Homological algebra and algebraic topology
Problem set 4

due: Tuesday Oct 1 in class.

Problem 1 (2pt)**.** Show that an exact sequence of abelian groups $0 \to \mathbf{Q} \to M \to N \to 0$ with the rationals on the left hand side always splits.

Problem 2 (3pt)**.** Let p be a prime, and denote by $\mathbf{Z}_{(p)} \subset \mathbf{Q}$ the subset of all fractions $\frac{n}{q}$ for which $p \nmid q$.

(1) Show that $\mathbf{Z}_{(p)}$ is a subring of $\mathbf{Q}$.
(2) Show that $\mathbf{Z}_{(p)}$ is flat as a $\mathbf{Z}$-module (abelian group).
(3) Show that if l is another prime and A is a finite abelian l-group then

$$\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} A \cong \begin{cases} A; & p = l \\ 0; & p \neq l. \end{cases}$$

Problem 3 (3pt)**.** Let $\tau = \sqrt{-5} \in \mathbf{C}$ and let $R = \mathbf{Z}[\tau] = \{a + b\tau \in \mathbf{C} \mid a,\ b \in \mathbf{Z}\}$. Let $M = (2, 1+\tau) < R$ be the ideal generated by the elements 2 and $1+\tau$, regarded as a (sub-)R-module of R.

(1) Show that M is not free.
(2) Show that M is projective.

Problem 4 (2pt+2pt)**.** Let R be a ring and M be an R-module. We say that M is **finitely generated** if there is a surjective homomorphism $p\colon R^n \to M$ for some $n \geq 0$. If n and p can be chosen such that $\ker(p)$ is also finitely generated, we call M **finitely presented**.

Denote by M^* the right R-module $\mathrm{Hom}_R(M, R)$, where the right action is given by $(f.r)(m) = f(m)r$.

(1) For R-modules M, N, consider the natural map

$$\begin{aligned} a\colon M^* \otimes_R N &\to \mathrm{Hom}_R(M, N) \\ f \otimes n &\mapsto (m \mapsto f(m)n). \end{aligned}$$

Show that a is an isomorphism if M is finitely presented and projective.

(2) **(bonus +2pt)** Show that finitely presented flat R-modules are projective.