1. Chain complexes

Definition. A sequence of abelian groups

$$\dots C_{-2},\ C_{-1},\ C_0,\ C_1, \dots$$

with homomorphisms $\partial_i \colon C_{i+1} \to C_i$ is called a (homological) **chain complex** if $\partial_{i-1} \circ \partial_i = 0$ for all $i \in \mathbf{Z}$.

A **cohomological chain complex** is almost the same thing, but with reversed grading: a sequence of abelian groups

$$\dots C^{-2},\ C^{-1},\ C^0,\ C^1, \dots$$

together with homomorphisms $d^i \colon C^{i-1} \to C^i$ such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbf{Z}$.

We will concentrate on homological chain complexes; all results hold analogously for cohomological chain complexes.

A chain complex (or just a sequence of abelian groups with homomorphisms) is called **bounded below** (**bounded above**) if $C_i = 0$ for $i \ll 0$ (resp. $i \gg 0$). It is called **non-negatively graded** (**non-positively graded**) if $C_i = 0$ for $i < 0$ (resp. $i > 0$).

Definition. We call the subgroup $Z_i(C_\bullet) = \ker(\partial_{i-1} \colon C_i \to C_{i-1}) < C_i$ the subgroup of ***i*-cycles** and the subgroup $B_i(C_\bullet) = \operatorname{im}(\partial_i \colon C_{i+1} \to C_i) < C_i$ the subgroup of ***i*-boundaries**.

Lemma 1.1. *For any chain complex $C_\bullet$, $B_i(C_\bullet)$ is a subgroup of $Z_i(C_\bullet)$.*

Definition. The *i*th **homology group** of a chain complex $C_\bullet$ is defined as the quotient group

$$H_i(C_\bullet) = Z_i(C_\bullet)/B_i(C_\bullet).$$

If $Z_n = B_n$ for all n (and thus $H_n = 0$), we call $C_\bullet$ **exact** or **acyclic**. An exact chain complex is more usually called **exact sequence**. An exact sequence of the form

$$0 \to A' \to A \to A'' \to 0$$

is often called a **short exact sequence**.

Lemma 1.2. *Let A, B, C be abelian groups and $f \colon A \to B$ and $g \colon B \to C$ homomorphisms.*

1. $0 \to A \xrightarrow{f} B$ *is exact iff f is injective.*
2. $A \xrightarrow{f} B \to 0$ *is exact iff f is surjective.*
3. $0 \to A \xrightarrow{f} B \to 0$ *is exact iff f is an isomorphism.*
4. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is exact iff f is injective, g is surjective, and* $\ker g = \operatorname{im} f$.

Definition. Let $A_\bullet$, $B_\bullet$ be sequences of abelian groups and homomorphisms (or chain complexes). A **map of sequences** (or **map of chain complexes**) is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial_n^A} & A_n & \xrightarrow{\partial_{n-1}^A} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial_n^B} & B_n & \xrightarrow{\partial_{n-1}^B} & B_{n-1} & \longrightarrow & \cdots
\end{array}$$

Lemma 1.3. *A map of chain complexes $f\colon C_\bullet \to D_\bullet$ induces maps*

$$\begin{aligned} Z(f)\colon &\quad Z_n(C_\bullet) \to Z_n(D_\bullet) &&\text{of } n\text{-cycles,}\\ B(f)\colon &\quad B_n(C_\bullet) \to B_n(D_\bullet) &&\text{of } n\text{-boundaries, and}\\ H_n(f) = f_*\colon &\quad H_n(C_\bullet) \to H_n(D_\bullet) &&\text{on homology.}\end{aligned}$$

Lemma 1.4 (Five-lemma)**.** *Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_3} & & \big\downarrow{\scriptstyle f_4} & & \big\downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram of abelian groups with exact rows. Then:

(1) *if f_2, f_4 are surjective and f_5 is injective then f_3 is surjective.*
(2) *if f_2, f_4 are injective and f_1 is surjective then f_3 is injective.*
(3) *in particular, if f_1, f_2, f_4, f_5 are isomorphisms then so is f_3.*

Definition. A short exact sequence of the form $0 \to A' \to A' \oplus A'' \to A''$, where the first map is the inclusion into the first summand and the second map is the projection onto the second, is called **split exact**.

See homework problem 1.2 for characterizations of split exact sequences.

