Lecture 1 PDE

1) Admin:

1) Lectures here every ~~day~~ Tuesday at 1 pm (Maybe one possible exception)

2) Course litterature

i) Lecture notes on the homepage.

ii) Possible books { Evans; Gilbarg Trudinger

iii) Have a good analysis book

3) 4 parts

Linear { 1) Basic theory ⟹ Assessed homework; 2) Variable coefficients ⟹ Assessed homework

Non-linear { 3) Generalizations to functional analysis ⟹ DAH; 4) Generalizations to viscosity solutions ⟹ DAH

⟹

Oral exam

Teaching philosophy

I ~~#~~ Ground our theory in analysis } Undergrad analysis

Our focus $\Longrightarrow$ $\Downarrow$

II Basic ~~Naive~~ approach to get some results } Undergrad

Our focus $\Longrightarrow$ $\Downarrow$ Analyze, generalize

III More sophisticated approach } Masters

$\Longrightarrow$ $\Downarrow$ Analyze generalize

IV Abstract approach } PhD

Normally one does I, II, III, IV in 4 different courses. Every course starts with ~~the new~~ new definitions and a new level of abstractness.

I want us to motivate the development of the theory. I ~~#~~ want us to have the feeling that we could create the theory if we had enough time. That it is natural.

Good things: Focus on research, understanding and the development of ideas

Bad things: Some of our discussions will be informal. We will however always prove our results. But you will need some mathematical maturity to distinguish the informal mumbo jumbo from the proofs.

What is a Partial Differential Equation (PDE)

An equation is an expression: ~~...~~

Expression in $\{u\}$ = Expression in known terms (*)

and a solution to the equation (*)

is an u s.t. (*) is true.

A PDE is an equation involving partial derivatives

Example 1: $|\nabla u(x)|^2 = f(x)$ (1) [Eikonal Eq.]

Given $f(x)$ find $u(x)$ s.t. (1) holds.

Example 2: Given $f(x)$ find $u(x)$ s.t.

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} = f(x)$$ [Laplace Eq.]

Example 3. Given an f find u s.t.

[Monge Ampere eq.]

$$\det\left[D^2 u(x)\right] = f(x)$$

$$D^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & & & \frac{\partial^2 u}{\partial x_n^2} \end{bmatrix}$$

We will focus on the Laplace equation for the first part of the course.

Main problem (Dirichlet): Given a domain (open connected set) D and a function $f(x) \in C(D)$ defined on D and a function $g(x) \in C(\partial D)$ find a function $u(x)$ that are two times continuously differentiable s.t.

$$\Delta u(x) = f(x) \quad \text{in } D \quad \left[\text{For all points } x \in D\right]$$

$$u(x) = g(x) \quad \text{on } \partial D \quad \left[\text{For all points } x \in \partial D\right]$$

Questions:

1) Does the Dirichlet problem have a solution

Answer: No!

Need some refinement in the statement

2) Is the solution unique

Answer: No!

Need some refinement.

3) Does the solution have any good properties?

Answer: Yes!

We will not try to motivate why the Dirichlet problem is important (Why is finding the solutions to a sixth degree polynomial important?) But we will accept that it is one of the fundamental equations of mathematics, physics, and engineering.

How do we start to approach a new problem?

We have all these objects flying around D [a domain] f & g [functions] and we want to find a u s.t.

$$\Delta u(x) = \overset{=0}{\cancel{f(x)}} \quad \text{for all } x \in \overset{\mathbb{R}^n}{\cancel{D}} \quad \left[\begin{array}{l}\text{infinitely} \\ \text{many eq.}\end{array}\right]$$

$$\cancel{u(x) = g(x) \quad x \in \partial D \quad \{ \text{— '' —} \}}$$

How do we do, where do we start?

First step: Find something simpler!

Let us try to solve

$$\Delta u(x) = 0 \qquad \text{in } \mathbb{R}^n = D \qquad (1)$$

Still very difficult. We don't know how to solve PDE yet? What do we know?

The closest thing we know how to do is ODE, So let us try to find solutions to

$$\Delta u(x) = 0 \quad \text{in } \mathbb{R}^n \qquad (1)$$

that only depend on one variable.

