Self-Similarity in Singularity-Formation

The (rather complicated) non-linear second-order ODE ($\alpha > 1$)

$$h''\left(2\alpha h - \frac{(\alpha+1)^2}{4}z^2\right) = (\alpha - 1)\left[h'^2 + \alpha h - \frac{3}{4}(\alpha + 1)zh'\right] \tag{1}$$

arises when studying singularity formation for closed curves moving in the plane in such a way that (locally) their height $y(t,x)$, as a function of x, and time t, satisfies

$$\ddot{y}\left(1 + y'^2\right) - y''(1 - \dot{y}^2) = 2\dot{y}y'\dot{y}' \tag{2}$$

where · denotes $\frac{\partial}{\partial t}$ and ' denotes $\frac{\partial}{\partial x}$; this can be verified by inserting the Ansats

$$y(t,x) = y_0 - \hat{t} + \hat{t}^\alpha h\left(\frac{x}{\hat{t}^{(\alpha+1)/2}}\right) + \cdots \tag{3}$$

into (2) and convincing oneself that the terms remaining after using (1) are of higher order in

$\hat{t} \coloneqq t_0 - t$ (going to zero from above; i.e. being > 0 and $\to 0$)

compared to those leading to (1).

- *Show* that if $h(z)$ satisfies the (much simpler, first order) equation

$$h'^2 + 2\alpha h - (\alpha+1)zh' = 0 \tag{4}$$

,it also solves (1).

- *Solve* (4) via

$$h(z) = z^2\left(\frac{(1+\alpha)^2}{8\alpha} - \frac{v^2}{2\alpha}\right) \tag{5}$$

in terms of a (to be determined) function $v(z)$.

-*Derive* that

$$z^2 \cdot \frac{\left|v - \frac{(\alpha+1)}{2}\right|^{(\alpha+1)}}{\left|v - \frac{(\alpha-1)}{2}\right|^{(\alpha-1)}} = constant\ (=: E) \tag{6}$$

and deduce the behavior of v for $z \to 0$ and $z \to \pm\infty$.

What does that imply for the *asymptotic behaviour* of $h(z)$?

-*Show* that for $\alpha = 2$ (4) is solved by

$$h(z(\zeta)) = \frac{\zeta^2}{2} + c\frac{\zeta^4}{4} \tag{7.1}$$

$$z(\zeta) = \zeta + c\frac{\zeta^3}{3} \tag{7.2}$$

-*Verify* that if h satisfies (1), so does

$\tilde{h}(z) := c^2 h(z/c)$

and that

$\frac{h'}{z} - \frac{2h}{z^2} = \frac{1}{\alpha}f(v)$

reduces (1) to a first-order ODE for f.