Definition. Let $f\colon A \to B$ be a homomorphism between abelian groups. Define its **cokernel** $\operatorname{coker}(f)$ to be the quotient group $B/\operatorname{im}(f)$ and its **coimage** $\operatorname{coim}(f)$ to be $A/\ker(f)$.

Lemma 1.5. *For any homomorphism $f\colon A \to B$ of abelian groups, we have:*

(1) $f\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ *is an isomorphism;*
(2) $0 \to \ker(f) \to A \xrightarrow{f} B \to \operatorname{coker}(f) \to 0$ *is exact.*

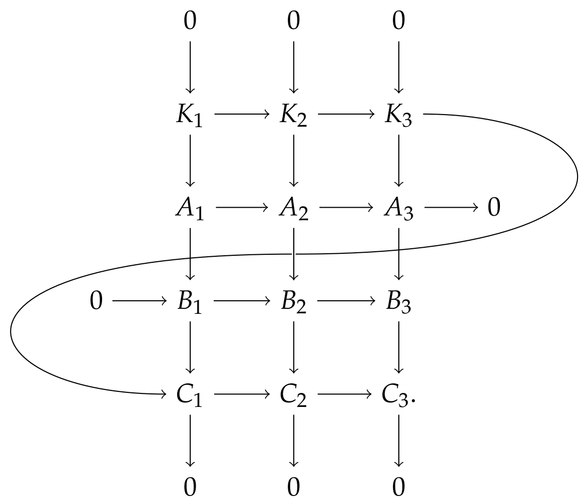
Lemma 1.6 (Snake lemma)**.** *Given a diagram of abelian groups*

$$\begin{array}{ccccccccc} & & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & & \end{array} \tag{1.7}$$

with exact rows. Let K_i denote the kernel of $A_i \to B_i$ and C_i its cokernel. Then there is a "snake homomorphism" $K_3 \to C_1$ such that the sequence

$$K_1 \to K_2 \to K_3 \to C_1 \to C_2 \to C_3$$

is exact:



If $A_1 \to A_2$ *is injective then so is* $K_1 \to K_2$*, and if* $B_2 \to B_3$ *is injective then so is* $C_2 \to C_3$.

Furthermore, the snake map is natural, *meaning that if we have a map* $(A_i, B_i) \to (A'_i, B'_i)$ *of diagrams of the type* (1.7) *then the following square commutes:*

$$\begin{array}{ccc} K_3 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ K'_3 & \longrightarrow & C'_1. \end{array}$$

Theorem 1.8. *Let* $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} C_\bullet \to 0$ *be a short exact sequence of chain complexes (meaning* $0 \to A_n \to B_n \to C_n \to 0$ *is exact for each* $n \in \mathbf{Z}$*). Then there is a* ***connecting homomorphism*** $\delta_n \colon H_{n+1}(C_\bullet) \to H_n(A_\bullet)$ *such that the following long sequence is exact:*

$$\cdots \xrightarrow{p_*} H_{n+1}(C_\bullet) \xrightarrow{\delta_n} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{p_*} H_n(C_\bullet) \xrightarrow{\delta_{n-1}} H_{n-1}(A_\bullet) \xrightarrow{i_*} \cdots.$$

The homomorphism δ *is natural: given a map of short exact sequences of chain complexes* $(A_\bullet, B_\bullet, C_\bullet) \to (A'_\bullet, B'_\bullet, C'_\bullet)$*, the following square commutes:*

$$\begin{array}{ccc} H_{n+1}(C_\bullet) & \xrightarrow{\delta} & H_n(A_\bullet) \\ \downarrow & & \downarrow \\ H_{n+1}(C'_\bullet) & \xrightarrow{\delta} & H_n(A'_\bullet). \end{array}$$

2. Categories and functors

Definition. A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{ob}(\mathcal{C})$ of **objects**;
- for each pair of objects $X, Y \in \mathrm{ob}(\mathcal{C})$, a set of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(X, Y)$;

- for each object $X \in \mathrm{ob}(\mathcal{C})$, an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ called **identity morphism**;
- for each three objects $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, a map

  $$\circ\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}(X, Z), \quad (g, f) \mapsto g \circ f$$

  called **composition**.

