The Fourier Transform on $\mathbb{R}$

(Stein, Shakarchi)

Introduction

The theory of Fourier series applies to integrable functions on the circle, or equivalently, to periodic functions on $\mathbb{R}$. For every such f, the Fourier coefficients are given by

$$(1) \qquad a_n = \int_0^1 f(x) e^{-2\pi i n x} dx$$

and then in the appropriate sense we have

$$(2) \qquad f(x) = \sum_{-\infty}^{+\infty} a_n e^{2\pi i n x}.$$

We develop an analogous theory for functions on the entire real line which are non-periodic. These functions are suitably „small" at infinity (soon we will make it more precise).

Given a suitable function on $\mathbb{R}$, the analogous object associated to f will in fact be another function $\hat{f}$ on $\mathbb{R}$, called the Fourier transform of f.

Roughly speaking, the Fourier transform is a continuous version of the Fourier coefficients, in the sense that in (1) we change the domain of integration $(0,1)$ to $\mathbb{R}$, and replace the discrete variable $n \in \mathbb{Z}$ with a continuous variable $\xi \in \mathbb{R}$:

$$(3) \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx.$$

Next we look at (2), then we expect to get the Fourier inversion formula

$$(4) \qquad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi .$$

We will study under which assumptions on f, (4) actually holds.

<u>Remark</u>: Suppose f is supported in a finite interval contained in $I = [-\frac{L}{2}, \frac{L}{2}]$ and we expand f in a Fourier series on I. Then, letting L tend to infinity, we are led to (4).

Homework!

Integration of functions on the real line.

Given a continuous function f on the real line, we want to extend the notion of the integral of f in $\mathbb{R}$. Assuming f is continuous, the integrals $\int_{-N}^{N} f(x)\,dx$ exist, so let

$$(1) \qquad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx.$$

But the limit may not exist; for example take $f = 1$ or $f(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$.

So we need a condition on f which will guarantee the existence of the integral.

<u>Definition:</u> A function f defined on $\mathbb{R}$ is said to be of <u>moderate decrease</u> if f is continuous and there exist a constant $A > 0$, s.t.

$$|f(x)| \le \frac{A}{1+x^2}, \quad \forall\, x \in \mathbb{R}.$$

<u>Note</u>: Such functions are bounded, and decay at infinity as fast as $\frac{1}{x^2}$.

<u>Examples</u>:

$$f(x) = \frac{1}{1+|x|^n}, \quad n \ge 2$$

$$f(x) = e^{-a|x|}, \quad a > 0$$

We denote by $\mathcal{M}(\mathbb{R})$ the set of functions of moderate decrease on $\mathbb{R}$. Under the usual additions and multiplications by scalars, $\mathcal{M}(\mathbb{R})$ forms a vector space over $\mathbb{C}$.

Claim: If $f \in \mathcal{M}(\mathbb{R})$, then the limit in (1) exists and we may define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx.$$

Proof: Define $I_N = \int_{-N}^{N} f(x)\,dx$, and we show that $\{I_N\}$ is a Cauchy sequence in $\mathbb{C}$.

Let $M > N > 0$, then

$$|I_M - I_N| \le \left| \int_{N \le |x| \le M} f(x)\,dx \right| \le A \int_{N \le |x| \le M} \frac{1}{x^2}\,dx$$

$$= 2A\left(\frac{1}{N} - \frac{1}{M}\right) < \frac{2A}{N} \to 0 \text{ as } N \to \infty$$

Note: We have also proved that

$$\int_{|x| \ge N} f(x)\,dx \to 0 \quad \text{as } N \to \infty.$$

Remark: We see that assuming

$$|f(x)| \le \frac{A}{1+|x|^{1+\varepsilon}}$$

for some $\varepsilon > 0$, would be another way of defining functions of moderate decrease.

