Matematiska Institutionen
KTH

Exam to the course Discrete Mathematics, SF2736, January 17, 2014, 08.00-13.00.

Observe:

1. Nothing else than pencils, rubber, rulers and papers may be used.
2. Bonus marks from the homeworks will be added to the sum of marks on part I. The maximum number of marks on part I is 15.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1. (a) (1p) Find $\gcd(1111, 1234)$.
   (b) (2p) Find $739^{962} (\text{mod } 360)$.
2. (3p) Are there any graphs G with 231 vertices and 234 edges, and containing exactly two cycles. The graph is assumed to have no multiple edges or loops. (A loop is an edge ending in the same vertex.)
3. (3p) Find the number of colorings of a necklace with seventeen beans. The beans are either white or black. The necklace can be rotated and flipped.
4. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$
   (a) (1.5p) Find the number of injective maps from A to B such that
   $$|\{x \in A \mid f(x) \in \{1, 2, 3\}\}| = 2.$$
   (b) (1.5p) Find the number of surjective maps from B to A such that $f(1) \neq f(2)$.

   Note. The answer must, besides explanations, be given as an integer.
5. (a) (1p) Find a group G that has exactly three non-trivial and distinct subgroups H_1, H_2 and H_3 such that $H_1 \subseteq H_2 \subseteq H_3$.
   (b) (1p) Find a group G' that has exactly three non-trivial subgroups H'_1, H'_2 and H'_3 such that $H'_1 \cap H'_2 = H'_1 \cap H'_3 = H'_2 \cap H'_3$.
   (c) (1p) Find a group G'' with two non-trivial distinct subgroups H''_1 and H''_2 such that for any two elements a and b of G'' and for the cosets aH''_1 and bH''
   $$aH''_1 \cap bH''_2 \neq \emptyset \qquad \Longrightarrow \qquad aH''_1 \subseteq bH''_2.$$

Part II

6. (3p) There are 14 girls and 15 boys in a class. Three teams shall be selected. How many distinct combination of teams can be found if each team consists of exactly five children, of which at least one child is a girl.

7. (4p) The graph G is bipartite with two sets of vertices X and Y, (no edges between vertices of X and no edges between vertices of Y). The graph G has an Euler circuit and an Hamilonian cycle. All vertices of X have the same valency (degree) and all vertices of Y have the same valency (degree). Which are the possibilities for the 3-tuples $(|X|, |Y|, |E|)$?

8. (4p) Let $\mathcal{S}_{14}$ denote the set of permutations of the elements in the set $\{1, 2, \ldots, 14\}$. Find the number of elements φ in $\mathcal{S}_{14}$ such that

$$\varphi^{12} = (1\ \ 2)(\ 3\ \ 4)(5\ \ 6)(7\ \ 8).$$

How many of the solutions to the equation above are odd permutations?

Part III

9. We consider codes of length 8 over the alphabet Z_3, i.e., subsets C of Z_3^8, the direct product of eight identical copies of the ring Z_3.

   (a) (1p) Give an upper bound for the size of a 2-error-correcting code C of length 8 over the alphabet Z_3.

   (b) (1p) Generalize the concept linear binary error-correcting code to linear error-correcting codes over the alphabet Z_3.

   (c) (3p) Find a linear 2-error-correcting code C of length 8 over the alphabet Z_3. The more words in C the more marks.

10. (5p) There are k containers C_1, C_2, ..., C_k, and each container C_i contains k marbles in the color c_i. For every integer $k \geq 2$, the number of distinct samples of size k you can get using marbles from the containers is equal to $\binom{a}{b}$, for some positive integers $a = a(k)$ and $b = b(k)$. Find $a(k)$ and $b(k)$.