Matematiska Institutionen
KTH

Solutions of the exam to the course Discrete Mathematics, SF2736, January 17, 2014, 08.00-13.00.

Observe:

1. Nothing else than pencils, rubber, rulers and papers may be used.
2. Bonus marks from the homeworks will be added to the sum of marks on part I. The maximum number of marks on part I is 15.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1. (a) (1p) Find $\gcd(1111, 1234)$.

   Solution. The Euclidian algorithm gives

   $$\begin{array}{rcrcr} 1111 & = & 1234 & - & 123 \\ 1234 & = & 10 \cdot 123 & + & 4 \\ 123 & = & 31 \cdot 4 & - & 1 \end{array}$$

   and thus
   ANSWER: 1.

   (b) (2p) Find $739^{962} (\text{mod } 360)$.

   Solution. As $360 = 5 \cdot 2^3 \cdot 3^2$ we get that the value of the Eulerian φ-function in the point 360 is equal to

   $$\varphi(360) = 360(1 - \frac{1}{5})(1 - \frac{1}{2})(1 - \frac{1}{3}) = 12 \cdot 4 \cdot 2 = 96.$$

   Thus by the theorem of Euler

   $$739^{962} \equiv_{360} (739^{96})^{10} \cdot 739^2 \equiv_{360} 1^{10} \cdot 19^2 \equiv_{360} 1 \cdot 361 \equiv_{360} 1$$

   ANSWER: 1.

2. (3p) Are there any graphs G with 231 vertices and 234 edges, and containing exactly two cycles. The graph is assumed to have no multiple edges or loops. (A loop is an edge ending in the same vertex.)

   Solution. Deleting one edge in each cycle gives a graph that has no cycles. Such graphs are forests. A forest consists of one or more trees, and each tree has exactly one more vertex than the number of edges in the tree. A graph without cycles, with 231 vertices and 232 edges can thus not exist.

ANSWER: No.

3. (3p) Find the number of colorings of a necklace with seventeen beans. The beans are either white or black. The necklace can be rotated and flipped.

Solution. We use the lemma of Burnside. We enumerate the elements of the group G consisting of all rotations and flips of the necklace. We count the number of colorings of the necklace that are fixed by these operations:

$g \in G$	$\lvert\text{Fix}(g)\rvert$
Id.	2^{17}
$\varphi = (1\ 2\ 3\ \ldots\ 17)$	2^1
$\varphi^2 = (1\ 3\ 5\ \ldots\ 16)$	2^1
$\vdots$	$\vdots$
$\varphi^{16} = (1\ 17\ 16\ \ldots\ 2)$	2^1
$\psi_1 = (1)(2\ 17)(3\ 16)\cdots(9\ 10)$	2^9
$\vdots$	$\vdots$
$\psi_{17} = (17)(1\ 16)(2\ 15)\cdots(8\ 9)$	2^9

ANSWER:

$$\frac{1}{34}(2^{17} + 16 \cdot 2 + 17 \cdot 2^9).$$

4. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$

(a) (1.5p) Find the number of injective maps from A to B such that

$$|\{x \in A \mid f(x) \in \{1, 2, 3\}\}| = 2.$$

Solution. We first find two elements a, b of A that maps to elements in the set $\{1, 2, 3\}$. They can be chosen in $\binom{5}{2} = 10$ distinct ways. The number of injective maps from $\{a, b\}$ to $\{1, 2, 3\}$ is equal to $3 \cdot 2 = 6$. The number of injective maps from the set $A \setminus \{a, b\}$ to the set $\{4, 5, 6, 7\}$ is equal to $4 \cdot 3 \cdot 2 = 24$. We thus get by using the principle of multiplication the

ANSWER: $10 \cdot 6 \cdot 24 = 1440$.

(b) (1.5p) Find the number of surjective maps from B to A such that $f(1) \neq f(2)$.

