

Homological algebra and algebraic topology

Problem set 1

due: Tuesday Sept 10 in class.

Problem 1 (2p). Verify that the following sequence is a chain complex and compute its homology groups:

$$\dots \xrightarrow{6} \mathbf{Z}/12\mathbf{Z} \xrightarrow{4} \mathbf{Z}/12\mathbf{Z} \xrightarrow{6} \mathbf{Z}/12\mathbf{Z} \xrightarrow{4} \mathbf{Z}/12\mathbf{Z} \xrightarrow{6} \dots$$

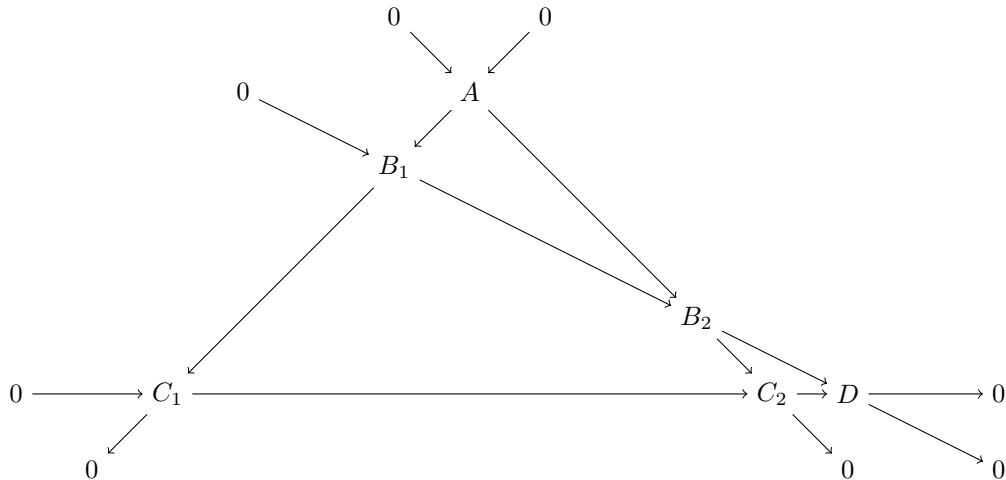
Problem 2 (3p). Let

$$0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0$$

be an exact sequence of abelian groups. Show that the following are equivalent:

- (1) There exists a homomorphism $q: A \rightarrow A'$ such that $q \circ i = \text{id}_{A'}$.
- (2) There exists a homomorphism $j: A'' \rightarrow A$ such that $p \circ j = \text{id}_{A''}$.
- (3) There is an isomorphism $\phi: A \rightarrow A' \oplus A''$ such that $\phi \circ i: A' \rightarrow A' \oplus A''$ is the inclusion into the first coordinate and $p \circ \phi^{-1}: A' \oplus A'' \rightarrow A''$ is the projection onto the second coordinate.

Problem 3 (3p). Given the following commutative diagram of abelian groups:



Assume that $0 \rightarrow A \rightarrow B_i \rightarrow C_i \rightarrow 0$ and $0 \rightarrow C_1 \rightarrow C_2 \rightarrow D \rightarrow 0$ are exact. Show, using a diagram chase, that $0 \rightarrow B_1 \rightarrow B_2 \rightarrow D$ is also exact.

Problem 4 (2p). Let $0 \rightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \rightarrow 0$ be an exact sequence of abelian groups, and let B be another abelian group. Then we obtain a sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(A'', B) & \rightarrow & \text{Hom}(A, B) & \rightarrow & \text{Hom}(A', B) \rightarrow 0 \\ & & f & \mapsto & f \circ p & & \\ & & & & g & \mapsto & g \circ i. \end{array}$$

Is this sequence a chain complex? Is it exact for any choice of A, A', A'' , and B ?