Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2008
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Homework assignment 4

This exercise set is due December 2, 2008

1. Consider the diffeomorphism Q_λ of the plane given by

$$x_1 = e^x - \lambda$$
$$y_1 = -\frac{\lambda}{2} \arctan y$$

where λ is a parameter.

a. Find all fixed points and periodic points of period 2 for Q_λ.

b. Classify each of these periodic points as sinks, sources, or saddles.

c. If the point is a saddle, identify and sketch the stable and unstable manifolds.

2. Let

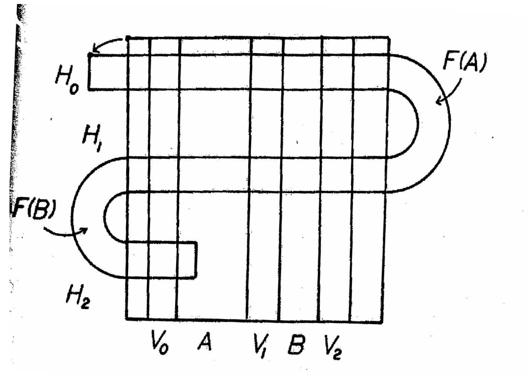
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Construct a Markov partition for the corresponding map L_A of the torus.

3. Consider the map F on D defined geometrically as in the picture. Assume that F linearly contracts vertical lenghts and linearly expands horizontal lengths in S exactly as in the case of Smale's horseshoe. Let

$$\Lambda = \{p \in D | F^n(p) \in S \text{ for all } n \in \mathbb{Z}\}.$$

Show that F on Λ is topologically conjugate to a two-sided subshift of finite type generated by a 3×3 matrix A. Identify A. Discuss the dynamics of F off Λ



4. *Linear automorphims of the sphere.* Let S^2 denote the two-dimensional sphere in $\mathbb{R}^3$, i.e.

$$S^2 = \{x \in \mathbb{R}^3 | \, |x| = 1\}.$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and define the map

$$F(x) = F_A(x) = \frac{Ax}{|Ax|}.$$

F_A os called a linear automorphism of S^2.

a. Prove that F maps $\mathbb{R}^3 - \{0\}$ onto S^2.

b. Prove that the restriction of F to S^2 is a diffeomorphims of the sphere.

c. Let $e_1 = (1, 0, 0)$ $e_2 = (0, 2, 0)$ $e_3 = (0, 0, 3)$. Prove that the $\pm e_j$ are the fixed points of F.

d. Compute the Jacobi matrices $DF(\pm e_j)$. Prove that $DF(\pm e_j)$ has an eigenvalue equal to 0 with corresponding eigenvector e_j.

e. Prove that each of the other vectors e_i, $i \neq j$, are also eigenvectors for $DF(\pm e_j)$. Evaluate the corresponding eigenvalues.

f. Conclude that $\pm e_1$ is a source, $\pm e_2$ is a saddle, and $\pm e_3$ is a sink.

g. Define $\phi : S^2 \to \mathbb{R}$ by $\phi(x) = |A^{-1}x|^2$. Prove that $\phi(F(x)) = \phi(x)$ if and only if $x = \pm e_j$ for some j. The function ϕ is called a gradient function since it decreases along the orbits of F except the fixed points. F itself is called *gradient like.*

h. Use the information above (including the gradient function) to sketch the phase portrait of F