Homological algebra and algebraic topology
Problem set 3

due: Tuesday Sept 24 in class.

Problem 1 (3pt)**.** An **equivalence of categories** is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $F \circ G \to \mathrm{id}_{\mathcal{D}}$, $G \circ F \to \mathrm{id}_{\mathcal{C}}$.

Show that an equivalence of categories sends products to products and coproducts to coproducts. That is, if X_i are a family of objects in $\mathcal{C}$ with coproduct X then $F(X)$ is the coproduct of $F(X_i)$ in $\mathcal{D}$, and similarly for products.

Problem 2 (2pt)**.** Let X be a partially ordered set and denote by $\mathcal{X}$ the associated category with $\mathrm{ob}(\mathcal{X}) = X$ and

$$\mathrm{Hom}_{\mathcal{X}}(x, y) = \begin{cases} \{*\}; & x \leq y \\ \emptyset; & \text{otherwise.} \end{cases}$$

Which pairs of objects x, y have a product (coproduct) in $\mathcal{X}$, and how can you describe such a product (coproduct) explicitly?

Problem 3 (3pt)**.** Let R^n denote the direct sum of n copies of a ring R, considered as a left or right R-module. Show that

$$R^n \otimes_R R^m \cong R^{mn}.$$

Problem 4 (2pt)**.** Show the following isomorphisms of abelian groups:

(1) $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.
(2) $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Q}) \cong \mathbf{Q}$.

Here $\mathrm{Hom}_{\mathbf{Z}}$ is a shorthand for $\mathrm{Hom}_{\mathrm{Mod}_{\mathbf{Z}}}$, i.e. homomorphisms of abelian groups.