These have to satisfy the following axioms:

(1) The composition $\circ$ is associative;
(2) For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\mathrm{id}_Y \circ f = f$ and $f \circ \mathrm{id}_X = f$.

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called an **isomorphism** (and the objects X, Y **isomorphic**) if there is another morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $g \circ g = \mathrm{id}_Y$. If such a g exists, it is unique and is denoted by f^{-1}.

We will often abuse notation and write $X \in \mathcal{C}$ for $X \in \mathrm{ob}(\mathcal{C})$, $f \in \mathrm{Hom}(X, Y)$ or even just $f\colon X \to Y$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and id for id_X. We will also use commutative diagrams to denote equalities between compositions of morphisms.

Definition. We use the following standard notations for familiar categories:

Set**:** The category of sets and functions;
Ab**:** The category of abelian groups and homomorphisms;
Top**:** The category of topological spaces and continuous maps.

Definition. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A **(covariant) functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, also called F; and
- for every $X, Y \in \mathrm{ob}(\mathcal{C})$, a function $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ denoted by $f \mapsto F(f)$ or $f \mapsto f_*$

satisfying $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and $(g \circ f)_* = g_* \circ f_*$.

A **contravariant functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- a function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, also called F; and
- for every $X, Y \in \mathrm{ob}(\mathcal{C})$, a function $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$ denoted by $f \mapsto F(f)$ or $f \mapsto f^*$

satisfying $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and $(g \circ f)^* = f^* \circ g^*$. ("It turns arrows around.")

Definition. A **natural transformation** $\eta\colon F \to G$ between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ consists of a morphism $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ for each object $X \in \mathcal{C}$ such that for each morphism $f\colon\ \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y). \end{array}$$

Natural transformations between contravariant functors are defined analogously.

A natural transformation $\eta\colon F \to G$ is called **natural isomorphism** (and F and G **isomorphic**, $F \simeq G$) if η_X is an isomorphism for all $X \in \mathcal{C}$.

Definition. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is another functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$, where $\mathrm{Id}_{\mathcal{C}}$, $\mathrm{Id}_{\mathcal{D}}$ denote the identity functors on $\mathcal{C}$ and $\mathcal{D}$, respectively.

Definition. Let $\mathcal{C}$ be a category and $(X_i)_{i\in I}$ a family of objects in $\mathcal{C}$, for some index set I. An object X together with morphisms $\iota_i\colon X_i \to X$ is called **coproduct** of the X_i, and is denoted by $\coprod_{i\in I} X_i$, if for each test object $Y \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(\iota_i,-)} \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i,Y)$$

is a bijection. The coproduct of only two objects is denoted by $X_1 \sqcup X_2$.

Similarly, an object X with morphism $\pi_i\colon X \to X_i$ is called **product** of the X_i, and is denoted by $\prod_{i\in I} X_i$, if for each test object $Y \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(Y,X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,\pi_i)} \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(Y,X_i)$$

is a bijection. The product of only two objects is denoted by $X_1 \times X_2$.

Lemma 2.1. *In an arbitrary category $\mathcal{C}$, (co-)products need not exist, but if they do, they are unique up to isomorphism.*

3. Rings and modules

Definition. A **ring** R is an abelian group together with a **unity** $1 \in R$ and an associative bilinear map $R \times R \to R$, $(x,y) \mapsto xy$, such that $1x = x1 = x$ for all $x \in R$. A ring is called **commutative** if $xy = yx$ for all $x, y \in R$.

A map $f\colon R \to S$ between rings is called a **ring homomorphism** or **map of rings** if it is linear, $f(1_R) = 1_S$, and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

Definition. A **left module** M over a ring R is an abelian group M together with a bilinear multiplication map $R \times M \to M$, $(r,m) \mapsto r.m$, such that $1.m = m$ and $(r_1r_2).m = r_1.(r_2.m)$ for all $m \in M$, $r_i \in R$.

A **right module** is an abelian group M with a bilinear multiplication map $M \times R \to M$, $(m,r) \mapsto m.r$, such that $m.1 = m$ and $m.(r_1r_2) = (m.r_1).r_2$ for all $m \in M$, $r_i \in R$.

When we just say "module", we agree to mean a left module.

A map $f\colon M \to N$ between two (left or right) R-modules M, N is an **R-module homomorphism** if it is a abelian group homomorphism and $f(r.m) = r.f(m)$ (resp. $f(m.r) = f(m).r$) for all $r \in R$, $m \in M$.

