Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2008
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Homework assignment 5

This exercise set is due Dec 16, 2008

1. (Robinsson, p. 473, 11.3.1) The map $g_a(x) = ax(1-x)$ for $a = \frac{19}{6}$ has a periodic 2 orbit at $q_1 = \frac{10}{19}$ and $q_2 = \frac{15}{19}$ and has fixed points at 0 and $p_a = \frac{13}{19}$.

   a) Find the Lyapunov exponents at the two fixed points 0 and p_a. Are the fixed points attracting or repelling?

   b) Find the Lyapunov exponent for the point q_1 for $g_{19/6}$. Is the orbit attracting or repelling?

   c) Find the Lyapunov exponent for the point $x_0 = g_{19/6}(0.5)$. Why is this the correct answer?. Hint: The map has a negative Schwarzian derivative. Also, we do not start at the critical point but at the image of the critical point. Compare and read Section 9.4 in Robinsson's book which we have not covered in class.

   d) What values are there for Lyapunov exponents for x_0 in $[0, 1]$.

2. a) Explain why the rotation $R_\alpha(x) = x + \alpha \,(\text{mod} 1)$ preserves Lebesgue measure on $[0, 1]$.

   Assume in the following parts b) and c) that α is irrational

   b) Prove that for each trigonometrical polynomial

$$P_N(t) = \sum_{\nu=-N}^{N} a_\nu e^{2\pi\nu i t}$$

   we have that

$$\frac{1}{n}\sum_{j=0}^{n-1} P_N(R_\alpha^j(x)) = \frac{1}{n}\sum_{j=0}^{n-1} P_N(x + j\alpha) \to \int_0^1 P_N(t)\, dt = a_0.$$

   as $n \to \infty$. Hint: It is enough to prove this for the monomials $e^{2\pi\nu i t}$. Explain why!

c) Use the Weierstrass approximation theorem to conclude that for continuous functions f and all x

$$\frac{1}{n}\sum_{j=0}^{n-1} f(R_\alpha^j(x)) \to \int_0^1 f(t)\,dt.$$

3. Consider the map

$$f(x) = \begin{cases} 4x & \text{for } \ 0 \le x \le \frac{1}{4} \\ -\frac{3}{2}x + \frac{11}{8} & \text{for } \ \frac{1}{4} < x \le \frac{3}{4} \\ 2x - \frac{5}{4} & \text{for } \ \frac{3}{4} < x \le 1. \end{cases}$$

a) Draw the graph of f. Also, explain why f is an expanding map that has a Markov partition.

b) Give the transition matrix $\mathbf{M}$ on masses on the subintervals, and find the invariant masses $\mathbf{m}^*$.

c) Give the densities ρ_i that correspond to the invariant masses $\mathbf{m}$. Sketch the graph of the densisty function ρ^* that takes on the values ρ_i^*.

d) What is the set of all possible periods for f?