Matematiska Institutionen
KTH

**Solutions to homework number 3 to SF2736, fall 2013.**

Please, deliver this homework at latest on Monday, December 2.

The homework must be delivered individually, and, in general, just hand-written notes are accepted. You are free to discuss the problems below with your class mates, but you are not allowed to copy the solution of another student.

1. (0.2p) Let $\mathrm{H}(n,k)$ denote the number of ways to partition a set with $n$ elements into $k$ subsets of the same size. Derive a formula for $\mathrm{H}(n,k)$.

   **Solution.** We first note that $k$ must divide $n$, that is, $n = kd$ for some integer $d$, the size of the subsets. If we regard the subsets as labeled then the number of partitions is the multinomial coefficient

   $$\binom{n}{d, d, \ldots, d} = \frac{n!}{d!\, d! \cdots d!} = \frac{n!}{(d!)^k}$$

   There are $k!$ ways to label the subsets, thus

   **ANSWER:**

   $$\mathrm{H}(n,k) = \begin{cases} n!/(k!(d!)^k) & \text{if} \quad n = kd, \\ 0 & \text{else.} \end{cases}$$

2. (a) (0.1p) Find a formula for the number of words of length $n+k$ you can form by using $n$ a's and $k$ b's and such that no two a's are adjacent.

   **Solution.** First put the b:s in a row, then we place the a:s in the spaces between the b:s, or in front of, or after all b:s, and at most one a in each such spaces. We thus have to select $n$ out of $k+1$ such spaces between the b:s. Consequently

   **ANSWER:** Zero unless $n \leq k+1$ and in that case

   $$\binom{k+1}{n}$$

(b) (0.2p) Find the number of words of length 21 you can form by using eight a's, seven b's and six c's such that no two a's are adjacent.

**Solution.** We first form words by using the b:s and the c:s. Such words have length 13, choosing the position for the seven b:s in such words gives

$$\binom{13}{7}$$

such words. Then as in the previous problem we put the a:s in the spaces between the letters in such words. So

**ANSWER:**

$$\binom{13}{7}\binom{14}{8}$$

3. Find the number of equivalence relations $\sim$ on the set $\{1, 2, 3, \ldots, 7\}$ such that

(a) (0.1p) $1 \sim 2$ and $3 \sim 4$. **Remark.** The answers to the two problems above must, beside explanations, be given as an integer, that is, as an element in $Z$.

**Solution.** We can regard 1 and 2 as the same element, as well as 3 and 4. There is a one to one correspondence between the equivalence relations on the set $M = \{12, 34, 5, 6, 7\}$ and the number of ways to partition the set $M$ into subsets, subsets constituting the equivalence classes induced by the equivalence relation.

The number of ways to partition a set with $n$ elements into $k$ non-empty subsets the Stirling number denoted by $\mathrm{S}(n, k)$. Hence the answer is given by

$$\mathrm{S}(5,1) + \mathrm{S}(5,2) + \mathrm{S}(5,3) + \mathrm{S}(5,4) + \mathrm{S}(5,5).$$

Using the recursion $\mathrm{S}(n, k) = \mathrm{S}(n-1, k-1) + k\,\mathrm{S}(n-1, k)$, and the values

$$\mathrm{S}(4,2) = 7, \qquad \mathrm{S}(4,3) = \binom{4}{2} = 6$$

we get

$$\mathrm{S}(5,2) = 1 + 2 \cdot 7 = 15, \qquad \mathrm{S}(5,3) = 7 + 3 \cdot 6 = 25$$

and hence

**ANSWER:**

$$1 + 15 + 25 + \binom{5}{2} + 1 = 52.$$

(b) (0.2p) $1 \not\sim 2$, $1 \not\sim 3$ and $3 \not\sim 2$. **Remark.** The answers to the two problems above must, beside explanations, be given as an integer, that is, as an element in $Z$.

**Solution.** We use the principle of inclusion exclusion. Let $A$ denote the family of equivalence relations such that $1 \sim 2$, $B$ the set of equivalence relations such that $1 \sim 3$ and $C$ the family, or set, of equivalence relations such that $2 \sim 3$. The answer is then given by

$$\sum_{k=1}^{7} \mathrm{S}(7,k) - |A \cup B \cup C|.$$

As in the previous problem we get

$$|A| = |B| = |C| = \sum_{k=1}^{6} \mathrm{S}(6,k).$$

We get from the recursion for Stirling numbers

$$\mathrm{S}(6,2) = 1 + 2 \cdot 15 = 31, \qquad \mathrm{S}(6,3) = 15 + 3 \cdot 25 = 90,$$

$$\mathrm{S}(6,4) = 25 + 4 \cdot 10 = 75, \qquad \mathrm{S}(6,5) = \binom{6}{2} = 15,$$

Thus

$$|A| = \sum_{k=1}^{6} \mathrm{S}(6,k) = 1 + 31 + 90 + 75 + 15 + 1 = 213.$$

We also get that $A \cap B = A \cap C = B \cap C = A \cap B \cap C$, and

$$|A \cap B \cap C| = \sum_{k=1}^{5} \mathrm{S}(5,k) = 52$$

We also need $\mathrm{S}(7,k)$, for $k = 1, 2, \ldots, 6$. We get

$$\mathrm{S}(7,2) = 1 + 2 \cdot 31 = 63, \qquad \mathrm{S}(7,3) = 31 + 3 \cdot 90 = 301,$$

$$S(7,4) = 90 + 4 \cdot 75 = 390, \qquad S(7,5) = 75 + 5 \cdot 15 = 150.$$

Thus

**ANSWER:**

$$(1 + 63 + 301 + 390 + 150 + \binom{7}{2} + 1) - 3 \cdot 138 + 3 \cdot 52 - 52 = 372.$$

4. (0.2p) In the class we recently proved the following formula by deriving the number of positive integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 10$ in two distinct ways:

$$\binom{9}{3} = \binom{13}{3} - 4\binom{12}{2} + 6\binom{11}{1} - 4.$$

Generalize this formula to a "new" equality for binomial coefficients.

**Solution.** The left side of the equality above was obtained by first distributing a "one" to each $x_i$, $i = 1, 2, 3, 4$, and then use the formula for the number of ways to distribute $10 - 4$ identical objects into four distinct boxes. The right side was obtained using the principle of inclusion-exclusion, as is described below.

So we consider the number of positive integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = k. \tag{1}$$

where $k \geq n$.

We distribute a "one" to each $x_i$. The remaining $k - n$ identical "ones" are then distributed to the $y_i$:s. The number of non-negative solutions to

$$y_1 + y_2 + \cdots + y_n = k - n,$$

is then given by

$$\binom{k - n + n - 1}{n - 1}.$$

Now to the inclusion exclusion way to find the number of positive solutions of the given equation. The number of solutions to Equation (1) when $t$ of the indeterminates $x_i$ are equal to zero is equal to

$$\binom{k + n - t - 1}{n - t - 1}$$

There are

$$\binom{n}{t}$$

choices of $t$ indeterminates $x_i$. Inclusion exclusion then also gives that the number of positive integer solutions to Equation (1) is

$$\binom{k+n-1}{n-1} + \sum_{t=1}^{n} (-1)^t \binom{n}{t} \binom{k+n-t-1}{n-t-1}$$

Consequently

**ANSWER:**

$$\binom{k-1}{n-1} = \sum_{t=0}^{n} (-1)^t \binom{n}{t} \binom{k+n-t-1}{n-t-1}$$