We use what we know to build new knowledge. Lets assume that u only depend on the radial variable ~~$u(|x|)$~~ $u(x) = h(r)$ where $r = |x|$.

Then we may calculate

$$\frac{\partial}{\partial x_i} = \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r} + \text{angular derivatives} = \frac{x_i}{r}\frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial x_i^2} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{x_i}{r^2}\frac{\partial r}{\partial x_i}\frac{\partial}{\partial r} + \frac{x_i}{r}\frac{\partial}{\partial x_i}\frac{\partial}{\partial r}$$

$$= \frac{1}{r}\frac{\partial}{\partial r} - \frac{x_i^2}{r^3}\frac{\partial}{\partial r} + \frac{x_i^2}{r^2}\frac{\partial^2}{\partial r^2} \quad \text{+ radial}$$

So

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} = \frac{n}{r}\frac{\partial}{\partial r} - \frac{\sum x_i^2}{r^3}\frac{\partial}{\partial r} + \frac{\cancel{\sum x_i^2}}{\cancel{r^2}}\frac{\partial^2}{\partial r^2} \quad \text{+ radial}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \quad \text{+ radial}$$

So

$$\Delta u = 0 \quad \Rightarrow \quad h''(r) + \frac{n-1}{r}h'(r) = 0$$

$$\Rightarrow r^{-(n-1)}\frac{\partial}{\partial r}\left(r^{n-1}h'(r)\right) = 0 \quad \Rightarrow \quad r^{n-1}h'(r) = a$$

$$\Rightarrow h(r) = \frac{a}{r^{n-2}} + b \quad n \geq 3$$

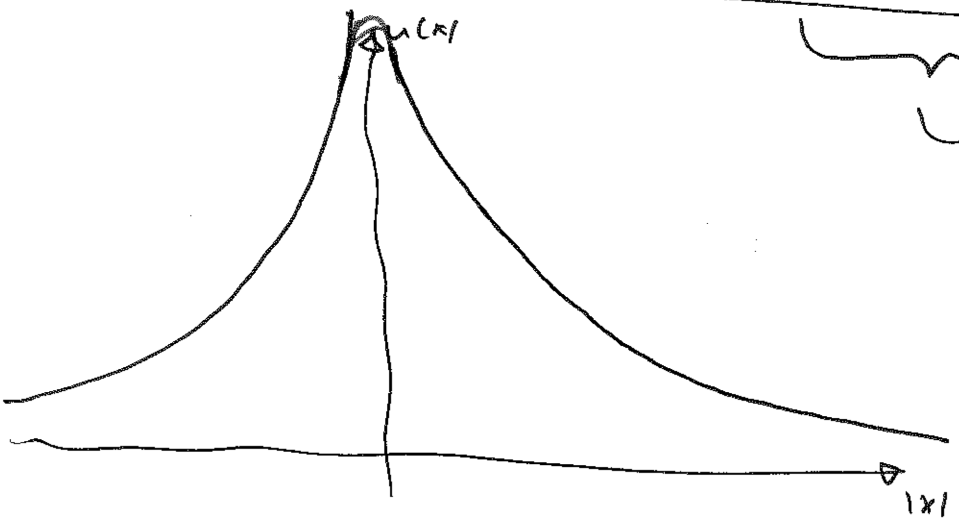
$h(r) = a\log r + b$

Lemma: Let

$$u(x) = \begin{cases} \dfrac{a}{|x|^{n-2}} + b & n \geq 3 \\ a \ln(|x|) + b & n = 2 \end{cases}$$

Then $\Delta u = 0$ in $\mathbb{R}^n \setminus \{0\}$

we have a singularity at $|x| = 0$.