The category of left R-modules and R-module homomorphisms is denoted by Mod_R.

Definition. The **product** of a family $(M_i)_{i\in I}$ of R-modules, denoted by $\prod_{i\in I} M_i$, is the module whose underlying abelian group is the product groups, and the R-module structure is given by $r.((m_i)_{i\in I}) = (r.m_i)_{i\in I}$. The **direct sum** of the family, denoted by $\bigoplus_{i\in I} M_i$, is the submodule of families $(m_i)_{i\in I}$ where all but finitely many $m_i = 0$.

An R-module M is called **free** if it is isomorphic to an (arbitrarily indexed) direct sum of copies of R.

Lemma 3.1. *The direct product is a product in* Mod_R *in the category-theoretic sense, and the direct sum is a coproduct.*

Definition. Let R be a ring, M a right R-module, and N a left R-module. The **tensor product** $M \otimes_R N$ is the abelian group obtained as follows. Denote by $\mathrm{Fr}(M \times N)$ the free abelian group with generators pairs (m,n) with $m \in M$, $n \in N$. Then

$M \otimes_R N$ is the quotient of $\mathrm{Fr}(M \times N)$ with respect to an equivalence relation $\sim$ given by:

- $(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$
- $(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$
- $(m.r, n) \sim (m, r.n)$

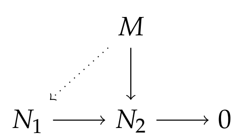
We denote the equivalence class of (m, n) in $M \otimes_R N$ by $m \otimes n$.

Proposition 3.2. *In the context of the previous definition, let T be an abelian group. Denote by $\mathrm{Bil}(M, N; T)$ the set of all bilinear homomorphisms $f\colon M \times N \to T$ with $f(m.r, n) = f(m, r.n)$. Then there is a natural isomorphism*

$$\mathrm{Bil}(M, N; T) \cong \mathrm{Hom}_{\mathbf{Z}}(M \otimes_R N, T).$$

Definition (and lemma)**.** An R-module M is called **projective** if it satisfies the following equivalent conditions:

(1) For each diagram in Mod_R



with exact row, a lift (dotted arrow) exists such that the resulting diagram commutes.

(2) There is an R-module N such that $M \oplus N$ is free.

(3) Every shot exact sequence $0 \to N_1 \to N_2 \to M \to 0$ splits.

(4) The functor $\mathrm{Hom}_R(M, -)$ maps exact sequences to exact sequences (the functor "is exact").

Lemma 3.3. *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of right R-modules, and let M be a left R-module. Then the sequence of abelian groups*

$$N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$

is exact. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of left R-modules, and let M be another left R-module. Then the sequence of abelian groups

$$0 \to \mathrm{Hom}_R(N'', M) \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N', M)$$

is exact.

Definition. A left R-module M is called **flat** if the functor $- \otimes_R M$ from right R-modules to abelian groups is exact. A right R-module is flat if the functor $M \otimes_R -$ from left R-modules to abelian groups is exact.

Lemma 3.4. *Free modules are projective. Projective modules are flat. Not every flat module is projective, and not every projective module is free.*

4. Resolutions and derived functors

Definition. Let R be a ring. A nonnegatively graded chain complex $P_\bullet$ of R-modules together with a map $\epsilon P_0 \to M$ (the "augmentation") is called a **projective resolution** of M if

- For every $i \geq 0$, P_i is projective;

- The extended chain complex $\cdots \to P_1 \to P_0 \xrightarrow{\epsilon} M$ is exact.

Proposition 4.1. *Every R-module M has a projective resolution.*

Corollary 4.2. *If R is a principal ideal domain then every R-module has a projective resolution of length* 2*:*

$$0 \to P_1 \to P_0 \to M \to 0$$

Definition. Let $C_\bullet$, $D_\bullet$ be nonnegatively graded chain complexes of R-modules and let $f, g\colon C_\bullet \to D_\bullet$ be two chain maps. A **chain homotopy** from f to g is a sequence of R-linear maps $h_n\colon C_{n-1} \to D_n$ such that

$$g - f = h \circ \partial^C + \partial^D \circ h.$$

If such a chain homotopy exists, we call f and g **chain homotopic** and write $f \simeq g$.

