Solutions Assignment week 2 SF2705.

4. Assignment for the 4th of February:
Let $f(x)$ be a 2π periodic function such that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for some sequence c_n such that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. We want to find a $2\pi-$periodic solution, $y(x)$, to the following differential equation

$$y'(x) + ay(x) = f(x) \tag{1}$$

where $a \in \mathbb{R}$ and $a \neq 0$.

1. For any $N \in \mathbb{N}$ find a $2\pi-$periodic solution, $y_N(x)$, to

$$y_N'(x) + ay_N(x) = S_N(f)(x).$$

2. Carefully prove that there exists a $2\pi-$periodic function $y(x)$ such that $y_N(x) \to y(x)$ uniformly on $[0, 2\pi]$. That $y(x)$ is continuously differentiable and that $y(x)$ solves (1).

Solution: All the references are to W. Rudin *Principles of mathematical analysis.*
Part 1. We guess that $y_N(x) = \sum_{n=-N}^{N} a_n e^{inx}$ for some constants a_n. Then

$$y_N'(x) + ay_N(x) = \sum_{n=-N}^{N} ina_n e^{inx} + a\sum_{n=-N}^{N} a_n e^{inx} = \sum_{n=-N}^{N} (in+a)a_n e^{inx}. \tag{2}$$

Identifying coefficients in the right side of (2) and $S_n(f)(x) = \sum_{n=-N}^{N} c_n e^{inx}$ we directly see that $y_N(x)$ is a solution if $a_n = \frac{c_n}{a+in}$. Since $a \neq 0$ is a real number the denominator $a + in \neq 0$ so a_n is well defined.

It only remains to prove that y_N is 2π periodic. This follows from the fact that e^{inx} is $2\pi-$periodic (Theorem 8.7 in Rudin) and the following simple calculation

$$y_N(x) = \sum_{n=-N}^{N} a_n e^{inx} = \left\{ \begin{array}{c} \text{Thm 8.7} \\ \text{in Rudin} \end{array} \right\} = \sum_{n=-N}^{N} a_n e^{in(x+2\pi)} = y_N(x+2\pi).$$

This finishes the proof of part 1.

Part 2: We need to show the following things

- There exists a function $y(x)$ such that $\lim_{N\to\infty} y_N(x) = y(x)$ for every x.
- $y(x)$ is continuously differentiable.
- $y(x)$ is 2π periodic.
- $y'(x) + ay(x) = f(x)$.

It is worth to notice that the first part is extraordinaryily abstract. We need to prove the **existence** of an object. In order to do so we will have to rely on the analysis we know. Theorem 7.8 in Rudin states that if $y_N(x)$ is a Cauchy sequence of functions then there is a $y(x)$ such that $\lim_{N\to\infty} y_N(x) = y(x)$ uniformly in x.

By a "Cauchy sequence of functions" we mean that for every $\epsilon > 0$ there should exist an $N_\epsilon > 0$ such that

$$N, M > N_\epsilon \Rightarrow |y_N(x) - y_M(x)| < \epsilon \text{ for every } x. \tag{3}$$

In order to prove the existence of an N_ϵ such that (3) holds we may assume that $N > M$ and calculate

$$|y_N(x) - y_M(x)| = \left| \sum_{n=-N}^{N} a_n e^{inx} - \sum_{n=-M}^{M} a_n e^{inx} \right| = \left| \sum_{n=-N}^{-M-1} a_n e^{inx} + \sum_{n=M+1}^{N} a_n e^{inx} \right| \leq$$

$$\leq \left\{ \begin{array}{c} \text{the triangle} \\ \text{inequality} \end{array} \right\} \leq \sum_{n=-N}^{-M-1} \left| a_n e^{inx} \right| + \sum_{n=M+1}^{N} \left| a_n e^{inx} \right| = \tag{4}$$

$$= \left\{ \begin{array}{c} \text{using } |e^{iy}| = 1 \\ \text{and } a_n = \frac{c_n}{a+in} \end{array} \right\} = \sum_{n=-N}^{-M-1} \left| \frac{c_n}{a+in} \right| + \sum_{n=M+1}^{N} \left| \frac{c_n}{a+in} \right| \leq \sum_{n=-N}^{-M-1} |c_n| + \sum_{n=M+1}^{N} |c_n|$$

where we used that $|a + in| = \sqrt{a^2 + n^2} \geq 1$ for $n > N_\epsilon > 0$. Notice that (4) reduces the statement that $y_N(x)$ is a Cauchy sequence to a statement about c_n - since we have an assumption on the coefficients c_n this is a good thing.

