1. Separation av variabler:

$$dx \underset{y\neq\pm3}{=} \frac{6\,dy}{y^2-9} = \left(\frac{1}{y-3} - \frac{1}{y+3}\right)dy$$

$$\Rightarrow \; x + \gamma = \ln\left|\frac{y-3}{y+3}\right| \; ; \; \frac{y-3}{y+3} = \pm e^{x+\gamma} = \beta e^x, \; \beta \neq 0$$

$$(y-3) = \beta e^x (y+3) \; , \; y(1-\beta e^x) = 3(\beta e^x + 1)$$

$$\Rightarrow \qquad y(x) = 3\,\frac{1+\alpha e^x}{1-\alpha e^x}, \; \alpha \in \mathbb{R}$$

($y(x) \equiv 3$ också löser $6y' = y^2 - 9$)

eller $y(x) \equiv -3$

(som är den sökta speciella lösningen)

2. Allmänna lösningen till den homogena ekvationen

$$y'' + y' + y = 0 \; : \; \lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

$$y_H(x) = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

En partikulär lösning till den inhomogena ekvationen är $y_p(x) = \sin x - \cos x$ (via Ansatz $C\sin x + D\cos x$)

$\Rightarrow$ den allmänna lösningen till den inhomogena ekvationen är

$$y(x) = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + \sin x - \cos x$$

3. $\dot{\vec{x}} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{x} \quad : \quad \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-4)(\lambda+1)$

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \vec{0} \qquad \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \vec{x}(t) = A e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + B e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

4. $e^{-s} = \int_0^\infty e^{-st}\delta(t-1)\,dt \overset{!}{=} \int_0^\infty e^{-st}(y''-y)\,dt$

$$= s^2 Y(s) - s\cancel{y}(0) - y'(0) - Y(s)$$

$$\Downarrow$$

$$Y(s) = \frac{s+e^{-s}}{s^2-1} \qquad = (s^2-1)Y(s) - s$$

$$(y(0)=1,\ y'(0)=0)$$

$$\Rightarrow y(t) = \cosh t + \sinh(t-1)\,u(t-1)$$

$$\Big(y' = \sinh t + \cosh(t-1)\,u(t-1)$$

$$y'' = \cosh t + \sinh(t-1)\,u(t-1) + \cosh(t-1)\,\delta(t-1) \Big)$$

$$y''-y = \cosh(t-1)\,\delta(t-1) = \delta(t-1)\ ,\ y(0)=1\ ,\ y'(0)=0$$

5. $\int_{-\infty}^{+\infty} e^{-i\omega t}\underbrace{\delta(5t-3)}_{=\frac{1}{5}\delta(t-\frac{3}{5})}\,dt = \frac{1}{5}e^{-i\omega\frac{3}{5}}$

$$\int_{-\infty}^{+\infty} e^{-i\omega t}e^{-|t-2|}\,dt \underset{\tilde{t}=t-2}{=} e^{-2i\omega}\underbrace{\int_{-\infty}^{+\infty} e^{-i\omega\tilde{t}}e^{-|\tilde{t}|}\,d\tilde{t}}_{\overset{(\text{Beta, S. 318, F32})}{=} \frac{2}{\omega^2+1}}$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-i\omega t}x(t)\,dt = \frac{1}{5}e^{-i\frac{3}{5}\omega} + \frac{2e^{-2i\omega}}{\omega^2+1}$$

6. $0 \overset{!}{=} 2x^2 + 2y^2 - 3 \underset{(2x^2 - y \overset{!}{=} 0)}{=} 2y^2 + y - 3 = 2\left(y^2 + \frac{1}{2}y - \frac{3}{2}\right)$

$\Rightarrow y_\pm = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{3}{2}}$

Eftersom $y_- < 0$
(inga reella x som uppfyller $2x^2 = y_-$),
dom kritiska punkter är $(y_+ = 1)$: $\vec{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 1 \end{pmatrix}$

Jacobi-matrisen $J = \begin{pmatrix} 4x & 4y \\ 4x & -1 \end{pmatrix}$ är

$$J_\pm = \begin{pmatrix} \pm 2\sqrt{2} & 4 \\ \pm 2\sqrt{2} & -1 \end{pmatrix}$$ i $\vec{x}_1$ respektive $\vec{x}_2$