For convenience, we let $\varepsilon = 1$

Proposition 1: The integral of a function of moderate decrease satisfies the following properties:

(i) Linearity; $\forall f, g \in \mathcal{M}(\mathbb{R})$, $\forall a, b \in \mathbb{C}$

$$\int (a f(x) + b g(x))\,dx = a \int f(x)\,dx + b \int g(x)\,dx$$

(ii) Translation invariance: $\forall h \in \mathbb{R}$

$$\int f(x-h)\,dx = \int f(x)\,dx$$

(iii) Scaling under dilations: $\forall \delta > 0$

$$\delta \int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

(iv) Continuity: $f \in \mathcal{M}(\mathbb{R})$, then

$$\int |f(x-h) - f(x)|\,dx \longrightarrow 0 \quad \text{as} \quad h \to 0.$$

Proof: (i) is clear.

(ii) consider the difference.

$$\int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx = \int_{-N-h}^{N-h} f(x)\,dx - \int_{-N}^{N} f(x)\,dx =$$

$$= \int_{-N-h}^{-N} f(x)\,dx + \int_{N-h}^{N} f(x)\,dx;$$

we have.

$$\left| \int_{-N-h}^{-N} f(x)\,dx + \int_{N-h}^{N} f(x)\,dx \right| \leq \frac{A}{1+N^2}, \text{ hence.}$$

$$\int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx \longrightarrow 0 \quad \text{as } N \to \infty.$$

(iii) a change of variable gives us

$$\delta\int_{-N}^{N} f(\delta x)\,dx = \int_{-\delta N}^{\delta N} f(x)\,dx, \quad \delta>0 \text{ let } N\to\infty.$$

(iv) Assume $|h|\leq 1$. For any $\varepsilon>0$ $\exists N=N_\varepsilon$ s.t.

$$\int_{|x|\geq N} |f(x)|\,dx \leq \frac{\varepsilon}{4} \quad \text{and} \quad \int_{|x|\geq N} |f(x-h)|\,dx \leq \frac{\varepsilon}{4}$$

f is uniformly continuous in the interval $[-N-1, N+1]$, hence $\sup\limits_{|x|\leq N} |f(x-h)-f(x)|\to 0$ as $h\to 0$.

So we can take $h=h_\varepsilon$ small s.t.

$$|f(x-h)-f(x)|\leq \frac{\varepsilon}{4N}, \quad \forall x\in[-N,N], \cancel{\forall h}$$
$$\forall h, |h|\leq h_\varepsilon$$

Then

$$\int_{-\infty}^{\infty} |f(x-h)-f(x)|\,dx \leq \int_{-N}^{N} |f(x-h)-f(x)|\,dx$$

$$+\int_{|x|\geq N} |f(x-h)|\,dx + \int_{|x|\geq N} |f(x)|\,dx \leq$$

$$\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.$$

Definition of Fourier transform

For $f \in \mathcal{M}(\mathbb{R})$, we define its Fourier transform by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

where $\xi \in \mathbb{R}$.

$|f(x) e^{-2\pi i x \xi}| = |f(x)|$, so the integral makes sense, and $|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)| dx$, $\forall \xi \in \mathbb{R}$, $\hat{f}$ is bounded. Moreover $\hat{f}$ is continuous and tends to 0 as $|\xi| \to \infty$. (Exercise)

Assuming $f \in \mathcal{M}(\mathbb{R})$ does <u>not</u> imply $\hat{f} \in \mathcal{M}(\mathbb{R})$. So we need to strengthen our assumptions on f, to assure that $\hat{f} \in \mathcal{M}(\mathbb{R})$ and the integral $\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$ makes sense.

For that we introduce the space of Schwartz functions.

The Schwartz space

The Schwartz space on $\mathbb{R}$ consists of all infinitely differentiable functions f so that f and all its derivatives $f', f'', \dots$ are rapidly decreasing in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty, \quad \forall k, \ell \in \mathbb{Z}, \ k, \ell \geq 0$$

Such functions form a vector space over $\mathbb{C}$, denoted by $S(\mathbb{R})$.

It is obvious that $S(\mathbb{R}) \subsetneq M(\mathbb{R})$,

$f(x) = \frac{1}{1+x^2} \notin S(\mathbb{R})$.

$$f \in S(\mathbb{R}) \Rightarrow f'(x) \in S(\mathbb{R}), \ x f(x) \in S(\mathbb{R}).$$

<u>Conclusion:</u> S is closed under differentiation and multiplication by polynomials.