If $f\colon C_\bullet \to D_\bullet$ and $g\colon D_\bullet \to C_\bullet$ are chain maps with chain homotopies $\mathfrak{g} \circ f \simeq \mathrm{id}_{C_\bullet}$ and $f \circ g \simeq \mathrm{id}_{D_\bullet}$, we call f and g **chain homotopy equivalences** and the chain complexes $C_\bullet$ and $D_\bullet$ **chain homotopy equivalent**.

Proposition 4.3. *If $f \simeq g$ then $f_* = g_*\colon H_*(C_\bullet) \to H_*(D_\bullet)$.*

Theorem 4.4. *Let $f\colon M \to N$ be a morphism of R-modules, $P_\bullet \to M$ a chain complex where all P_i are projective, and $N_\bullet \to N \to 0$ be an exact complex. Then*

(1) *The exists a chain map $f_\bullet\colon P_\bullet \to N_\bullet$ making the following ladder commute:*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & N_1 & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

(2) *Any two such extensions $f_\bullet$, $g_\bullet$ are chain homotopic.*

Corollary 4.5. *Any two projective resolutions of M are chain homotopy equivalent.*

Definition. Let R, S be two rings and $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ a (covariant or contravariant) functor. We call F **additive** if the induced map on Hom-sets

$$\mathrm{Hom}_R(M, N) \xrightarrow{F} \mathrm{Hom}_S(F(M), F(N)) \quad (\text{resp. } \mathrm{Hom}_S(F(N), F(M)))$$

is a homomorphism of abelian groups.

Let F be an additive covariant functor as above. Then we call F

- **left exact** if $0 \to F(M') \to F(M) \to F(M'')$ is exact;
- **right exact** if $F(M') \to F(M) \to F(M'')$ is exact;
- **exact** if it is right and left exact, i. e. if $0 \to F(M') \to F(M) \to F(M'') \to 0$ is exact

for all choices of exact sequences $0 \to M' \to M \to M'' \to 0$ of R-modules.

Similarly, if F is contravariant, we call it

- **left exact** if $0 \to F(M'') \to F(M) \to F(M')$ is exact;
- **right exact** if $F(M'') \to F(M) \to F(M')$ is exact;
- **exact** if it is right and left exact, i. e. if $0 \to F(M'') \to F(M) \to F(M') \to 0$ is exact

for all choices of exact sequences $0 \to M' \to M \to M'' \to 0$ of R-modules.

Definition (and lemma)**.** Let $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ be a covariant right exact functor, M an R-module, and $P_\bullet \to M$ a projective resolution of M. Define the n**th left derived functor** $L_nF\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ by

$$(L_nF)(N) = H_n(F(P_\bullet)).$$

Similarly, if F is a contravariant left exact functor, define the n **right derived functor** $R^nF\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ by

$$(R^nF)(N) = H^n(F(P_\bullet)).$$

This is independent of the choice of resolution and extends to a functor by defining it on morphisms as follows: if $f\colon M \to M'$ is a morphism of R-modules, extend it to a morphism $f_\bullet\colon P_\bullet \to P'_\bullet$ by Thm. 4.4 and set

$$L_n(F)(f) = H_n(F(f_\bullet));$$

similarly for right derived functors.

Lemma 4.6. *If F is covariant right exact then $L_0F = F$. If F is contravariant left exact then $R^0F = F$.*

Lemma 4.7. *If R is a principal ideal ring and $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ a right exact covariant or left exact contravariant functor. Then $L_nF = 0$ (resp. $R^nF = 0$) if $n \geq 2$.*

Lemma 4.8. *Let F be a covariant left exact functor. Then $L_nF = 0$ for all $n \geq 1$ if and only if F is exact.*

