1) $y_H'' + y_H = 0$: $y_H(x) = A\cos x + B\sin x$

Ansatz $y(x) = C\sin 2x$ $\left(y' = 2C\cos 2x,\ y'' = -4C\sin 2x\right)$

$y'' + y = -3C\sin 2x \overset{!}{=} \sin 2x \Rightarrow C = -\frac{1}{3}$

$y(x) = A\sin x + B\cdot\cos x - \frac{1}{3}\sin 2x$

2) $\begin{vmatrix} 2-\lambda & a \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 - 2a \Rightarrow \lambda_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2 + 2a}$

$a = -\frac{1}{8} + \frac{\epsilon^2}{2} > -\frac{1}{8}$: $\lambda_\pm = \frac{3}{2} \pm \epsilon$

$\begin{pmatrix} +\frac{1}{2} \mp \epsilon & -\frac{1}{8} + \frac{\epsilon^2}{2} \\ 2 & -\frac{1}{2} \mp \epsilon \end{pmatrix} \vec{V}_\pm \overset{!}{=} \vec{0} \Rightarrow \vec{X} = \vec{V}_+ e^{\left(\frac{3}{2}+\epsilon\right)t} + \vec{V}_- e^{\left(\frac{3}{2}-\epsilon\right)t}$

$a = -\frac{1}{8} - \frac{\omega^2}{2} < -\frac{1}{8}$: $\lambda_\pm = \frac{3}{2} \pm i\omega$

$\begin{pmatrix} \frac{1}{2} \mp i\omega & -\frac{1}{8} - \frac{\omega^2}{2} \\ 2 & -\frac{1}{2} \mp i\omega \end{pmatrix} \vec{V}_\pm = 0$ complex! note: $\vec{V}_- \parallel \vec{V}_+^*$ (choose $= \vec{V}_+^*$)

$\vec{X} = e^{3/2\, t}\Big(\underbrace{\vec{V}_+ e^{i\omega t} + \vec{V}_+^* e^{-i\omega t}}_{\text{real!}}\Big)$

3) $\int_{-\infty}^{+\infty} e^{-i\omega t}\,\delta(7t-2)\,dt = \frac{1}{7} e^{-\frac{2i\omega}{7}}$

$\delta(7t-2) = \frac{1}{7}\delta\left(t - \frac{2}{7}\right)$

Beta 317

$\int_{-\infty}^{+\infty} e^{-i\omega t}\, 5\sin(t+1)\,dt = 5e^{i\omega}\,\mathcal{FT}(\sin(\tilde{t})) = 5e^{i\omega}\frac{\pi}{i}\left(\delta(\omega-1) - \delta(\omega+1)\right)$

Beta 318

$\int_{-\infty}^{+\infty} e^{-i\omega t} e^{-|t+3|}\,dt = e^{3i\omega}\,\mathcal{FT}\left(e^{-|\tilde{t}|}\right) = e^{3i\omega}\frac{2}{1+\omega^2}$

$\Rightarrow \int_{-\infty}^{+\infty} x(t)\, e^{-i\omega t}\,dt = \frac{1}{7}e^{-\frac{2i\omega}{7}} + \frac{5\pi}{i} e^{i\omega}\left(\delta(\omega-1) - \delta(\omega+1)\right) + \frac{2e^{3i\omega}}{\omega^2+1}$

4) $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(b_n \sin(nx) + a_n \cos(nx)\right)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx$

5) $u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a} y\right) \cos\left(\frac{n\pi}{a} x\right)$

$A_0 = \frac{1}{ab} \int_0^a f(x)\,dx$, $A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \cos\left(\frac{n\pi}{a} x\right) dx$

(cp. page 474/475 of Zill/Wright)

6) $v' = \frac{dv}{dz}$, $\int -\frac{dz}{z} = \int \frac{v\,dv}{v^2 - 2v + 3/4} = \frac{3}{2} \int \frac{dv}{v - 3/2} - \frac{1}{2} \int \frac{dv}{v - 1/2}$

$-\ln|z| = \frac{3}{2} \ln|v - 3/2| - \frac{1}{2} \ln|v - 1/2| + \text{const.}$

$\frac{E}{z^2} = \frac{|v - 3/2|^3}{|v - 1/2|}$

7) $z(x,y) = u(f_1(x) + f_2(y)) \Rightarrow z_x = f_1' u'$, $z_y = f_2' u'$, $z_{xy} = f_1' f_2' u''$

$z_{xx} = f_1'' u' + f_1'^2 u''$, $z_{yy} = f_2'' u' + f_2'^2 u''$

$u(-f_3(z)) = z$ $\qquad u' = -\frac{1}{f_3'}$, $u'' = -f_3''/f_3'^3$

$\Rightarrow f_1''(f_2'^2 + f_3'^2) + f_2''(f_3'^2 + f_1'^2) + f_3''(f_1'^2 + f_2'^2) = 0 \quad (\sim)$

(on $f_1(x) + f_2(y) + f_3(z) = 0$)

In all 4 cases (a–d) one can straightforwardly show that $(\sim)$ holds, with the 3 functions $f_i(x_i)$ as follows

a) $\ln x - \ln y + \ln \tan z = 0$

b) $x^2 + y^2 - (\cosh z)^2 = 0$ $\quad (f_1' = 2x, f_1'' = 2, f_2' = 2y, f_2'' = 2, \ldots)$

c) $\ln \sinh x + \ln \sinh y - \ln \sin z = 0$

d) $\ln \cos x - \ln \cos y + z = 0$

(e.g. $f_1' = -\tan x$, $f_1'' = \frac{-1}{(\cos x)^2}$, $f_2' = +\tan y$, $f_2'' = \frac{+1}{(\cos y)^2}$, $f_3' = 1$)

a, c, d satisfy $f_i'^2(x_i) = a_i + b_i e^{\gamma f_i} + c_i e^{-\gamma f_i}$ $(i = 1,2,3)$ for certain constants a_i, b_i, c_i, γ (which make $(\sim)$ satisfied)