

Homological algebra and algebraic topology

Problem set 2

due: Tuesday Sept 17 in class.

Problem 1 (3pt). Assume that the following diagram of abelian groups has exact rows and columns. Can you determine the missing entries and maps? Give short reasonings.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & & \longrightarrow & & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{Z}/3\mathbf{Z} & \longrightarrow & & \longrightarrow & \mathbf{Z}/2\mathbf{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Problem 2 (4pt). Let C_\bullet be a chain complex and denote by $C_\bullet/2C_\bullet$ the chain complex which in degree n is the quotient groups $C_n/2C_n$. Consider the sequence

$$0 \rightarrow C_\bullet \xrightarrow{2} C_\bullet \rightarrow C_\bullet/2C_\bullet \rightarrow 0.$$

- (1) Show that this sequence is exact if all C_n are free abelian groups.
- (2) Give an example of such a C_\bullet that has a nontrivial connecting homomorphism $\delta_n: H_{n+1}(C_\bullet/2C_\bullet) \rightarrow H_n(C_\bullet)$.

Problem 3 (3pt). Let \mathcal{C} be a category. Denote by $\text{Ar}(\mathcal{C})$ the following category: an object in $\text{Ar}(\mathcal{C})$ is a morphism $X_1 \rightarrow X_2$ in \mathcal{C} . A morphism in $\text{Ar}(\mathcal{C})$ from $X_1 \rightarrow X_2$ to $Y_1 \rightarrow Y_2$ is a commutative diagram

$$\begin{array}{ccc}
 X_1 & \longrightarrow & Y_1 \\
 \downarrow & & \downarrow \\
 X_2 & \longrightarrow & Y_2.
 \end{array}$$

- (1) Show that $\text{Ar}(\mathcal{C})$ is indeed a category.
- (2) Show that $F: \text{Ar}(\text{Ab}) \rightarrow \text{Ar}(\text{Ab})$ with $F(X_1 \xrightarrow{f} X_2) = (\ker(f) \hookrightarrow X_1)$ defines a functor.