Solutions Assignment week 8 SF2705.

Assignment for the 1st of April: Let $f^j(x)$ be a sequence of integrable functions on the circle such that $\int_{-\pi}^{\pi}|f^j(x)|^2dx = 1$. Furthermore let $u^j(x,t)$ solve the heat equation on the circle with initial data $u^j(x,0) = f^j(x)$.

Prove that for any $t_0 > 0$ there exists a function $u_{t_0}(x)$, and a subsequence u^{k_j}, such that

$$\int_{-\pi}^{\pi} \left|u^{k_j}(x,t_0) - u_{t_0}(x)\right|^2 dx \to 0 \qquad \text{as } j \to \infty.$$

Does it necessarily exist a function $f_0(x)$ such that

$$\int_{-\pi}^{\pi} |f^{k_j}(x) - f_0(x)|^2 dx \to 0?$$

Instead of a proof: I will not write a formal proof of the assignment. A proof would be to formal and instead I will try to reason myself to a solution. However, a formal proof can easily be constructed from the following discussion. But in discussing my way towards a proof I will have the chance to implicitly remark on the informal thinking behind the process of writing a proof.

We know that the space of square integrable functions is isometric to $l^2(\mathbb{Z})$. Therefore the question is about convergence properties of $l^2(\mathbb{Z})$. Furthermore the question has two parts: *i)* relates the convergence in l^2 to the heat equation and *ii)* is just a statement about the convergence in $l^2(\mathbb{Z})$ (it has nothing really to do with the heat equation).

It is therefore reasonable to start with the second part of the question - since there we don't have to deal with the heat equation. That is, can we construct a sequence f^j of integrable functions such that $\int_{-\pi}^{\pi}|f^j(x)|^2dx = 1$ such that f^j does not have any subsequence converging in the mean square sense?

If we formulate this in the $l^2(\mathbb{Z})-$sense, which is equivalent by Parseval's identity, we look for a sequence $A_j \in l^2(\mathbb{Z})$, say $A_j = (..., a_{-1}^j, a_0^j, a_1^j, a_2^j, ...) \in l^2(\mathbb{Z})$ such that $\sum_{n=-\infty}^{\infty} |a_n^j|^2 = 1$, but it does not exist any A_0 such that $\lim_{j\to\infty} A_j = A_0$ where the limit is interpreted in the $l^2(\mathbb{Z})-$sense ($\lim_{j\to\infty}\sum_{n=-\infty}^{\infty}|a_n^j - a_n^0|^2 \neq 0$ for all $A_0 \in l^2(\mathbb{Z})$).

We immediately observe that this is related to Bolzano-Weierstrass Theorem. Bolzano-Weierstrass Theorem says that, for any $n \in \mathbb{N}$, if $v^j \in \mathbb{R}^n$ is a bounded sequence then v^j has a convergent subsequence. So the second part of the exercise states that the Bplzano-Weierstrass Theorem does not hold in $l^2(\mathbb{Z})$. And the only thing that distinguishes $l^2(\mathbb{Z})$ from a space $\mathbb{R}^n$ is that $l^2(\mathbb{Z})$ is infinite dimensional - we therefore need to use the infinite dimensionality of $l^2(\mathbb{Z})$ to construct our counterexample.

One way to use the infinite dimensionality of $l^2(\mathbb{Z})$ is to let A_j be defined by

$$a_n^j = \begin{cases} \frac{1}{\sqrt{j}} & \text{if } 1 \le n \le j \\ 0 & \text{else,} \end{cases}$$

another way would be to define A_j by

$$a_n^j = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{else.} \end{cases}$$

Notice that in both cases we use that we have an infinite dimensional vector-space $l^2(\mathbb{Z})$.

In both cases we get $|A_j| = \sum_{n=-\infty}^{\infty}|a_n^j|^2 = 1$. Clearly, in both cases, $\lim_{j\to\infty} a_n^j = 0$ for any $n \in \mathbb{Z}$. This implies that any limit, if one exists, A_0 of the sequence A_j must satisfy $A_0 = 0$. But then $\|A_j - A_0\| = \|A_j\| = 1$ which contradicts that $\lim_{k\to\infty}\|A_{j_k} - A_0\| = 0$ for some subsequence. We may conclude that A_j does not have any convergent subsequences.

