# Homological algebra and algebraic topology
# Problem set 5

due: Tuesday Oct 8 in class.

**Problem 1** (3pt)**.** Let $R$ be the polynomial ring $\mathbf{Q}[x, y]$. Construct free resolutions of the following $R$-modules:

(1) $M = \mathbf{Q}$ with $R$-action $\mathbf{Q}[x, y] \times \mathbf{Q} \to \mathbf{Q}$ given by $x.a = 0 = y.a$ for all $a \in \mathbf{Q}$.
(2) The ideal $(x, y) \subset R$.

**Problem 2** (2pt)**.** Let $R = \mathbf{Z}[x]$ and define a covariant functor $F\colon \mathrm{Mod}_R \to \mathrm{Ab}$ by

$$F(M) = \{x.m \mid m \in M\} \subseteq M.$$

On maps $f\colon M \to N$, we define $F(f)\colon F(M) \to F(N)$ by $F(f) = f|_{F(M)}$. Verify that this is actually a functor. Is $F$ left/right exact?

**Problem 3** (3pt)**.** Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of $R$-modules and let $P'_\bullet \to M'$, $P''_\bullet \to M''$ be projective resolutions. Show that there is a projective resolution $P_\bullet \to M$ with $P_n = P'_n \oplus P''_n$ for all $n \geq 0$ such that the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_1 & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_2} & P''_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_0 & \xrightarrow{\iota_1} & P_0 & \xrightarrow{\pi_2} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' & \longrightarrow & 0
\end{array}$$

commutes and has exact rows and columns, where $\iota_1\colon P'_n \to P'_n \oplus P''_n$ denotes the inclusion into the first summand and $\pi_2\colon P'_n \oplus P''_n \to P''_n$ the projection onto the second summand.

*Hint:* the middle vertical maps are *not* of the form $(x, y) \mapsto (\partial' x, \partial'' y)$!

**Problem 4** (2pt)**.** Let $R$ be a ring and $F\colon \mathrm{Mod}_R \to \mathrm{Ab}$ a right exact functor. Using the result of Problem 3 and the homology long exact sequence, show that there is a long exact sequence

$$\cdots \to (L_2F)(M) \xrightarrow{L_2F(p)} (L_2F)(M'') \to (L_1F)(M')$$
$$\xrightarrow{L_1F(i)} (L_1F)(M) \xrightarrow{L_1F(p)} L_1F(M'') \to F(M') \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(M'') \to 0.$$