Matematiska Institutionen
KTH

Solutions to homework number 2 to SF2736, fall 2013.

Please, deliver this homework at latest on Monday, November 25.

1. (0.1) Let $A = \{1, 2, \ldots, 7\}$ and let $\mathcal{R}$ be the following relation on A:

$$\mathcal{R} = \{(1, 2), (2, 3), (3, 5), (7, 6), (7, 7)\}.$$

Find the size $|\mathcal{R}'|$ of the smallest equivalence relation $\mathcal{R}'$ that contains $\mathcal{R}$.

Solution. Equivalence classes of an equivalence relation $\mathcal{R}''$ that contain $\mathcal{R}$ must be combinations of the equivalence classes

$$C_1 = \{1, 2, 3, 5\}, \qquad C_4 = \{4\}, \qquad C_7 = \{6, 7\}$$

The smallest one must have the classes described above. As $a\mathcal{R}'b$ if and only if a and b belongs to the same equivalence class we get that

$$|\mathcal{R}'| = 4^2 + 1^2 + 2^2 = 21.$$

2. (0.2p) Let $\mathcal{M}$ be a set of size 7. Find the number of equivalence relations $\mathcal{R}$ on $\mathcal{M}$ such that $|\mathcal{R}| = 29$. (Some elementary combinatorics is needed for the solution.)

Solution. Assume the number of elements in the equivalence classes are x_i, for $i = 1, 2, \ldots, t$, respectively. Then

$$\begin{cases} x_1 + x_2 + \cdots + x_t = 7 \\ x_1^2 + x_2^2 + \cdots + x_t^2 = 29 \end{cases}$$

Some trial and error gives that there is just one integer solution to the above system $x_1 = 5$ and $x_2 = 2$, or vice versa. Every partition of $\mathcal{M}$ defines an equivalence relation. Thus the number of equivalence relation satisfying the conditions above is

$$\binom{7}{2} = 21.$$

3. (0.2p) Are there any relations $\mathcal{R}$ of size 15 on the set $\{1, 2, \ldots, 6\}$ such that $\mathcal{R}$ is both transitive and symmetric. (A solution with a correct answer but with an incorrect proof will give zero points.)

   Solution. A relation on a set $\mathcal{M}$ which is both symmetric and transitive partitions the set $\mathcal{M}$ into classes

   $$C_a = \{x \in \mathcal{M} \mid x\mathcal{R}a\}.$$

   The proof that these classes are either pairwise disjoint or identical, is similar to the proof of the same fact for equivalence relations.

   If $a\mathcal{R}b$, then by symmetry and transitivity we obtain

   $$\begin{cases} a\mathcal{R}b \\ b\mathcal{R}a \end{cases} \quad \Longrightarrow \quad a\mathcal{R}a.$$

   Thus if $|C_a| > 1$ the class C_a contributes with $|C_a|^2$ elements to $\mathcal{R}$.

   Assume there are s classes containing just one element which is not related to itself, and assume the other classes contain x_i elements, $i = 1, 2, \ldots, t$, respectively. Then

   $$\begin{cases} x_1 + x_2 + \cdots + x_t + s = 6 \\ x_1^2 + x_2^2 + \cdots + x_t^2 = 15 \end{cases}$$

   Checking all cases we note that the above system has no feasible solutions.

4. (0.1p) Show that if A is a countable infinite set of the real numbers then its complement is infinite but not countable infinite.

   Solution. If $A = \{a_1, a_2, \ldots\}$ is a countable set and its complement B also were countable, i.e., $B = \{b_1, b_2, \ldots\}$ then its union

   $$A \cup B = \{a_1, b_1, a_2, b_2, \ldots\}$$

   is countable. But as R is not countable we then get a contradiction.

5. (0.1p) Is the family of equivalence relations on a countable infinite set always an uncountable infinite set?

Solution. The number of subsets containing a fixed element (for example containing the element 1 of the set of natural numbers) of an infinite countable set is a uncountable infinite set. The family of equivalence relations that give arise to two equivalence classes, of which one contains the fixed element, is thus an uncountable infinite set. This family of equivalence relations is a subset of the set of all equivalence relations.

6. (0.3p) Can you to any uncountable infinite set $\mathcal{M}$ find an uncountable infinite family $\mathcal{F}$ consisting of pairwise disjoint uncountable infinite subsets to $\mathcal{M}$?

Solution. We first solve the problem for $R \times R$, where R is the set of real numbers. Simply, let for every real number α

$$M_\alpha = \{(\alpha, \beta) \mid \beta \in R\},$$

which is an infinite uncountable set. Trivially

$$\alpha \neq \alpha' \qquad \Longrightarrow \qquad M_\alpha \cap M_{\alpha'} = \emptyset,$$

and

$$R \times R = \bigcup_{\alpha \in R} M_\alpha.$$

Now $R \times R$ has the same cardinality as R, which we get convinced of by the bijective map ψ

$$(a_{-k}a_{-k-1} \dots a_{-1}a_0 \,.\, a_1a_2a_3 \dots, b_{-n}b_{-n-1} \dots b_{-1}b_0 \,.\, b_1b_2b_3 \dots)$$

$$\longleftrightarrow$$

$$b_{-n}0b_{-n-1}0 \dots 0b_{-k+1}a_{-k}b_{-k}a_{-k-1}b_{-k-1} \dots a_{-1}b_{-1}a_0b_0 \,.\, a_1b_1a_2b_2a_3b_3 \dots$$

where we assumed $n \geq k$.

If the set $\mathcal{M}$ is uncountable infinite then there is an injective map φ from R to $\mathcal{M}$. Let $\mathcal{N} = \mathrm{Im}(\varphi)$, then φ is a bijective map from R to $\mathcal{N}$. Then, the following family is an infinite uncountable set of infinite uncountable sets, the union of which is $\mathcal{M}$:

$$((\mathcal{M} \setminus \mathcal{N}) \cup \varphi(\psi(M_0))) \cup (\bigcup_{\alpha \in R \setminus \{0\}} \varphi(\psi(M_\alpha))).$$