Definition. Let R be a ring, M a right R-module, and N a left R-module. Define $\mathrm{Tor}_n^R(M,N)$ to be the nth left derived functor of the functor $- \otimes_R N\colon {}_R\mathrm{Mod} \to \mathrm{Ab}$, applied to M:

$$\mathrm{Tor}_n^R(M,N) = [L_n(- \otimes_R N)]\,(M).$$

Let M and N be left modules. Define $\mathrm{Ext}_R^n(M,N)$ to be the nth right derived functor of the functor $\mathrm{Hom}_R(-,N)$, applied to M:

$$\mathrm{Ext}_R^n(M,N) = [R^n\,\mathrm{Hom}(-,N)]\,(M)$$

Proposition 4.9. *(symmetric of Tor) The functor Tor_n^R coincides with the nth left derived functor of the functor $M \otimes_R -\colon \mathrm{Mod}_R \to \mathrm{Ab}$, applied to N:*

$$\mathrm{Tor}_n^R(M,N) = [L_n(M \otimes_R -)]\,(N).$$

5. Homology of spaces

Definition. Denote by Top the category of topological spaces and continuous maps. We also write Top_* for the category of **pointed spaces**. Its objects are pairs (X, x_0) where X is a topological spaces and $x_0 \in X$. Morphisms from (X, x_0) to (Y, y_0) in Top_* are continuous maps $f\colon X \to Y$ such that $f(x_0) = y_0$.

Definition (recollection)**.** Two maps $f,\ g\colon X \to Y$ are called **homotopic** ($f \simeq g$) if there exists a **homotopy** between them, i.e. a map $H\colon X \times [0,1] \to Y$ with $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We call two spaces X and Y **homotopy equivalent** if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

5.1. **Cones, mapping cones, and suspensions.**

Definition. Let X be a space. Its **(unreduced) cone** is the space

$$CX = X \times [0,1]/\sim,$$

where $(x,1) \sim (x',1)$ for all $x,\ x' \in X$. If x_0 is a fixed base point of X, we also denote its **reduced cone** by $C^{\mathrm{red}}X$; it is defined by

$$CX = X \times [0,1]/sim,$$

where $(x,1) \sim (x',1)$ as before but also $(x_0,t) = (x_0,t')$ for all $t,\ t' \in [0,1]$.

Lemma 5.1. *A map $f\colon X \to Y$ is homotopic to a constant map ("null-homotopic") iff it extends to a map $\tilde{f}\colon CX \to Y$ from the unreduced cone on X to Y.*

A pointed map $f\colon (X,x_0) \to (Y,y_0)$ is homotopic to the constant map with value y_0 via a homotopy that does not move x_0 iff it extends to a map $\tilde{f}\colon C^{\mathrm{red}}X \to Y$ from the reduced cone on X to Y.

Definition. Given a map $f\colon A \to X$, define its **(unreduced) mapping cone** by

$$C_f = (A \times [0,1] \sqcup X)/\sim,$$

where $(a,1) \sim (a',1)$ for all $a,\ a' \in A$ and $(a,0) \sim f(a)$ for $a \in A$. Similarly, if f is a pointed map with $f(a_0) = x_0$, the **reduced mapping cone** C_f^{red} is obtained by adding

$$(a_0,t) \sim (a_0,t') \sim x_0$$

to the equivalence relation, for all $t,\ t' \in [0,1]$.

Lemma 5.2. *Let $f\colon A \to X$, $g\colon X \to Y$ be maps. Then g extends to $\tilde{g}\colon C_f \to Y$ iff the composite $g \circ f$ is homotopic to a constant map.*

If all maps are pointed then g extends to $\tilde{g}\colon C_f^{\mathrm{red}} \to Y$ iff the composite $g \circ f$ is homotopic to the constant map with value y_0 via a homotopy that does not move x_0.

Definition. The **unreduced suspension** SX of a space X is the unreduced mapping cone of the unique map $X \to *$; the **reduced suspension** ΣX of a pointed space X is the reduced mapping cone of the unique pointed map $X \to *$.

Remark 5.3. For "good" spaces X and base points $x_0 \in X$, the quotient maps $CX \to C^{\mathrm{red}}X$, $SX \to \Sigma X$, and, for based maps $A \to X$, $C_f \to C_f^{\mathrm{red}}$, are homotopy equivalences. "Good" here means "well-pointed", which is implied for instance if x_0 has a contractible neighborhood in X.

5.2. **The Eilenberg-Steenrod axioms.** Let R be a ring, A an R-module, and

$$H_n\colon \mathrm{Top} \to \mathrm{Mod}_R$$

be a sequence of functors. We write $\tilde{H}_n(X) = \ker\left(H_n(X) \to H_n(*)\right)$, where the map is induced by the unique map $X \to *$.

