

### Homework assignment 3

*This exercise set is due November 18, 2008*

*The Adding Machine* (Misiurewicz) The purpose of this set of exercises is to construct and analyze a continuous map of  $I$  which has exactly one periodic point of period  $2^j$  for each  $j$  and no other periodic points. The construction of the map relies on the notion of the double of a map, a topic discussed in class and in Robinson, Example 10.1.17, p. 375-376. Start with  $f_0 = 1/3$ . Let  $f_1(x)$  denote the double of  $f_0(x)$ , i.e.  $f_1(x)$  is obtained from  $f_0(x)$  by the following procedure:

1.  $f_1(x) = \frac{1}{3}f_0(3x) + \frac{2}{3}$  if  $0 \leq x \leq 1/3$ .
2.  $f_1(2/3) = 0$ ;  $f_1(1) = 1/3$ .
3.  $f_1$  is continuous and linear on the intervals  $1/3 \leq x \leq 2/3$  and  $2/3 \leq x \leq 1$ .

That is, the graph of  $f_1$  is obtained from  $f_0$  as shown in Figure 10.1.9, Robinson p. 376. Inductively, we define  $f_{n+1}$  to be the double of  $f_n$ . Finally let  $F(x) = \lim_{n \rightarrow \infty} f_n(x)$ .

1. Prove that  $f_{n+j}(x) = f_n(x)$  for all  $j \geq 1$  and  $x \geq 1/3^n$ . Conclude that if we define  $F(0) = 1$ , then  $F(x)$  is a well-defined continuous map of  $I$ .
2. Prove that  $f_n(x)$  has a unique periodic orbit  $2^j$  for each  $j \leq n$ . Prove that each of these periodic orbits is repelling, if  $j < n$ .
3. Prove that  $f_n(x)$  has no other periodic orbits.
4. Prove that  $F(x)$  has a unique periodic orbit of period  $2^j$  for each  $j$  and no other periodic orbits. Show that this periodic orbit is repelling.

Recall the construction of the Cantor Middle-Third set. Let  $A_0 = (\frac{1}{3}, \frac{2}{3})$  be the middle third of the unit interval  $I$ . Let  $I_0 = I \setminus A_0$ . Let  $A_1 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$  be the middle third of the two intervals in  $I_0$ . Let  $I_1 = I_0 \setminus A_1$ . Inductively, let  $A_n$  denote the middle third of the intervals in  $I_{n-1}$  and let  $I_n = I_{n-1} \setminus A_n$ . Finally, let

$$I_\infty = \bigcap_{n \geq 0} I_n.$$

$I_\infty$  is the classical middle-thirds Cantor set.

5. Show that the periodic points of period  $2^j$  for  $F$  lie in the union of intervals which comprise  $A_j$ .
6. Prove that  $F(I_n) = I_n$ .
7. Prove that if  $x \in A_n$  and  $x$  is not periodic, then there exists  $k > 0$  such that  $F^k(x) \in I_n$ .
8. Prove that  $I_\infty$  is invariant under  $F$ .
9. Prove that, if  $x \notin I_\infty$  and  $x$  is not periodic, then the orbit of  $x$  tends to  $I_\infty$  or eventually lies in  $I_\infty$ .

Thus all of the non-periodic points for  $F$  are attracted to the set  $I_\infty$ . Thus to understand the dynamics of  $F$ , it suffices to understand the dynamics of  $F$  on  $I_\infty$ . For each point  $p \in I_\infty$ , we attach an infinite sequence of 0's and 1's,  $S(p) = (s_0 s_1 s_2 \dots)$ , according to the rule:  $s_0 = 1$  if  $p$  belongs to the left component of  $I_0$ ;  $s_0 = 0$  if  $p$  belongs to the right component. Note that this is slightly different from our coding for the quadratic map! Now  $p$  belongs to some component of  $I_{n-1}$ , and  $I_n$  is obtained by removing the middle third of this interval. Therefore we may set  $s_n = 1$  if  $p$  belongs to the left hand interval in  $I_n$  and  $s_n = 0$  otherwise.

Let  $\Sigma_2$  be the set of all sequences of 0's and 1's. Define the adding machine  $A : \Sigma_2 \rightarrow \Sigma_2$  by  $A(s_0 s_1 s_2 \dots) = (s_0 s_1 s_2 \dots) + (100 \dots) \bmod 2$ , i.e.  $A$  is obtained by adding 1 mod 2 to  $s_0$  and carrying the result. For example,  $A(110\overline{110} \dots) = (001\ 110\ \overline{110} \dots)$  and  $A(11\overline{1} \dots) = (00\overline{0} \dots)$ .

10. Let  $d$  be the usual distance of  $\Sigma_2$ . Prove that  $S : I_\infty \mapsto \Sigma_2$  is a topological conjugacy between  $F$  on  $I_\infty$  and  $A$  on  $\Sigma_2$ .
11. Prove that  $A$  has no periodic points.
12. Prove that every orbit of  $A$  is dense in  $\Sigma_2$ .

Since  $\Sigma_2$  has no proper closed invariant subsets under  $A$ ,  $\Sigma_2$  is an example of a *minimal set*.