Solution. The total number of surjective maps from B to A is $5!\mathrm{S}(7, 5)$, while those when $f(1) = f(2)$ is equal to $5! \cdot \mathrm{S}(6, 5)$. Hence the answer is given by

$$5!(\mathrm{S}(7, 5) - \mathrm{S}(6, 5)).$$

Any division of a set of size 6 into 5 non-empty subsets results in one set of size 2 and the remaining four subsets of size 1. Thus

$$\mathrm{S}(6,5) = \binom{6}{2} = 15.$$

Using the recursion formula $\mathrm{S}(n,k) = \mathrm{S}(n-1,k-1) + k \cdot \mathrm{S}(n-1,k)$ we get

$$\mathrm{S}(7,5) = \mathrm{S}(6,4) + 5 \cdot \mathrm{S}(6,5) = \left(\mathrm{S}(5,3) + 4\binom{5}{2}\right) + 5 \cdot 15 =$$

$$\left(\mathrm{S}(4,2) + 3\binom{4}{2}\right) + 115 = 7 + 18 + 115 = 140.$$

Thus
ANSWER: $120(140 - 15) = 15000$

5. (a) (1p) Find a group G that has exactly three non-trivial and distinct subgroups H_1, H_2 and H_3 such that $H_1 \subseteq H_2 \subseteq H_3$.

Solution. We consider the group $G = (Z_{16}, +)$. Let

$$H_3 = \langle 2 \rangle = \{0, 2, 4, \ldots, 14\}.$$

Any element a in the set $G \setminus H_1$ generates G, (as $\gcd(a, 16) = 1$). Hence all other non-trivial subgroups to G_1 must be subgroups to H_1. Similarly, every element b such that

$$b \in H_1 \setminus \{0, 4, 8, 12, \}$$

generates H_1. Similar arguments give the
ANSWER. The group $(Z_{16}, +)$ has the following three non-trivial subgroups

$$\{0, 8\} \subseteq \{0, 4, 8, 12\} \subseteq \{0, 2, 4, 6, 8, 10, 12, 14\}.$$

(b) (1p) Find a group G' that has exactly three non-trivial subgroups H_1', H_2' and H_3' such that $H_1' \cap H_2' = H_1' \cap H_3' = H_2' \cap H_3'$.

Solution. The group $(Z_2, +) \times (Z_2, +)$, has the following three non-trivial subgroups that satisfy the given condition

$$H_1' = \{(0,0), (0,1)\}, \quad H_2' = \{(0,0)(1,0)\}, \quad H_3' = \{(0,0)(1,1)\}$$

(c) (1p) Find a group G'' with two non-trivial distinct subgroups H_1'' and H_2'' such that for any two elements a and b of G'' and for the cosets aH_1'' and bH''

$$aH_1'' \cap bH_2'' \neq \emptyset \qquad \Longrightarrow \qquad aH_1'' \subseteq bH_2''.$$

Solution. The group $G'' = (Z_8, +)$ with

$$H_1'' = \{0, 4\}, \qquad H_2'' = \{0, 2, 4, 6\}$$

has as easily checked the desired property.

Part II

6. (3p) There are 14 girls and 15 boys in a class. Three teams shall be selected. How many distinct combination of teams can be found if each team consists of exactly five children, of which at least one child is a girl.

   Solution. Label the teams as the a-team, the b-team and the c-team. The total number of possible labeled teams, without the restrictions that each team must contain at least one girl, is equal to

   $$\binom{29}{5,5,5,14}$$

   Some combinations of teams are not allowed, those with no girl in team a, b and/or c. Denote by X the set of selections into teams such that the x-team has no girl. The number of selections into labeled teams with at least one girl in each team is then

   $$\binom{29}{5,5,5,14} - |A \cup B \cup C|.$$

   We get,

   $$|A| = |B| = |C| = \binom{15}{5}\binom{24}{5,5,14}$$

   as we first can choose the five boys to the x-team and then choose 5 and 5 children to the other two teams. Similarly

   $$|A \cap B| = |A \cap C| = |B \cap C| = \binom{15}{5}\binom{10}{5}\binom{19}{5}$$

   and

   $$|A \cap B \cap C| = \binom{15}{5,5,5}.$$

   From the principle of inclusion-exclusion we thus get, as we are searching the number of selections into unlabeled teams, the

   ANSWER:

   $$\frac{1}{3!}\left(\binom{29}{5,5,5,14} - 3\cdot\binom{15}{5}\binom{24}{5,5,14} + 3\binom{15}{5}\binom{10}{5}\binom{19}{5} - \binom{15}{5,5,5}\right)$$

7. (4p) The graph G is bipartite with two sets of vertices X and Y, (no edges between vertices of X and no edges between vertices of Y). The graph G has an Euler circuit and an Hamiltonian cycle. All vertices of X have the same valency (degree) and all vertices of Y have the same valency (degree). Which are the possibilities for the 3-tuples $(|X|, |Y|, |E|)$?

