PDE 10

Last week we proved that

Theorem: Let $f \in C_c^\alpha(B_{2R}(0))$ for some $\alpha \in (0,1)$ and define

$$u(x) = \int_{B_{2R}(0)} N(x-y) f(y)\, dy$$

then there exists a constant $C_{\alpha,n}$ s.t.

$$\sup_{x,y \in B_R(0)} \frac{\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right|}{|x-y|^\alpha} \leq C_{\alpha,n} \left([f]_{C^\alpha(B_{2R}(0))} + \frac{\|f\|_{C^0(B_R)}}{R^\alpha} \right).$$

Proposition: Assume that

$$\Delta u = f \quad \text{in} \quad B_{2R}(0)$$

$$\sup_{B_{2R}(0)} |u| < \infty$$

Then

$$(\ast) \begin{cases} \sup\limits_{x \in B_R(0)} |D^2 u(x)| \le C_{n,\alpha} \left([f]_{C^\alpha(B_{2R}(0))} + \|f\|_{C^0} + \sup\limits_{B_{2R}} |u| \right) \\[2ex] \sup\limits_{x,y \in B_R(0)} \dfrac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \le C_{n,\alpha} \left([f]_{C^\alpha(B_{2R}(0))} + \|f\|_{C^0} + \sup\limits_{B_{2R}} |u| \right) \end{cases}$$

Proof: We may write $u(x) = v(x) + h(x)$ where

$$v(x) = \int_{B_{2R}} N(x-\xi) f(\xi) \, d\xi$$

$$\Delta h = 0 \quad \text{in} \quad B_{2R}(0)$$

$$h = u - v \quad \text{on} \quad \partial B_{2R}(0)$$

Then v satisfies the estimates in $(\ast)$ by the previous theorem.

Moreover $\sup_{B_2} |v(x)| \leq C \sup |f| \int_{B_4} |N(x-\xi)|\, d\xi \leq C \sup |f|$

so

$$\sup_{\partial B_2(0)} |h| \leq \sup_{\partial B_2} |u| + C \sup_{B_2} |f|$$

and by the maximum principle

$$\sup_{B_2} |h| \leq \sup_{\partial B_2} |u| + C \sup_{B_2} |f|$$

Now

$$\sup_{B_1(0)} |D^2 h| \leq C_2 \|h\|_{C^1(B_2)} \leq C \left(\sup_{B_2} |u| + \sup_{B_2} |f| \right)$$

So the first estimate holds for h and thus for $u = v + h$ by the triangle inequality.

Also

$$\sup |D^3 h| \leq C_3 \|h\|_{L^1} \qquad \text{so}$$

$$\sup \frac{|D^2 h(x) - D^2 h(y)|}{|x-y|^\alpha} = \left\{ \begin{matrix} \text{mean value} \\ \text{theorem} \end{matrix} \right\} \leq \frac{|D^3 h(\xi)|}{|x-y|^\alpha} \leq C \left(\sup|u| + \ldots \right)$$

for some ξ on the line from x to y

The second estimate in $(*)$ follows for h and thus for u by the triangle ineq.

~~PDE 10.~~

~~Last week we proved that~~

Theorem: Let Ω be a domain and $u(x)$ a solution to

$$\Delta u(x) = f(x).$$

Where $f(x) \in C^{\alpha}_{loc}(\Omega)$ $\underbrace{}_{\alpha \in (0,1)}$ and that for any subset $K \subset\subset \Omega$ $f(x)$ satisfies

$$\sup_{x,y \in K} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} = [f]_{C^{\alpha}(K)} \leq \frac{C_{\alpha,f}}{\operatorname{dist}(K,\partial\Omega)^{2+\alpha}}$$

and

$$\sup_{x \in K} |f(x)| = \|f\|_{C(K)} \leq \frac{C_{0,f}}{\operatorname{dist}(K,\partial\Omega)^{2}}.$$

Then there exists a constant $C_{n,\alpha}$ depending only on the dimension n, $\alpha \in (0,1)$ such that

$$\sup_{x \in K} |D^2 u(x)| \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_{\Omega} |u|}{\operatorname{dist}(K,\partial\Omega)^2}$$