We assume that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. By definition this means that $C_N = \sum_{n=-N}^{N} |c_n| < C$ for some constant C and every $N \geq 0$. Also $C_{N+1} - C_N = |c_{N+1}| + |c_{-N-1}| \geq 0$ so $\{C_N\}_{N=0}^{\infty}$ forms a bounded increasing sequence. Bounded increasing sequences converge (Theorem 3.14 in Rudin) so $\lim_{N\to\infty} C_N$ exists. Convergent sequences are Cauchy (Theorem 3.11 in Rudin) which by definition means that for every $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that

$$M, N > N_\epsilon \Rightarrow |C_N - C_M| < \epsilon. \tag{5}$$

Noticing that

$$|C_N - C_M| = \left| \sum_{n=-N}^{N} |c_n| - \sum_{n=-M}^{M} |c_n| \right| = \sum_{n=-N}^{-M-1} |c_n| + \sum_{n=M+1}^{N} |c_n|$$

we have shown that for every $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that

$$M, N > N_\epsilon \Rightarrow \sum_{n=-N}^{-M-1} |c_n| + \sum_{n=M+1}^{N} |c_n| < \epsilon. \tag{6}$$

Using (6) in (4) we may conclude that $y_N(x)$ forms a Cauchy sequence. It follows that there exists a function $y(x)$ such that $y_N(x) \to y(x)$ uniformly in x.

In order to show that $y(x)$ is continuously differentiable we argue similarly. Theorem 7.12 in Rudin states that if $y'_N(x)$ converges uniformly and $y'_N(x)$ is continuous then the limit function is also continuous. It is clear that $y'_N(x)$ is continuous since

$$y'_N(x) = \sum_{n=-N}^{N} \frac{inc_n}{in + a} e^{inx}$$

and e^{inx} is continuous so a finite sum of such terms has to be continuous.

We thus only need to show that y'_N converges uniformly. To show that we argue as before (why not try - it worked before!) and show that $y'_N(x)$ forms a Cauchy sequence uniformly in x. In particular we assume that $N > M$ and calculate

$$\begin{aligned} |y'_N(x) - y'_M(x)| &= \left| \sum_{n=-N}^{N} \frac{inc_n}{in+a} e^{inx} - \sum_{n=-M}^{M} \frac{inc_n}{in+a} e^{inx} \right| = \left| \sum_{n=-N}^{-M-1} \frac{inc_n}{in+a} e^{inx} + \sum_{n=M+1}^{N} \frac{inc_n}{in+a} e^{inx} \right| \leq \\ &\leq \left\{ \begin{array}{l} \text{the triangle} \\ \text{inequality} \end{array} \right\} \leq \sum_{n=-N}^{-M-1} \left| \frac{inc_n}{in+a} e^{inx} \right| + \sum_{n=M+1}^{N} \left| \frac{inc_n}{in+a} e^{inx} \right| \leq \\ &\leq \left\{ \begin{array}{l} \text{using} \\ \text{that } |e^{iy}| = 1 \end{array} \right\} \leq \sum_{n=-N}^{-M-1} \left| \frac{inc_n}{in+a} \right| + \sum_{n=M+1}^{N} \left| \frac{inc_n}{in+a} \right| \leq \sum_{n=-N}^{-M-1} |c_n| + \sum_{n=M+1}^{N} |c_n| \end{aligned} \tag{7}$$

where we used that

$$\left| \frac{in}{in+a} \right| = \left(\frac{n^2}{n^2 + a^2} \right)^{\frac{1}{2}} \leq \frac{1}{2}$$

in the last inequality.

Using (5) and (7) we can conclude, as before that for every $\epsilon > 0$ there exists an N_ϵ such that

$$M, N > N_\epsilon \Rightarrow |y'_N(x) - y'_M(x)| < \epsilon$$

for all x. The uniform convergence of $y'_N(x)$ to some function $\tilde{y}'(x)$ follows. Observe that it is not obvious that $\tilde{y}'(x) = y'(x)$ - that needs to be proved. To that end we define, for $x \in [-\pi, \pi]$,

$$\tilde{y}(x) = \sum_{n=-\infty}^{\infty} \frac{c_n}{ni + a} + \int_0^x \tilde{y}'(t) dt.$$

By the fundamental theorem of calculus this makes sense (that is the derivative of $\tilde{y}$ equals $\tilde{y}'(x)$).

