Matematiska Institutionen
KTH

Exam to the course Discrete Mathematics, SF2736, March 13, 2014, 08.00-13.00.

Observe:

1. Nothing else than pencils, rubber, rulers and papers may be used.
2. Bonus marks from the homeworks will be added to the sum of marks on part I. The maximum number of marks on part I is 15.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1. (3p) Find all solutions in the ring Z_{56} to the system of equations

$$\begin{cases} 4x + 7y = 5 \\ 3x + 2y = 8 \end{cases}$$

2. (3p) Solve, by using the technique with generating functions, the recursion

$$a_n = 3a_{n-1} + 2^n, \qquad n = 1, 2, \ldots, \qquad a_0 = 1.$$

3. (3p) Find the number of ways to distribute 18 identical marbles in the boxes no. 1, no. 2, ..., no. 6, such that the total number of marbles distributed in box no. 1 to no. 4 is twice as many as the number of marbles distributed in the boxes no. 5 and no. 6.
4. (3p) Find the number of cyclic subgroups to the group $(Z_2, +) \times (Z_3, +) \times (Z_4, +)$.
5. (3p) Show that a graph with $2n-3$ vertices cannot be connected if n of the vertices have valency 1 and the remaining vertices have either valency 2 or 3.

Part II

6. (3p) How many arrangements are there of $a, a, a, b, b, b, c, c, c, d, d, d$ without three consecutive letters the same.

7. Let $\mathcal{S}_5$ denote the group of all permutations of the elements in the set $\{1, 2, 3, 4, 5\}$ and let $\mathcal{M}$ denote the following subset of $\mathcal{S}_5$:

$$\mathcal{M} = \{(1\ \ 2\ \ 3), (2\ \ 4), (2\ \ 5), (3\ \ 5)\}.$$

   (a) (2p) Find the least, according to size, subgroup H of $\mathcal{S}_5$ that contains the set $\mathcal{M}$.

   (b) (1p) Is there a subset of $\mathcal{M}$ that gives the same result as in the problem above.

8. (5p) Find all positive integers n less than 1 000 such that $12^{41} \equiv 1(\text{mod } n)$ and $7^5 \equiv 1(\text{mod } n)$.

Part III

9. (a) (1p) Explain why the two parity-check matrices $\mathbf{H}$ and $\mathbf{H}'$ below define the same 1-error-correcting code:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{H}' = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

   (b) (2p) Assume that the 1-error-correcting code C of length $n = 2^k - 1$ is defined by its parity-check matrix $\mathbf{H}$ of size $k \times n$. Every permutation φ of the set of coordinate positions induces a permutation of the set of words of C, by

$$\varphi: \ \bar{c} = (c_1, c_2, \ldots, c_n) \quad \mapsto \quad \varphi(\bar{c}) = (c_{\varphi^{-1}(1)}, c_{\varphi^{-1}(2)}, \ldots, c_{\varphi^{-1}(n)}).$$

   Explain why if $\varphi(C) = C$ then there is a $k \times k$-matrix $\mathbf{A}$ such that column number i of $\mathbf{H}' = \mathbf{AH}$ is equal to column number $\varphi^{-1}(i)$ of $\mathbf{H}$. (No formal proofs are needed, good explanations are enough.)

   (c) (2p) Find the number of distinct linear 1-error-correcting codes C of length 15 and size $|C| = 2^{11}$.

10. (5p) Let $\chi(G)$ denote the chromatic number of a graph G and let $\bar{G}$ denote the complement of G, (so we assume that has no multiple edges or loops). Show that

$$\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n},$$

where n denotes the number of vertices of G.