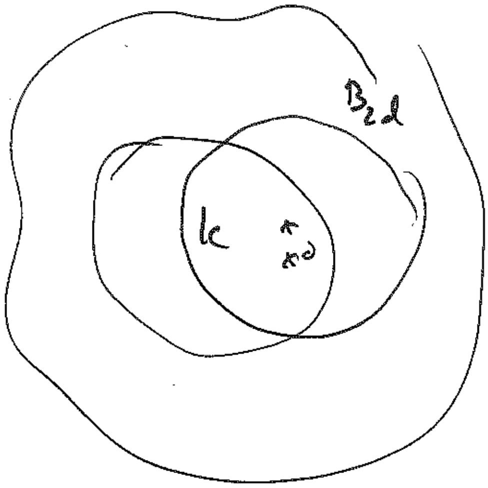
$$\sup_{x,y \in K} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^{\alpha}} \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_{\Omega}|u|}{\operatorname{dist}(K;\partial\Omega)^{2+\alpha}}.$$

Proof: The idea of the proof is to use scaling. Let K be a compact set ~~We def~~ and $d = \frac{\operatorname{dist}(K, \partial\Omega)}{4}$.

Then for any $x^0 \in K$ we have that

$$v(x) = u(dx + x^0)$$

is defined in $B_2(0)$



$$\Delta v(x) = d^2 \Delta u(dx + x^0) = d^2 f(dx + x^0)$$

$$\sup_{B_2(0)} |v(x)| = \sup_{B_{2d}(x^0)} |u| \le \sup_{\Omega} |u|.$$

Using the proposition we can deduce

$$|D^2 v(0)| \le \sup_{x \in B_1(0)} |D^2 v| \le C_{\alpha,n} \left(d^2 \left[f(d \cdot) \right]_{C^\alpha(B_2)} + d^2 \|f\|_{C^0} + \sup |u| \right)$$

But $|D^2 v(0)| = d^2 |D^2 u(x^0)|$ so

$$\sup_{x,y\in B_2(0)} \frac{|f(dx+x^0)-f(dy+x^0)|}{|x-y|^\alpha} = \sup_{x,y\in B_d(x^0)} d^\alpha \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq$$

$$\leq d^\alpha \sup_{x,y\in K} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

Thus $$\underbrace{d^{\alpha+2}[f]_{C^\alpha(K)}}_{C_{\alpha,f}} \geq d^2\,[f(d\cdot+x^0)]_{C^\alpha(B_2(0))}$$

and $$d^2 \sup_{B_2(0)} |f(dx+x^0)| = d^2 \sup_{B_{2d}(0)} |f(x)| \leq d^2 \sup_K |f(x)| = C_{0,f}.$$

Thus

$$|D^2u(x^0)| \leq \frac{1}{d^2}|D^2v(0)| \leq C_{n,\alpha}\left(\frac{C_{\alpha,f}+C_{0,f}+\sup|u|}{d^2}\right).$$

The estimate for

$$\sup_{x,y}\frac{|D^2u(x)-D^2u(y)|}{|x-y|^\alpha}$$ is done similarly.

We are starting to get lots of notation so we need to clean it up ~~and~~ in order to better be able to formulate results.

Most of the theory we do can be formulated in Banach spaces

Definition: We say that a set A is a linear space over $\mathbb{R}$ if there are two operations defined on A "+" and multiplication by $a \in \mathbb{R}$ s.t

Additive group

a) $u, v \in A \Rightarrow u+v = v+u$ (commutative)

b) $u, v, w \in A \Rightarrow (u+v)+w = u+(v+w)$

c) There exist a special element $0 \in A$ s.t $u+0=u$.

d) For every $u \in A$ there exist $v \in A$ s.t. $u+v=0$.

⋮

MORE CRAP.

In particular if A consists of all continuous, (differentiable, integrable etc.) functions on a domain. Then A is a linear space under the normal addition and multiplication by real numbers.

Definition 2. A norm on $\|\cdot\|$ linear space A is a function $A \longmapsto \mathbb{R}$ s.t. for $u,v \in A$, $a \in \mathbb{R}$

1) $\|u\| \geq 0$ with equality iff $u=0$
2) $\|u+v\| \leq \|u\|+\|v\|$ (Triangle ineq.)
3) ~~For any~~ $\|a u\| = |a| \, \|u\|$.

Definition 3 A normed linear space that is complete (say every Cauchy sequence is convergent) is called a Banach space

Example : Let A be the set of two times differentiable functions on Ω such that

C^α

$$\|u\|_{C^{2,\alpha}(\Omega)} = \sup_{x\in\Omega} |u(x)| \cancel{+ \sup_{x\in\Omega} |\nabla u(x)| + \sup_{x\in\Omega} |D^2 u(x)|}$$
$$+ \sup_{x,y\in\Omega} \frac{|D^2 u(x) - D^2 u(y)|}{|x+y|^\alpha} < \infty .$$

Then A is a Banach space with the norm $\|\cdot\|_{C^{2,\alpha}(\Omega)}$. We denote that space $C^{2,\alpha}(\Omega)$.