To prove the first part of the assignment we need to understand what is going on in that part. Somehow, what makes the convergence failing is that the major contribution of $\sum_{n=-\infty}^{\infty}|a_n^j|^2$ comes from large n. In particular, in both our examples of non-converging A_j we have that for any $N > 0$ and $\epsilon > 0$ there exists a j_ϵ such that $\sum_{n>|N|}|a_n^j|^2 > 1-\epsilon$ for all $j > j_\epsilon$. So, we may hypothesize that, convergence fails because the mass "leaks out to infinity".

Now if we consider the sequence of solutions u^j to the heat equation with initial data f^j we see from equation (8) on page 119 in Stein-Shakarchi that

$$u^j(x,t) = \sum_{n=-\infty}^{\infty} a_n^j e^{-4\pi^2n^2t} e^{2\pi i n x},$$

where a_n^j are the Fourier coefficients of f^j. So if $t_0 > 0$ then the Fourier coefficients of $u^j(x,t_0)$ are $a_n^j e^{-4\pi^2n^2t_0}$. Notice that the factor $e^{-4\pi^2n^2t_0}$ goes to zero very fast as $n^2 \to \infty$ so somehow that u^j solves the heat equation forces the Fourier coefficients to die off very fast. That is the Fourier coefficients for large n does not add much to the integral $\int_{-\pi}^{\pi}|u^j(x,t_0)|^2dx = 2\pi\sum_{n=-\infty}^{\infty}|a_n^j e^{-4\pi^2n^2t_0}|^2$ (where we used Parseval's identity).

We need to use that intuition to create a subsequence u^{j_k} that converges (in the mean square sense). How do we do that?

We intuitively think that the Fourier coefficients of u^j for large n, say for $|n| > N$, should not influence the convergence much. So let us assume that the Fourier coefficients of f^j satisfies $a_n^j = 0$ for $|n| > N$ and all j. This is crazy and absolutely unjustified - but when we are just playing with ideas we are allowed to do whatever we want to see where that leads us. In this case we may use the Bolzano-Weierstrass Theorem and conclude that there exists a subsequence such that

$$(a^{j_k}_{-N}, a^{j}_{-N+1}, ..., a^{j}_{0}, a^{j}_{1}, a^{j}_{2}, ..., a^{j}_{N}) \to (a^{0,N}_{-N}, a^{0,N}_{-N+1}, ..., a^{0,N}_{0}, a^{0,N}_{1}, ..., a^{0,N}_{N}).$$

So under this, ridiculous assumption, we can find a subsequence $j_{k,N}$ such that

$$u^{j_{k,N}} = \sum_{n=-N}^{N} a_n^{j_{k,N}} e^{-4\pi^2 n^2 t} e^{2\pi i n x} \to u^{0,N} = \sum_{n=-N}^{N} a_n^{0,N} e^{-4\pi^2 n^2 t} e^{2\pi i n x}, \tag{1}$$

where the last equality defines $u^{0,N}$.

Let us pick such a sub-sequence $j_{k,1}$ (that is we choose $N = 1$), and then a subsequence of $j_{k,1}$ which we denote by $j_{k,2}$ where (1) holds for $N = 2$ and inductively a subsequence of $j_{k,N-1}$ denoted by $j_{k,N}$ where (1) holds for N.

Since $\{u^{j_{k,N}}\}_{k=1}^{\infty}$ is a subsequence of $\{u^{j_{k,N-1}}\}_{k=1}^{\infty}$ we see that $a_n^{0,N-1} = a_n^{0,N}$ for $n = -N+1, -N+2, ..., 0, 1, 2, ..., N-1$. This defines a unique element $(..., a^0_{-1}, a^0_0, a^0_1, a^0_2, ...) \in l^2(\mathbb{Z})$ such that $a^0_n = a^{0,N}_n$ for all $N \geq |a|$ and a corresponding function $u^0 = \sum_{n=-\infty}^{\infty} a^0_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}$.

