Lecture 8

Theorem: If $f(x)$ is differentiable at x^0, then $f(x)$ is continuous at x^0

Proof: If $f(x)$ is differentiable at x^0 then

$$\lim_{h \to 0} \frac{f(x^0+h) - f(x^0)}{h} \text{ exsists} \implies \left| \frac{f(x^0+h)-f(x^0)}{h} \right| \text{ is bounded}$$

$$\implies |f(x^0+h) - f(x^0)| \leq C|h| \overset{\text{Sandwish Thm}}{\implies} \lim_{y \to x^0} |f(y) - f(x^0)| = 0$$

$\implies f$ is continuous.

$\boxtimes$

We know that the reverse is false, say $f(x) = |x|$ is continuous but not differentiable.

The question is "How far from differentiable can a continuous function be?"

Observe that this is again something that needs to be interpeted, how do we measure "far from differentiable."

Does a continuous function have to be differentiable anywhere.

How could such a function be constructed?

Naturally, we will use Fourier series to prove

Theorem: If $0 < \alpha < 1$, then the function

$$f_\alpha(x) = f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i 2^n x}$$

is continuous but not differentiable anywhere.

Before we prove it we will try to understand what is going on. If

$$g(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$ (assume all the convergence you want)

$$g'(x) = \sum_{n=-\infty}^{\infty} \underbrace{in}_{\text{extra term}} a_n e^{inx}$$

so the Fourier series for g' have worse convergence properties and we should be able to find some a_n, such that $\sum_{-\infty}^{\infty} |a_n| < \infty$ (say $a_n = \frac{1}{n^2}$ but $\sum_{-\infty}^{\infty} |n| |a_n|$ diverges, ($|n||a_n| = \frac{1}{|n|}$ which diverges).

The problem is that even if $\sum_{-\infty}^{\infty} |n| |a_n| = \infty$ that doesn't mean that g' isn't continuous, even less that it doesn't exsist.

But we see that taking derivatives worsens the convergence properties - this shows that we are onto something.

Idea: 1) The more terms that are zero the more effect does the $|n|$ in $\sum |n||a_n|$ get. So having $e^{i2^n x}$ means that most a_n are zero (all but when $n = 2^k$ for some k)

2) Instead of focusing on $\sum_{n=-\infty}^{\infty} |n||a_n|$ diverges, which is rather loosely connected to the function g' we will argue by contradiction and assume that $g'(x^0)$ exsists and try to deduce properties of the convergence of the fourier series of g' and then show that f_a doesn't satisfy these properties.

This might not be a function so we have to be carefull in our formulation of this

Principle for argument of contradiction:

When we make an argument by contradiction we add an extra assumption (the assumption that leads to the contradiction).

This gives us extra axioms to work with which should simplify the proof.

Definition: Remember that we defined

$\sigma_N(g) = F_N * g$, where F_N is the

Fejér kernel:

$$F_N(x) = \frac{1}{N} \frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} := \frac{D_0(x) + D_1(x) + \dots + D_{N-1}(x)}{N}$$

We define the delayed means

$$\Delta_N(g) = 2\sigma_{2N}(g) - \sigma_N(g)$$

We make this definition here because it works. Most likely someone tried to do the proof for D_N - but got stuck so he changed to Δ_N which has better properties

Lemma: Let g be any continuous function that is differentiable at x_0. Then

$$\Delta_N(g)'(x_0) = O(\ln N).$$

Observe that this is well defined since $\Delta_N(g)$ is a trig poly. But $\Delta_N(g')$ might not be well defined since g' might not be defined for any $x \neq x_0$

About the proof.

The proof is rather standard. We have an explicit expression $\Delta_N(g)'(x_0)$ that we want to estimate - as you will see we also have the technique for it.

Proof:

$$D_x(\Delta_N(g))'(x_0) = \int_{-\pi}^{\pi} \underbrace{F_N'(x_0-t)}_{\substack{\text{this is a}\\ \text{trig poly so}\\ \text{the derivative}\\ \text{is well defined}}} g(t)\,dt = \left\{\begin{array}{l}\text{Change of variables}\\ x_0 - t \Rightarrow t\end{array}\right\} =$$

$$= \int_{-\pi}^{\pi} F_N'(\cancel{x}\,t)\, g(\cancel{t}\,x_0-t)\,dt - \int_{-\pi}^{\pi} \underbrace{F_N'(t)}_{\boxed{\substack{\text{The derivative of a}\\ \text{cont. function on}\\ \text{periodic function has}\\ \text{integral zero}}}} \underbrace{g(x_0)}_{\text{constant}}\,dt =$$

$$= \int_{-\pi}^{\pi} F_N'(t) \underbrace{\Big(g(x_0-t) - g(x_0)\Big)}_{|g(x_0-t)-g(x_0)| \leq C|t|} dt \Rightarrow |(\Delta_N g)'(x_0)| \leq C \int_{-\pi}^{\pi} |F'(t)||t|\,dt$$

[Observe that we use the same idea as before, (when we proved that $S_N(f)(x^0) \to f(x^0)$ for functions differentiable at x_0).