Can we analyze the singularity? Let us define in $\mathbb{R}^3$

$$u_\delta(x) = \begin{cases} -\dfrac{3}{8\pi\delta} + \dfrac{1}{8\pi\delta^3}|x|^2 & |x| \leq \delta \\ -\dfrac{1}{4\pi}\dfrac{1}{|x|} & |x| > \delta \end{cases}$$

Cuts out the singularity

Then

$$\Delta u_\delta = \left.\begin{cases} +\dfrac{3}{4\pi\delta^3} & |x| < \delta \\ ? & |x| = \delta \\ 0 & x > \delta \end{cases}\right\} = \chi_{B_\delta}(x)$$

So we can solve

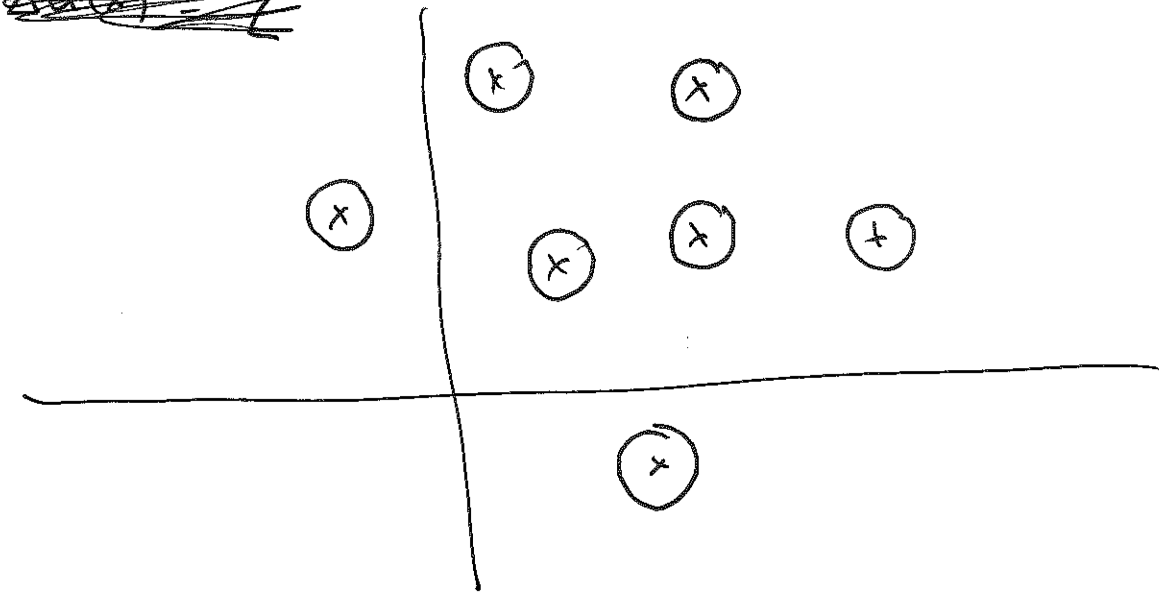
$$\Delta u_\delta(x^i) = f(x^i)$$

for any finite set of points

By writing $u(x) = \sum_{j=1}^{N} u_\delta(x-x^j)\, f(x^j)\, \frac{4\pi\delta^3}{3}$

by choosing $\delta < \min_{i,j} |x^i - x^j|$ since then

$\cancel{\Delta u(x) = \sum}$



But $\sum u_\delta(x-x^j)\, f(x^j)\, \text{Volume}\left(B_\delta(x^j)\right)$

looks like a Riemann sum so what if we define

$$u^\delta(x) = \int_{\mathbb{R}^3} f(\xi)\, u_\delta(x-\xi)\, d\xi$$

Would we get something like a solution? $\Delta u = f$?

This is something that we can calculate

$$|\Delta u^\delta(x) - f(x)| = \left\{\begin{matrix}\text{Diff} \\ \text{under} \\ \text{the} \\ \text{integral}\end{matrix}\right\} = \left| \int_{\mathbb{R}^3} f(\xi) \underbrace{\Delta u_\delta(x-\xi)}_{\substack{= \frac{3}{4\pi\delta^3} \quad x-\xi \in B_\delta \\ =0 \quad x-\xi \notin B_\delta}} d\xi - f(x) \right|$$

$$= \left| \int_{B_\delta(x)} \frac{3}{4\pi\delta^3} f(\xi)\, d\xi - f(x) \right| = \left| \frac{3}{4\pi\delta^3} \int_{B_\delta} |f(\xi) - f(x)|\, d\xi \right| \le \sup_{\xi \in B_\delta(x)} |f(\xi) - f(x)|$$