Now let us get rid of the ridiculous assumption. We should be able to do that since the contribution of the Fourier coefficients for $|n| > N$ should contribute with less than ϵ when N is large enough. That is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u^{j_{k,N}} - u^0|^2 dx = \sum_{n=-N}^{N} e^{-8\pi^2 n^2 t} |a_n^{j_{k,N}} - a^0_n|^2 + \sum_{|n|>N} e^{-8\pi^2 n^2 t_0} |a_n^{j_{k_n}} - a^0_n|^2.$$

Using that $a_n^{j_{k,N}} \to a^0_n$ as $k \to \infty$ for $|n| \leq N$ we can conclude that there exists a $k_{\epsilon,N}$ such that if $k > k_{\epsilon,N}$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u^{j_{k,N}} - u^{0,N}|^2 dx \leq \epsilon + \sum_{|n|\geq N} e^{-8\pi^2 n^2 t_0} |a_n^{j_{k,N}}|^2 \leq \epsilon + C e^{-8\pi^2 N^2 t_0},$$

where we used that $|a_n^{j_{k,N}}|, |a^0_n| \leq 1$ (why?). But that implies that there exists a N_ϵ, depending only on ϵ and t_0 but not on k_ϵ, such that $Ce^{-8\pi^2 N^2 t_0} < \epsilon$ if $N > N_\epsilon$. We may thus conclude that there exists an N_ϵ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u^{j_{k,N}} - u^0|^2 dx < 2\epsilon \tag{2}$$

if $N > N_\epsilon$ and $k > k_{\epsilon,N_\epsilon}$.

Now we choose our final sub-sequence in the following way. For each $m \geq 1$ there exists an $N_{1/m}$ and a $k_{1/m,N_{1/m}}$ such that (2) holds with $\epsilon = 1/m$. That is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u^{j_{k,N}} - u^0|^2 dx < \frac{2}{m}$$

for all $k > k_{1/m,N_{1/m}}$ and $N > N_{1/m}$. We can thus choose $j_m = j_{k_{1/m}+1,N_{1/m}+1}$, this is a subsequence of j that satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u^{j_m} - u^0|^2 dx < \frac{2}{m}.$$

Clearly this implies, by the sandwich Theorem, that $u^{j_m} \to u^0$ in the mean square sense.

Remark: The proof of this exercise was a little more difficult than I had hoped so it will not be marked. Instead you get a solution.

There is a simpler proof of the first part of the exercise. Let me briefly describe the idea. Notice that

$$D_x S_N(u^j(\cdot, t_0))(x) = \sum_{n=-N}^{N} 2\pi i n a_n^j e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

but

$$\sum_{n=-\infty}^{\infty} \left| 2\pi i a_n^j e^{-4\pi^2 n^2 t_0} \right| < \infty$$

since $t_0 > 0$. Hence the Fourier coefficients of $D_x u^j(x, t_0)$ are absolutely convergent and we may therefore conclude that $D_x S_N(u^j)(x)$ converges uniformly to a continuous function. As we showed in the first assignment (in the beginning of the course) $D_x S_n(u^j) \to D_x u^j$.

Moreover, since $|a_n^j| \leq 1$ we may conclude that $|D_x S_N(u^j)| \leq \sum_{n=-\infty}^{\infty} \left| 2\pi e^{-4\pi^2 n^2 t_0} \right|$. Therefore $D_x S_N(u^j)$ is uniformly bounded, and thus the derivatives $D_x u^j$ are uniformly bounded. This means that the sequence $\{u^j\}_{j=1}^{\infty}$ forms an equicontinuous sequence of functions and we may, by the Arzela-Ascoli Theorem, find a sub-sequence u^{j_k} that converges uniformly to some continuous function u^0. But we know that $u^{j_k} \to u^0$ uniformly implies that $u^{j_k} \to u^0$ in the mean square sense which proves the statement.

I did not want to use this argument for two reasons. First, this proof is logically not much simpler since the Arzela-Ascoli Theorem uses the rather complicated diagonalization argument that we did explicitly. Of course, one reason that we remember difficult theorems is not to have to do the argument again. But logically it is not a simplification to "hide" the diaginalization in a proof from a previous course.

Secondly, the proof that we gave does not use as much about the sequence. As a matter of fact we only used that for any $\epsilon > 0$ there exists an N_ϵ such that $\|u^j - S_N(u^j)\| < \epsilon$ for all $N > N_\epsilon$ and all j. But in the second argument based on the Arzela-Ascoli Theorem we used that the Fourier coefficients of the derivatives of u^j where absolutely convergent which is much stronger. Last lecture I tried to make a point of analyzing exactly what we use in our proofs and estimates. And from that perspective the first argument is better since it uses less and may thus be generalized to more situations.