An important example of a Schwartz function is the Gaussian, $f(x) = e^{-x^2}$. Indeed, for any polynomial $p(x)$, $p(x) e^{-x^2}$ is bounded, and the derivatives of f are of the form $p(x) e^{-x^2}$.

Another example is given in exercise 4, compactly supported smooth functions.

<u>Note:</u> $g(x) = e^{-|x|} \notin S(\mathbb{R})$, it is not differentiable at 0.

The Fourier transform on S

Let $f \in S(\mathbb{R})$,

$$\hat{f}(\xi) = \int f(x) e^{-2\pi i x \xi} dx.$$

We use the notation

$$f(x) \longrightarrow \hat{f}(\xi)$$

to mean that $\hat{f}$ denotes the Fourier transform of f.

<u>**Proposition 2:**</u> If $f \in S(\mathbb{R})$, then

<u>(i)</u> $f(x+h) \longrightarrow \hat{f}(\xi) e^{2\pi i h \xi}, \quad \forall h \in \mathbb{R}$

<u>(ii)</u> $f(x) e^{-2\pi i x h} \longrightarrow \hat{f}(\xi + h), \quad \forall h \in \mathbb{R}$

<u>(iii)</u> $f(\delta x) \longrightarrow \delta^{-1} \hat{f}(\delta^{-1} \xi), \quad \forall \delta > 0$

<u>(iv)</u> $f'(x) \longrightarrow 2\pi i \xi \hat{f}(\xi)$

<u>(v)</u> $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi} \hat{f}(\xi)$

<u>Proof:</u> (ii) is obvious from the definition, (i) and (iii) follow from Proposition 1, (i) is the translation invariace property for the integrals over $\mathbb{R}$. and (iii) is the property of scaling.

<u>(iv)</u> Integrating by parts gives

$$\int_{-N}^{N} f'(x) e^{-2\pi i x \xi} dx = \Big[f(x) e^{-2\pi i x \xi} \Big]_{-N}^{N} + $$

$$+ 2\pi i \xi \int_{-N}^{N} f(x) e^{-2\pi i x \xi} dx$$

Letting $N \to \infty$, we get (iv).

(v) show that $\hat{f}$ is differentiable, and find the derivative. Let $\varepsilon > 0$ and consider

$$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} - \widehat{(-2\pi i x f)}(\xi) =$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \left[\frac{e^{-2\pi i x h}-1}{h} + 2\pi i x \right] dx$$

$f \in S \Rightarrow \exists N = N_\varepsilon$ s.t.

$$\int_{|x| \geq N} |f(x)| dx \leq \varepsilon \quad \text{and} \quad \int_{|x| \geq N} |x| |f(x)| dx \leq \varepsilon$$

$|x| \leq N \Rightarrow \exists h_0$ s.t. $\forall h$, s.t. $|h| \leq h_0$ we have

$$\left| \frac{e^{-2\pi i x h}-1}{h} + 2\pi i x \right| \leq \frac{\varepsilon}{N}$$

Finally,

$$\left| \frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} - \widehat{(2\pi i x f)}(\xi) \right| \leq$$

$$\leq \int_{-N}^{N} \left| f(x) e^{-2\pi i x \xi} \left[\frac{e^{-2\pi i x h}-1}{h} + 2\pi i x \right] \right| dx + C\varepsilon \leq C_0 \varepsilon$$

ends the proof of proposition.

Theorem: If $f \in S(\mathbb{R})$ then $\hat{f} \in S(\mathbb{R})$.

Proof: With the method of induction, using (iv) and (v) in proposition, we can prove that

$$\frac{1}{(2\pi i)^k}\left(\frac{d}{dx}\right)^k\left[(-2\pi i x)^\ell f(x)\right] \longrightarrow \xi^k\left(\frac{d}{d\xi}\right)^\ell \hat{f}(\xi)$$

for any $k, \ell \in \mathbb{Z}$, $k, \ell \geq 0$

so $\xi^k\left(\frac{d}{d\xi}\right)^\ell \hat{f}(\xi)$ is the Fourier transform of a Schwartz function, hence it is bounded.