   Solution. Every second vertex in an Hamiltonian cycle belongs to X and every second vertex belongs to Y. Hence $|X| = |Y|$. Let k_x denote the valency of the vertices of X and k_y denote the valency of the vertices of Y. As the graph has an Euler circuit k_x and k_y must be even integers. The number $|E|$ of edges in G satisfies

   $$|X| \cdot k_x = |E| = |Y| \cdot k_y.$$

   Thus $k_x = k_y = 2k$ and $(|X|, |Y|, |E|) = (n, n, 2kn)$ for some integers $n \geq 2$ and $k = k_x/2$ such that $2 \leq 2k \leq n$.

   We now prove that for each such triple $(n, n, 2kn)$ there is a bipartite graph G that has both an Hamiltonian cycle and an Euler circuit, with $|X| = |Y| = n$ and such that every vertex has valency $2k$.

   We leet $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$ be the set of vertices. We let the set of edges be the set

   $$E = \{(x_i, y_{(i+j)(\bmod n)}) \mid i = 0, 1, \ldots, n-1, \quad j = 0, 1, \ldots, 2k-1\}.$$

   This graph has the Hamiltonian cycle

   $$y_0 x_0 y_1 x_2 \cdots y_{n-1} x_{n-1} y_0$$

   It follows that the graph is connected, and furthermore, as every vertex has even degree $2k$ we know that the graph has an Euler circuit.

   ANSWER: The triples in the set $\{(n, n, 2tn) \mid n \geq 2, 2 \leq 2t \leq n\}$

8. (4p) Let $\mathcal{S}_{14}$ denote the set of permutations of the elements in the set $\{1, 2, \ldots, 14\}$. Find the number of elements φ in $\mathcal{S}_{14}$ such that

   $$\varphi^{12} = (1\ \ 2)(\ 3\ \ 4)(5\ \ 6)(7\ \ 8).$$

   How many of the solutions to the equation above are odd permutations?

   Solution. We write φ as a product of disjoint cycles φ_i, $i = 1, 2, \ldots, k$, and observe that

   $$\varphi = \varphi_1 \varphi_2 \cdots \varphi_k \qquad \Longrightarrow \qquad \varphi^{12} = \varphi_1^{12} \varphi_2^{12} \cdots \varphi_k^{12}.$$

We also observe that if the cycle φ_i is either a 2-, 3-, 4-, 6- or an 12-cycle then $\varphi_i^{12} = \mathrm{Id}.$. If φ_i is an 8-cycle then we have the following

$$\varphi_i = (a_1\ \ a_2\ \ a_3\ \ \cdots\ \ a_8) \quad \Longrightarrow \quad \varphi_i^{12} = (a_1\ \ a_5)(a_2\ \ a_6)(a_3\ \ a_7)(a_4\ \ a_8).$$

If φ_i is an n-cycle where n is coprime to 12, then φ_i^{12} is an n-cycle. It remains to check 9- and 14-cycles. Such permutations raised to the power twelve will give a product of three 3-cycles or a product of two 7-cycles, respectively. So the only possibility is that one of the cycles, e.g. φ_1 is an 8-cycle on the elements in the set $\{1, 2, \ldots, 8\}$, and that the remaining cycles $\varphi_2, \ldots, \varphi_k$ are 2-, 3-, 4- and 6-cycles on the elements in the set $\{9, 10, ..., 14\}$.

Now we search for suitable combinations of these latter cycles. The following distinct cases can appear:

One 6-cycle: the number of 6-cycles on six elements is 5!, that is, 120.

One 4-cycle and one 2-cycle: the number of such combinations is

$$\binom{6}{2} \cdot 3 \cdot 2 = 90.$$

One 4-cycle: the number of 4-cycles equals the previous.

Two 3-cycles: the number of such combinations of permutations is

$$\frac{1}{2} \cdot \binom{6}{3} \cdot 2 \cdot 2 = 40.$$

One 3-cycle and one 2-cycle: the number of such combinations is

$$\binom{6}{3, 2, 1} \cdot 2 = 120.$$

One 3-cycle: the number of such combinations is

$$\binom{6}{3} \cdot 2 = 40.$$

Three 2-cycles: the number of such is

$$\binom{6}{2, 2, 2} = 90.$$

Two 2-cycles: the number of such is also 90

One 2-cycle, the number of such is $\binom{6}{2} = 15$

In total the number of suitable combinations of permutations of the elements in the set $\{9, 10, \ldots, 14\}$ is

$$120 + 90 + 90 + 40 + 120 + 40 + 90 + 90 + 15 = 695.$$

Now to the 8-cycle. We can let any of the elements 1, 2, ..., 8 be the element a_1, for example $a_1 = 1$. Then a_5 must be equal to 2. To a_2 we can choose any of the remaining six elements 3, 4, ..., 8. If e.g. $a_2 = 3$ then $a_6 = 4$. After the choice of a_2 it remains four elements to a_3 and then two elements to a_4. The number of possible 8-cycles is thus $6 \cdot 4 \cdot 2$, that is, 48.