Next, let $\epsilon > 0$ and $N_{\epsilon/2}$ be as in (3), then for $N > N_{\epsilon/2}$

$$\begin{aligned} |\tilde{y}(x) - y(x)| &= \left| \left(\tilde{y}(x) - \sum_{n=-N}^{N} \frac{c_n}{in+a} e^{inx} \right) - \left(y(x) - \sum_{n=-N}^{N} \frac{c_n}{in+a} e^{inx} \right) \right| \leq \\ &\leq \left\{ \begin{array}{l} \text{the triangle} \\ \text{inequality} \end{array} \right\} \leq \left| \tilde{y}(x) - \sum_{n=-N}^{N} \frac{c_n}{in+a} e^{inx} \right| + \left| y(x) - \sum_{n=-N}^{N} \frac{c_n}{in+a} e^{inx} \right| = \\ &= \left\{ \begin{array}{l} \text{by defi.} \\ \text{of } y \text{ and } \tilde{y} \end{array} \right\} = \left| \sum_{n=-\infty}^{\infty} \frac{c_n}{ni+a} + \int_0^x \left(\lim_{N\to\infty} \sum_{n=-M}^{M} \frac{inc_n}{in+a} e^{inx} dt \right) - \sum_{n=-N}^{N} \frac{c_n}{in+a} e^{inx} \right| + \end{aligned}$$

$$+\left|\lim_{M\to\infty}\sum_{n=-M}^{M}\frac{c_n}{in+a}e^{inx}-\sum_{n=-N}^{N}\frac{c_n}{in+a}e^{inx}\right|=$$

$$=\left|\lim_{M\to\infty}\sum_{M\geq|n|>N}\frac{c_n}{in+a}e^{inx}\right|+\left|\lim_{M\to\infty}\sum_{M\geq|n|>N}\frac{c_n}{in+a}e^{inx}\right|<\epsilon$$

where we used the Corollary to Theorem 7.16 in Rudin in the last equality. Thus $\tilde{y}(x)=y(x)$.

That $y(x)$ is $2\pi-$periodic follows from

$$y(x)-y(x+2\pi)=\lim_{N\to\infty}y_N(x)-\lim_{N\to\infty}y_N(x+2\pi)=$$

$$=\left\{\begin{array}{l}\text{since}\\ y_N(x)=y_N(x+2\pi)\end{array}\right\}=\lim_{N\to\infty}y_N(x)-\lim_{N\to\infty}y_N(x)=0.$$

It only remains to show that $y'(x)+ay(x)=f(x)$. To that end, given an $\epsilon>0$, we choose $N_{\epsilon/3}$ large enough so that $N>N_{\epsilon/3}$ implies that

$$\left|y(x)-\sum_{n=-N}^{N}\frac{c_n}{a+in}e^{inx}\right|<\frac{\epsilon}{3},$$

$$|a|\left|y'(x)-\sum_{n=-N}^{N}\frac{inc_n}{a+in}e^{inx}\right|<\frac{\epsilon}{3}$$

and

$$\left|f(x)-\sum_{n=-N}^{N}c_ne^{inx}\right|<\frac{\epsilon}{3}$$

uniformly in x, such a choice of $N_{\epsilon/3}$ is possible by the uniform convergence $y_N\to y$, $y_N'\to y'$ and $S_N(f)\to f$.

It follows that for any $\epsilon>0$ and x that

$$|y'(x)+ay(x)-f(x)|=$$

$$=\left|\left(y'(x)-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{inc_n}{a+in}e^{inx}\right)+a\left(y(x)-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{c_n}{a+in}e^{inx}\right)-\left(f(x)-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}c_ne^{inx}\right)+\right.$$

$$\left.+\left(\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{inc_n}{a+in}e^{inx}+\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{c_n}{a+in}e^{inx}-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}c_ne^{inx}\right)\right|\leq$$

$$\leq\left|y'(x)-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{inc_n}{a+in}e^{inx}\right|+|a|\left|y(x)-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{c_n}{a+in}e^{inx}\right|+\left|f(x)-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}c_ne^{inx}\right|+$$

$$+\left|\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{inc_n}{a+in}e^{inx}+\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}\frac{c_n}{a+in}e^{inx}-\sum_{n=-N_{\epsilon/3}}^{N_{\epsilon/3}}c_ne^{inx}\right|<$$

$$<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}+0=\epsilon,$$

where we used the triangle inequality in the first inequality.

Thus $|y'(x)-ay(x)-f(x)|<\epsilon$ for every $\epsilon>0$ and therefore $y'(x)-ay(x)=f(x)$. The proof is therefore done.