So if $f(x)$ is uniformly continuous, then at least formally

$$\lim_{\delta\to 0} \left| \Delta u^{\delta}(x) - f(x) \right| \leq \lim_{\delta \to 0} \sup_{|x-\xi|<\delta} |f(x) - f(\xi)| = 0.$$

But
$$\lim_{\delta\to 0} u^{\delta}(x) = \lim_{\delta\to 0} \int_{\mathbb{R}^3} f(\xi)\, u_\delta(x-\xi)\, d\xi \overset{\text{formally}}{=} \int_{\mathbb{R}^3} f(\xi) \left(-\frac{1}{4\pi} \frac{1}{|x-\xi|} \right) d\xi.$$

$$\boxed{\begin{array}{l} \underline{\text{Conjecture:}} \text{ Let } f \text{ be a uniformly continuous function on } \mathbb{R}^3 \text{ with compact support and define} \\[1em] \qquad u(x) = -\displaystyle\int_{\mathbb{R}^3} \frac{1}{4\pi |x-\xi|} f(\xi)\, d\xi \\[1em] \qquad \text{then } \Delta u(x) = f(x). \qquad\qquad (2) \end{array}}$$

Is the conjecture true?

Is $u(x)$ well defined, is the integral well defined?

To make sense of (2) we need to make sure that $u(x)$ has second derivatives.

We end this lecture by showing that $u(x)$ is well defined.

~~Lemma~~ Theorem: Assume that $g(\xi) \in C(\mathbb{R}^3 \setminus \{x\})$ has compact support in $\mathbb{R}^n$ and

$$|g(\xi)| \leq \frac{C}{|\xi - x|^p} \quad \text{for}$$

some $p < n$ then

$$\int_{\mathbb{R}^n} g(\xi)\, d\xi \quad \text{is well defined}$$

Proof: By definition the generalized integral

$$\int_{\mathbb{R}^n} g(\xi)\, d\xi \quad \text{is well defined if}$$

$$\lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{B_R(x) \setminus B_\varepsilon(x)} g(\xi)\, d\xi \quad \text{is well defined}$$

~~Clearly if R is large enough~~

Now if R_0 is so large that $g = 0$ in $B_{R_0}(x)$ it follows that

$$\lim_{R \to \infty} \int_{B_R \setminus B_\varepsilon} g(\xi)\, d\xi = \int_{B_{R_0} \setminus B_\varepsilon} g(\xi)\, d\xi \quad \text{for any } \varepsilon$$

so it is enough to show that

$$\lim_{\varepsilon \to 0} \int_{B_{R_0} \setminus B_\varepsilon} g(\xi)\, d\xi \quad \text{exists.}$$

But for $0 < \varepsilon_1 < \varepsilon_2$

$$\left| \int_{B_{R_0} \setminus B_{\varepsilon_1}} g(\xi)\, d\xi - \int_{B_{R_0} \setminus B_{\varepsilon_2}} g(\xi)\, d\xi \right| = \left| \int_{B_{\varepsilon_2} \setminus B_{\varepsilon_1}} g(\xi)\, d\xi \right| \leq$$

$$\leq \int_{B_{\varepsilon_2} \setminus B_{\varepsilon_1}} |g(\xi)|\, d\xi \leq \int_{B_{\varepsilon_2} \setminus B_{\varepsilon_1}} \frac{c}{|x-\xi|^p}\, d\xi = \left\{ \begin{array}{l} \text{polar} \\ \text{coordinates} \end{array} \right\} =$$

$$= c\omega_n \int_{\varepsilon_1}^{\varepsilon_2} r^{n-1-p}\, dr \leq \frac{c\,\omega_n (\varepsilon_2)^{n-p}}{n-p}.$$

It follows that $\int_{B_{R_0} \setminus B_\varepsilon} g(\xi)\, d\xi$ is

Cauchy in ε as $\varepsilon \to 0$ and thus

convergent.

Corollary: The function $u(x)$ in the conjecture is well defined.

Proof: Choose $g(\xi) = -\frac{1}{4\pi|x-\xi|} f(\xi)$ in the lemma.