Matematiska Institutionen
KTH

Solution to problem 8b of the typical exam, fall 2013.

Let H and K be subgroups of the abelian group G. We prove that for any two elements a and b of G such that $(a+H)\cap(b+K)\neq\emptyset$ it is true that

$$|(a+H)\cap(b+K)| = |H\cap K|.$$

Then it follows that the size of the given intersection divides the size of G, (as $H\cap K$ is a subgroup of G.)

Assume $h_0, h\in H$ and $k_0, k\in K$ are such that

$$\begin{cases} a + h_0 = b + k_0 \\ a + h = b + k \end{cases}$$

Then

$$h_0 - h = k_0 - k = g \in H\cap K.$$

Consequently

$$h = h_0 - g \in h_0 + H\cap K, \qquad \text{and} \qquad k = k_0 - g \in k_0 + H\cap K.$$

On the other hand, if $g\in H\cap K$, and $a+h_0 = b+k_0$, then also

$$a + h_0 + g = b + k_0 + g$$

where $h_0+g\in H$ and $k_0+g\in K$. Thus we can conclude that

$$(a+H)\cap(b+K) = \{a+h_0+g \mid g\in H\cap K\}.$$