We use that g is differentiable at x_0 to control the singularity of F_N'

See Thm 2.1 p. 82 in Stein - Shakarchi]

To estimate $\int_{-\pi}^{\pi} |F_N'|\,|t|\,dt$ we only need to estimate F_N' in a good way. But F_N is a function with a given expression so that should not be difficult.

Observe that we do not ~~est~~ need to calculate the integral, just ~~to~~ estimate it.

$$F_N(t) = \frac{1}{N}\,\frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \qquad \text{so}$$

$\left|\sin\left(\frac{Nt}{2}\right)\right| \leq CN|t|$

$$F_N' = \frac{\sin\left(\frac{Nt}{2}\right)\cos\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} - \frac{1}{N}\,\frac{\cos\left(\frac{t}{2}\right)\sin^2\left(\frac{Nt}{2}\right)}{\sin^3\left(\frac{\cancel{N}t}{2}\right)}$$

$\left|\sin\left(\frac{t}{2}\right)\right| > c|t|$

for $|t| \leq \pi$

so

$$|F_N'| \leq \left|\frac{\overset{1}{\cancel{\sin\left(\frac{Nt}{2}\right)}}\;\overset{1}{\cancel{\cos\left(\frac{Nt}{2}\right)}}}{\underset{c^2|t|^2}{\cancel{\sin^2\left(\frac{t}{2}\right)}}}\right| + \left|\frac{\overset{C\cancel{|t|}}{\sin\left(\frac{Nt}{2}\right)}}{\cancel{N}}\;\frac{\overset{1}{\cancel{\cos\left(\frac{t}{2}\right)\sin\left(\frac{Nt}{2}\right)}}}{\underset{c^3|t|^{\cancel{3}\,2}}{\sin^3\left(\frac{\cancel{N}t}{2}\right)}}\right| \leq$$

$$\leq \frac{C}{|t|^2}.$$

~~Bd~~ So

$$\int_{-\pi}^{\pi} |F_N'|\,|t|\,dt \le \int_{-\pi}^{\pi} \frac{c}{|t|}\,dt$$

Diverges, so we need to be more carefull, we know that F_N' should be a trig. poly so this should not diverge. As a matter of fact we should not have a singularity at the origin. We need to analyze things more carefully.

Notice that

$$F_N(t) = \sum_{n=-N}^{N} \underbrace{\frac{N-|n|}{N}}_{|\ |\le 1} e^{-int}.$$

so

$$|F_N'(t)| = \left| \underbrace{\sum_{n=-N}^{N}}_{2N \text{ terms}} \underbrace{\frac{N-|n|}{N}\, n}_{|\ |\le N} e^{-int} \right| \le C'N^2$$

So

$$|F_N'(t)| \le \min\left(\frac{c}{|t|^2},\ C'N^2\right) \quad \text{so}$$

$$\int_{-\pi}^{\pi} |F_N'(t)|\,|t|\,dt \le \int_{-\pi}^{\pi} \min\left(\frac{c}{|t|},\ C'N^2|t|\right) dt \le$$

$$\le C \underbrace{\int_{\pi > |t| > \frac{1}{N}} \frac{1}{|t|}\, dt}_{= C \ln|N|} + C' \underbrace{\int_{|t| < \frac{1}{N}} N^2 |t|\, dt}_{\le C'} = O(\ln(N)).$$

$\blacksquare$

So if f_α is differentiable at x_0

then $$\left|\Delta_{2N}(f_\alpha)'(x_0) - \Delta_N(f_\alpha)'(x_0)\right| = O(\ln(N))$$

But taking $2N = 2^n$ we get, since $f_\alpha = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i 2^n x_0}$

$$\left|\Delta_{2N}(f_\alpha)'(x_0) - \Delta_N(f_\alpha)'(x_0)\right| = \left| \underbrace{\left|2^{-n\alpha}\, 2^n\right|}_{= 2^{(1-\alpha)N}} \underbrace{\left|e^{i 2^n x_0}\right|}_{=1} \right| =$$

$$= 2^{(1-\alpha)\cancel{N}} = \cancel{2^{\log_2 N}} = (2^n)^{1-\alpha} = 2^{1-\alpha} N^{1-\alpha} \gtrsim O(\ln(N))$$

since $\frac{N^{1-\alpha}}{\ln(N)} \to \infty$ as $N \to \infty$.