Example: Let A be the set of functions s.t. $u \in C^{2,\alpha}(K)$ for any $K \subset\subset \Omega$ ~~and ... that~~ and there exists a C

$$\frac{\sup_{x\in K} |u(x)|}{\operatorname{dist}(K,\partial\Omega)^{l}} + l\,\frac{\sup_{x\in K} |\nabla u(x)|}{\operatorname{dist}(K,\partial\Omega)^{1+l}} + \frac{\sup_{x\in K} |D^2 u(x)|}{\operatorname{dist}(K,\partial\Omega)^{2+l}} + \frac{\sup_{x,y} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^{\alpha}}}{\operatorname{dist}(K,\partial\Omega)^{2+\alpha+l}} \leq C \qquad (1)$$

Then A is a Banach space under the norm

$\|u\|_{C^{2,\alpha}_{int,(l)}(\Omega)}$ = inf of constants C s.t. (1) holds for all $K \subset\subset \Omega$.

We can thus formulate our previous result

Thm: Assume that $\Delta u = f$ in Ω then

$$[D^2u]_{C^{\alpha}_{int,(2)}(\Omega)} \leq C_{n,\alpha}\left(\|f\|_{C^{\alpha}_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)}\right).$$

A change in perspective.

Before we have viewed a PDE, say

$$\left.\begin{aligned} Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} &= f(x) \quad \text{in } \Omega \\ u(x) &= g(x) \quad \text{on } \partial\Omega \end{aligned}\right\} (1)$$

as the problem to find ~~a~~ function u s.t. (1) is satisfied for every $x \in \Omega$.

However, we could formulate (or rather conceptualize) L as a mapping between Banach spaces. Given a $u \in C^2(\Omega)$ then $Lu(x) \in C(\Omega)$ so

$$L : C^2(\Omega) \to C(\Omega).$$

Is L invertible?

Thm: Let $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be a map from $C^2(\mathbb{R}^n) \to C(\mathbb{R}^n)$.

Then Δ is invertible as a mapping

$$A = \{u \in C^{2,\alpha}(\mathbb{R}^n);\ \lim_{x\to\infty} u(x) = 0\} \to C_c^{\alpha}(\mathbb{R}^n)$$

with inverse given by

$$\Delta^{-1} f = \int_{\mathbb{R}^n} N(x-\xi) f(\xi)\, d\xi.$$

A last thing - Algebras

One problem with algebraic definitions is that when we work with real structures, such as $C^{\alpha}(\Omega)$, the ~~space~~ Banach space has extra structure.
We may define multiplication between elements in C^{α} in the obvious way ($C^{\alpha}(\Omega)$ is an algebra)

Thm: Let $\varphi, \phi \in C^{\alpha}(\Omega)$ then

$$\varphi(x)\phi(x) \in C^{\alpha}(\Omega)$$

and

$$\underbrace{\sup_{x,y\in\Omega} \frac{|\varphi(x)\phi(x) - \varphi(y)\phi(y)|}{|x-y|^{\alpha}}}_{[\varphi(x)\phi(x)]_{C^{\alpha}(\Omega)}} \leq \|\varphi\|_{C(\Omega)}[\phi]_{C^{\alpha}} + \|\phi\|_{C(\Omega)}[\varphi]_{C^{\alpha}(\Omega)}$$

Proof:

$$|\varphi(x)\phi(x) - \varphi(y)\phi(y)| = \left|\left(\varphi(x)\phi(x) - \varphi(y)\phi(x)\right) - \left(\varphi(y)\phi(y) - \varphi(y)\phi(x)\right)\right|$$

$$\leq \underbrace{|\phi(x)|}_{\leq \|\phi\|_{C(\Omega)}}\underbrace{|\varphi(x) - \varphi(y)|}_{\leq [\varphi]_{C^{\alpha}}|x-y|^{\alpha}} + \underbrace{|\varphi(y)|}_{\|\varphi\|_{C(\Omega)}}\underbrace{|\phi(x) - \phi(y)|}_{\leq |x-y|^{\alpha}[\phi]_{C^{\alpha}(\Omega)}} \leq$$

$$\leq \left(\|\phi\|_{C(\Omega)}[\varphi]_{C^{\alpha}} + \|\varphi\|_{C(\Omega)}[\phi]_{C^{\alpha}}\right) \cdot |x-y|^{\alpha}.$$

Divide by $|x-y|^{\alpha}$ and take the supremum. ☒