problem - math is creative

Def: We denote by $\chi_{(a,b)}(x)$, the characteristic function of the set (a,b):

$$\chi_{(a,b)}(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b). \end{cases}$$

Then the question becomes: Does

$$(1) \quad \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(\langle n\gamma\rangle) \to (b-a) = \int_0^1 \chi_{(a,b)}(x)\,dx. \quad \text{as } N \to \infty$$

Now we have a functional, analytic, expression of the problem. However, $\chi_{(a,b)}(\langle n\gamma\rangle)$ is complicated and we rather deal with continuous functions since we can approximate them by trigonometric polynomials and thus reduce (1) to a statement about trigonometric polynomials.

$$② \quad \lim \frac{\#\{n;\ 1 \le n \le N,\ \langle n\gamma\rangle \in (a,b)\}}{N} = b-a \iff$$

$$③ \quad \frac{1}{N}\sum \chi_{(a,b)}(\langle n\gamma\rangle) \to \int_0^1 \chi_{(a,b)}(x)\,dx \quad \xLeftrightarrow[\text{by continuous}]{\text{approximation}}$$

$$④ \quad \frac{1}{N}\sum f(\langle n\gamma\rangle) \to \int_0^1 f(x)\,dx \quad \xLeftrightarrow[\text{trig poly}]{\text{approx by}} \quad P(x) = \sum_{|k|\le M} a_k e^{2\pi i k x}$$

$$⑤ \quad \frac{1}{N}\sum P(\langle n\gamma\rangle) \to \int_0^1 P(x)\,dx = \int_0^1 \sum_{|k|\le M} a_k e^{2\pi i k x}\,dx = \int_0^1 a_0\,dx$$

Notice that $\frac{1}{N}\sum_{n=1}^{N} p(\langle n\gamma\rangle) = \frac{1}{N}\sum_{n=1}^{N}\left(\sum_{|k|\le M} a_k e^{2\pi i k\langle n\gamma\rangle}\right)$

$$= a_0 + \underbrace{\frac{1}{N}\sum_{n=1}^{N}\left(\sum_{\substack{|k|\le M\\ |k|\neq 0}} a_k e^{2\pi i k\langle n\gamma\rangle}\right)}_{\text{Must be zero for any } a_k \text{ if (2) should hold.}}$$

Must be zero for any a_k if ~~[illegible]~~ (2) should hold.

So we should be able to reduce the problem to:

(6) Does $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\langle n\gamma\rangle} \longrightarrow 0$ as $N\to\infty$.

Since if (6) holds then by our approximations (4), (3) and (2) should work (there are of course some details to work out.)

<u>Take 5 to discuss the idea.</u>

It remains to prove (6) (I leave the other details to you).

Notice that

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{i2\pi nk\gamma}\right| = \left\{ \begin{array}{l} \sum_{n=1}^{N} z^{n} = \frac{z - z^{N+1}}{1-z} \\ = z\,\frac{1-z^{N}}{1-z} \end{array} \right\} = \left|\frac{1}{N}\, e^{i2\pi k\gamma}\,\frac{1-e^{2\pi i k N\gamma}}{1-e^{2\pi i k\gamma}}\right|$$

$\longrightarrow 0$

since $\left|1-e^{2\pi i k N\gamma}\right| \leq 2$ independent of N

and $\frac{e^{i2\pi k\gamma}}{1-e^{2\pi i k\gamma}} = \text{constant}$. since $e^{2\pi i k\gamma} \neq 1$.

Remark: Notice that we are that it is very simple to sum up $\sum_{n=1}^{N} e^{i2\pi i k\gamma}$

Thus $\frac{1}{N}\sum_{n=1}^{N} e^{i2\pi n k\gamma} \longrightarrow 0$ as $N \to \infty$

$\Longrightarrow$ (standard argument) $\frac{1}{N}\sum_{n=1}^{N} f(\langle n\gamma\rangle) \longrightarrow \int_0^1 f(x)\,dx$ for f cont.

$\Longrightarrow$ standard argument $\frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(\langle n\gamma\rangle) \longrightarrow \int_0^1 \chi_{(a,b)}\,dx = (b-a)$.