Matematiska Institutionen
KTH

Solutions to homework number 4 to SF2736, fall 2013.

Please, deliver this homework at latest on Monday, December 9.

The homework must be delivered individually, and, in general, just hand written notes are accepted. You are free to discuss the problems below with your classmates, but you are not allowed to copy the solution of another student.

1. (0.1p) Find the order of the product $\varphi \circ \psi$ of the permutations $\varphi = (1\ \ 3\ \ 2\ \ 7)(4\ \ 5\ \ 6)$ and $\psi = (1\ \ 2\ \ 5)(3\ \ 6)$.

   Solution. We get that
   $$\varphi \circ \psi = (1\ \ 7)(2\ \ 6)(3\ \ 4\ \ 5)$$
   so the order is

   Answer: $\mathrm{lcm}(2, 2, 3) = 6$.

2. (0.1) Let φ denote the permutation
   $$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 6 & 3 & 7 & 4 & 1 \end{pmatrix}$$
   of the set of elements in the set $\{1, 2, \ldots, 7\}$. Show that there is no permutation ψ such that $\psi\varphi\psi = \varphi^2$.

   Solution. We get that
   $$\varphi = (1\ \ 2\ \ 5\ \ 7)(3\ \ 6\ \ 4) = (1\ \ 7)(1\ \ 5)(1\ \ 2)(3\ \ 4)(3\ \ 6)$$
   and thus an odd permutation. Then the left side of the given equality is an odd permutation, but the right side not.

3. (0.2p) Find the number of permutations of order 4 in the group $\mathcal{S}_8$. (Some combinatorics is needed.)

Solution. As the order of an permutation is the least common multiple of the lengths of the mutually disjoint cycles of the permutation, we get that a permutation of order four must be a combination of disjoint cycles of order 1, 2 and 4, with at least one cycle of length 4.

We must consider four distinct cases:

Case 1. *Two cycles of length 4.* Any four elements can be ordered into 3! distinct cycles. Counting the number of unlabeled partitions into two subsets of size four we thus get in this case

$$\frac{1}{2}\binom{8}{4} \cdot 3! \cdot 3!.$$

Case 2. *One 4-cycle and two 2-cycles.* Similarly we get

$$\frac{1}{2}\binom{8}{4,2,2} \cdot 3!.$$

Case 3. *One 4-cycle and one 2-cycle.* As above we get

$$\binom{8}{4}\binom{4}{2} \cdot 3!.$$

Case 4. *One 4-cycle.* In this case we get

$$\binom{8}{4} \cdot 3!.$$

4. (0.2p) Show that the group $G = (Z_{23} \setminus \{0\}, \cdot)$ is cyclic by finding a generator of G.

   The answer is given by the sum of the above integers, which is 5460.

   Solution. We use trial and error, starting with the candidate 2. We know that an element g is a generator of the multiplicative group iff the order of g equals $o(g) = 22$. Furthermore, we know that the order of any element divides 22. So checking 2 gives

$$2^2 = 4 \neq 1, \qquad 2^{11} = 1,$$

thus the order of 2 is 11. All elements in the group $\langle 2 \rangle$ generated by 2 has either order 2 or order 11. So one strategy would be to check these elements and choose an element not in this group:

$$\langle 2 \rangle = \{2, 4, 8, 16, 9, 18, 13, 3, 6, 12, 1\}.$$

Indeed,

$$5^2 = 2 \neq 1 \qquad 5^{11} = 5^{10} \cdot 5 = 2^5 \cdot 5 = 9 \cdot 5 = -1 \neq 1$$

Answer: The element 5 has order 22 and generates the multiplicative group of Z_{23}.

5. (0.2p) Let $\mathcal{S}_{10}$ denote the group of all permutations of the set of elements $\{1, 2, \ldots, 10\}$. Find an Abelian subgroup of size 24 to $\mathcal{S}_{10}$

   Solution. Let $\varphi = (1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$, and $\psi = (7\ \ 8\ \ 9\ \ 10)$. Then φ and ψ commute and hence the following set of size 24

$$G = \{\varphi^i \psi^j \mid i = 1, 2, 3, 4, 5, 6,\ j = 1, 2, 3, 4\},$$

   consitutes a group. Note for example that

$$\varphi^i \psi^j \varphi^s \psi^k = \varphi^{i+s (\bmod 6)} \psi^{j+k (\bmod 4)}$$

   so the set is closed under multiplication.

6. (0.2p) Find a non-Abelian group of size 48 such that the order of its elements are either 1, 2, 3 or 6.

   Solution. The group $\mathcal{S}_3 \times (Z_3, +) \times (Z_3, +) \times (Z_3, +)$, where $\mathcal{S}_3$ is the group of all permutations of the elements in the set $\{1, 2, 3\}$, has the desired property, which can be easily verified.