$J_\pm$'s
Egenvärden : $\begin{vmatrix} \pm 2\sqrt{2} - \lambda & 4 \\ \pm 2\sqrt{2} & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda(1 \mp 2\sqrt{2}) \mp 10\sqrt{2} \overset{!}{=} 0$

i $\vec{x}_1$: $\lambda_\pm = \left(\sqrt{2} - \frac{1}{2}\right) \pm \sqrt{\left(\sqrt{2} - \frac{1}{2}\right)^2 + 10\sqrt{2}}$

$\Rightarrow$ instabil (eftersom $\lambda_+ > 0$)

i $\vec{x}_2$: $\lambda_\pm = \underbrace{-\left(\sqrt{2} + \frac{1}{2}\right)}_{<0} \pm \underbrace{\sqrt{\left(\sqrt{2} + \frac{1}{2}\right)^2 - 10\sqrt{2}}}_{\text{imaginär}}$

$\Rightarrow$ stabil (eftersom den reella delen av λ_+ och λ_- är negativ)

7. $(\vec{y}')^{\cdot} = -\dot{\psi} \sin\psi \begin{pmatrix} c \\ s \end{pmatrix} + \dot{\chi} \cos\psi \begin{pmatrix} -s \\ c \end{pmatrix}$

$c := \cos\chi$

$(\dot{\vec{y}})' = \psi' \cos\psi \begin{pmatrix} -s \\ c \end{pmatrix} - \chi' \sin\psi \begin{pmatrix} c \\ s \end{pmatrix}$

$s := \sin\chi$

$\Rightarrow \quad \dot{\psi} = \chi', \quad \psi' = \dot{\chi} \quad$ ((11) OK!)

$(\Rightarrow \ddot{\psi} = (\chi')^{\cdot} = (\dot{\chi})' = \psi'', \ \ddot{\chi} = (\psi')^{\cdot} = (\dot{\psi})' = \chi'')$

$2\cos\psi\cos\chi = \cos(\psi - \chi) + \cos(\psi + \chi) = \cos 2g + \cos 2f$

$2\cos\psi \sin\chi = \sin(\psi + \chi) - \sin(\psi - \chi) = \sin 2f + \sin 2g$

$\Rightarrow \vec{y}' = \frac{1}{2}\begin{pmatrix} \cos 2f \\ \sin 2f \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos 2g \\ \sin 2g \end{pmatrix} \overset{!}{=} \frac{1}{2}\begin{pmatrix} -\sin\ldots \\ \cos\ldots \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\sin\ldots \\ \cos\ldots \end{pmatrix}$

$\Rightarrow 2f(x+t) = n(x+t) - \pi/2, \quad 2g(x-t) = n(x-t) - \pi/2$

mod 2π)

$f = g = h/2 \Rightarrow \dot{\vec{y}}(x,0) = \sin\left(\frac{h(x)}{2} - \frac{h(x)}{2}\right)\begin{pmatrix} \vdots \end{pmatrix} = \vec{0}$

$\vec{y}'(x,0) = \cos\left(\frac{h(x)}{2} - \frac{h(x)}{2}\right) \cdot \begin{pmatrix} \cos h(x) \\ \sin h(x) \end{pmatrix}$

b) $y_i = T_i(t) X_i(x) \rightarrow \frac{\ddot{T}_i}{T_i}(t) \overset{!}{=} \frac{X_i''}{X_i}(x) = -\alpha_i^2$

$X_i = A_i \cos\alpha_i x + B_i \sin\alpha_i x, \quad T_i = C_i \cos\alpha_i t + D_i \sin\alpha_i t$

$\dot{T}_i(0) = \dot{T}_i(2\pi) \underset{(a)}{\overset{!}{=}} 0 \Rightarrow D_i = 0, \quad \alpha_i = n_i \in \mathbb{N}$

$(\alpha^2 = 0 \Rightarrow T = \text{const.}, \ X = \text{const.})$

$\vec{y}(x,t) = \vec{y}_0 + \sum_{n=1}^{\infty} \cos(nt)(\vec{A}_n \cos nx + \vec{B}_n \sin nx)$

$\vec{y}'(x,0) = \sum_{1}^{\infty} \underbrace{\cos(0)}_{=1} (\vec{B}_n \cos nx - \vec{A}_n \sin nx) n \overset{!}{=} \begin{pmatrix} \cos h(x) \\ \sin h(x) \end{pmatrix}$