It thus follows that

$$\Delta_N(f_\alpha)'(x_0) \neq O(\ln(N)) \quad \text{for every } x_0$$

and thus f' is not differentiable for any x_0.

Does $\sum_{n=-N}^{N} \hat{f}(n) e^{inx} \to f(x)$ as $N \to \infty$.

Not a well formulated question. In what sense does it converge

Pointwise convergence

For every $x \in [-\pi, \pi]$ and $\varepsilon > 0$, does it exsist an $N_\varepsilon > 0$ s.t. $N > N_\varepsilon \implies |S_N(f)(x) - f(x)| < \varepsilon$.

1st step

$\hat{f}(n) = 0 \implies f(x) = 0$ for every continuous point

PROOF

$(\varepsilon + \cos(x))^k = P_k(x) \approx \begin{cases} \infty & x \approx 0 \\ 0 & x \not\approx 0 \end{cases}$

$\implies \int_{-\pi}^{\pi} P_k(x) f(x)\, dx \approx f(0) \int_{-\pi}^{\pi} P_k$

$f = g$ and cont. $\iff \hat{f}(n) = \hat{g}(n)$ all n cont.

PROOF

$h = f - g \iff \hat{h}(n) = 0 \iff h = 0 \iff f = g$

$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \implies S_N(f)(x) \to f(x)$

PROOF

$\sum_{n=-N}^{N} \hat{f}(n) e^{inx}$ converges uniformly to a cont. function with the same Fourier coeff as $f(x)$

$f \in C^2 \implies S_N(f)(x) \to f(x)$

PROOF

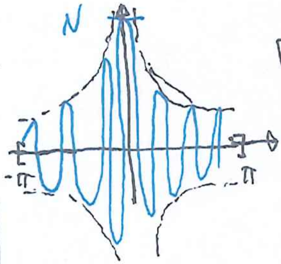
Integration by parts

Should we be satisfied?

No, the C^2 convergence result is good, but it doesn't settle the convergence properties.

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \sum_{n=-N}^{N} \frac{1}{2\pi} e^{inx} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy = \int_{-\pi}^{\pi} f(y) \underbrace{\left[\sum_{n=-N}^{N} \frac{1}{2\pi} e^{in(x-y)} \right]}_{D_N(x-y)} dy$$

$D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$



$D_N(x)$ not good since
i) oscillations
ii) singularity

This leads to "good kernels" s.a.

$F_N = \frac{1}{N} \sum_{n=1}^{N} D_n(x)$, $P_r = \sum r^{|n|} e^{inx}$

$F_N * f(x) \to f(x)$ as $N \to \infty$

$P_r * f(x) \to f(x)$ as $r \to 1$

at points of continuity

$f(x)$ continuous $\implies f(x)$ can be uniformly approximated by trig polynomials.

PROOF

$F_N * f$ is a trig poly and converges uniformly to f since F_N is "good"

Riemann-Lebesgue lemma

f integrable $\implies \hat{f}(n) \to 0$

Takes care of oscillations

$f(x)$ differentiable at $x_0 \implies S_N(f)(x_0) \to f(x_0)$

Takes care of singularity.

Mean square convergence

$\int_{-\pi}^{\pi} |S_N(f)(x) - f(x)|^2 dx \to 0$ as $N \to \infty$

essentially a "linear algebra" approach with basis e^{inx} and inner product

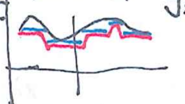
$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)}\, dx$

PROOF Mean square convergence

step 1

$\|f - S_N(f)\| \le \|f - \sum_{n=-N}^{N} b_n e^{inx}\|$ for all b_n

Step 2, approximate f by a continuous function g_ε



$\|f - g_\varepsilon\| \le \varepsilon$

Approximate g_ε by a trig poly $P_\varepsilon(x)$, $|g_\varepsilon - P_\varepsilon| \le \varepsilon$

$\|f - S_N(f)\| \le$

$\|f - g_\varepsilon\| \le \ldots$