The principle of multiplication thus gives that the total number of solutions to the given equation is

$$695 \cdot 48 = 33360.$$

In order to find the number of odd permutations we use the fact that a cycle of odd length is an even permutation and a cycle of even length is an odd permutation. As "odd times odd is even" and "odd times even is odd" we get when analyzing the possible combinations above that the number of odd permutations is equal to the sum of the following integers

$$0 + 90 + 0 + 40 + 0 + 40 + 0 + 90 + 0 = 260.$$

The number odd permutations φ that satisfies the equation is thus equal to

$$260 \cdot 48 = 12480$$

ANSWER: 33360 and 12480, respectively.

Part III

9. We consider codes of length 8 over the alphabet Z_3, i.e., subsets C of Z_3^8, the direct product of eight identical copies of the ring Z_3.

   (a) (1p) Give an upper bound for the size of a 2-error-correcting code C of length 8 over the alphabet Z_3.

   Solution. The number of words at distance at most 2 from a given code word $\bar{c}$ is equal to

   $$1 + 8 \cdot 2 + \binom{8}{2} 2^2 = 1 + 16 + 28 \cdot 4 = 129.$$

   As the total number of words with letters from Z_3 is 3^8 and spheres with centers at code words must be mutually disjoint we get that

   $$|C| \leq \frac{3^8}{129} = \frac{81 \cdot 81}{129} = \frac{6561}{129}.$$

   As $50 \cdot 129 = 6450 = 6561 - 111$ and so $50 < 6561/129 < 51$ we conclude that

   ANSWER: $|C| < 50$.

(b) (1p) Generalize the concept linear binary error-correcting code to linear error-correcting codes over the alphabet Z_3.

Solution. We may consider Z_3^8 as a vector space over Z_3. We define a linear code as a subspace of this vector space.

(c) (3p) Find a linear 2-error-correcting code C of length 8 over the alphabet Z_3. The more words in C the more marks.

Solution. We let C be the null space of a matrix $\mathbf{H}$ with digits in Z_3 and where we count as in the ring Z_3. The distance between any two words must be at least equal to five. Hence no linear combination of four or less columns in $\mathbf{H}$ may be equal to the zero column. The number of words of C will be a power of 3. The best we can do is then, by problem a), with a matrix $\mathbf{H}$ with 8 columns and 5 rows, as then the dimension of C will be 3, and thus the size of C equal to $3^3 = 27$. By some trial and error we find the following matrix satisfying these conditions

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 \end{bmatrix}$$

10. (5p) There are k containers C_1, C_2, ..., C_k, and each container C_i contains k marbles in the color c_i. For every integer $k \geq 2$, the number of distinct samples of size k you can get using marbles from the containers is equal to $\binom{a}{b}$, for some positive integers $a = a(k)$ and $b = b(k)$. Find $a(k)$ and $b(k)$.

Solution. The answer is given by the coefficient a_k of x^k in the polynomial

$$1 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots + x^{(k^2)} = (1 + x + x^2 + \cdots + x^k)^k$$

which is equal to the coefficient of x^k in the function

$$1 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots = (1 + x + x^2 + \cdots + x^k + x^{k+1} + \cdots)^k.$$

The above equality can be expressed as

$$1 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots = (1 - x)^{-k}.$$

We differentiate both sides k times and get the equality

$$a_kk! + a_{k+1}(k + 1)!x + \cdots = (-k)(-1)\cdots(-k - (k - 1))(-1)(1 - x)^{-2k}.$$

Substituting x by 0 we get

$$a_k = \frac{k(k + 1)\cdots(2k - 1)}{k!} = \binom{2k - 1}{k}$$

ANSWER: $a(k) = 2k - 1$ and $b(k) = k$ (or $b(k) = k - 1$).