Then $(H_n)_{n\in\mathbf{Z}}$ is called a **homology theory with coefficients in** A if the following axioms hold:

- **homotopy:** if $f \simeq g$ then $H_n(f) = H_n(g)$ for all $n \in \mathbf{Z}$.
- **additivity:** if $X = \coprod_{i\in I} X_i$ then $\bigoplus_{i\in I} H_n(X_i) \cong H_n(X)$; the isomorphism is given by the canonical inclusions $X_i \hookrightarrow X$.
- **dimension:** $H_n(*) = \begin{cases} 0; & n \neq 0 \\ A; & n = 0. \end{cases}$ In particular, $H_n(X) \cong \tilde{H}_n(X)$ for $n \neq 0$.

exactness: Let $f\colon A \to X$ be a map and $g\colon X \to C_f$ be the standard inclusion. Then there is a natural long exact sequence

$$\cdots \to H_n(A) \xrightarrow{f_*} H_n(X) \xrightarrow{g_*} \tilde{H}_n(C_f) \to H_{n-1}(A) \to \cdots$$

Mayer-Vietoris: Let $X = U \cup V$, where U and V are open subsets of X, and $Z = U \cap V$. Then there is a long exact sequence

$$\cdots \to H_n(Z) \xrightarrow{i_* - j_*} H_n(U) \oplus H_n(V) \xrightarrow{p_* + q_*} H_n(X) \to H_{n-1}(Z) \to \cdots,$$

where the map $i\colon Z \hookrightarrow U$, $j\colon Z \hookrightarrow V$, $p\colon U \hookrightarrow X$, $q\colon V \hookrightarrow X$ are all the standard inclusions.

Theorem 5.4. *For every ring R and every R-module A, there exists (up to equivalence of functors) precisely one homology theory with coefficients in A.*

5.3. **Beginning calculations.** For simplicity, let $R = \mathbf{Z}$, $A = \mathbf{Z}$.

Lemma 5.5. *If X is discrete then* $H_n(X) \cong \begin{cases} 0; & n \neq 0 \\ \bigoplus_{x \in X} \mathbf{Z}; & n = 0. \end{cases}$

Lemma 5.6. *Denote by $\mathbf{S}^k$ the standard k-dimensional sphere. Then*

$$\tilde{H}_n(\mathbf{S}^k) \cong \begin{cases} 0; & n \neq k \\ \mathbf{Z}; & n = k. \end{cases}$$

Lemma 5.7. *For any pointed space X, $H_{n+1}(\Sigma X) \cong \tilde{H}_n(X)$.*

Lemma 5.8. *Let $\mathbf{D}^{n+1}$ be the $(n+1)$-dimensional disk, which has $\mathbf{S}^n$ as boundary. There is no continuous function $\mathbf{D}^{n+1} \to \mathbf{S}^n$ which is the identity, or even homotopic to the identity, on $\mathbf{S}^n$.*

Corollary 5.9 (Brouwer's fixed point theorem)**.** *Every continuous self-map of $\mathbf{D}^n$ has a fixed point.*

5.4. **Mapping degrees.** A map $f\colon \mathbf{S}^n \to \mathbf{S}^n$ gives a homomorphism of homology groups $H_n(\mathbf{S}^n) \cong \mathbf{Z}$, so it's multiplication by a number d, called the mapping degree of f, $\deg(f)$.

Lemma 5.10. *If $f\colon \mathbf{S}^n \to \mathbf{S}^n$ is homotopic to a constant map then $\deg(f) = 0$.* □

Lemma 5.11. $\deg(\mathrm{id}) = 1$

Lemma 5.12. $\deg(f \circ g) = \deg(f)\deg(g)$.

Lemma 5.13. *If $f \in O(n+1)$ then $\deg(f) = \det(f)$.*

Corollary 5.14. *The map $x \mapsto -x$ on $\mathbf{S}^n$ has degree $(-1)^{n+1}$. (This map is called the antipodal map.)*

Corollary 5.15. *If $f\colon \mathbf{S}^n \to \mathbf{S}^n$ has no fixed points then $\deg f = (-1)^{n+1}$.*

Theorem 5.16 (Hairy ball theorem)**.** *Let n be even and $f\colon \mathbf{S}^n \to \mathbf{R}^{n+1}$ be a continuous map such that $f(x) \perp x$ for all x. Then $f(x) = 0$ for some $x \in \mathbf{S}^n$.*