Our aim is to show that if $u \in C^{2,\alpha}_{int}(\Omega)$ and

$$Lu(x) = \sum a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} + cu = f(x) \quad \text{in } \Omega$$

then

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \leq C\left(\|f\|_{C^{\alpha}_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)}\right)$$

That is for any $K \subset\subset \Omega$

$$\boxed{\sum_{j=0}^{2} \left(\operatorname{dist}(K,\partial\Omega)^j \sup_K |D^j u|\right)} + \overbrace{\sup_{x,y\in K} \operatorname{dist}(K,\partial\Omega)^{2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha}}^{[D^2u]_{C^\alpha_{int,(2)}}} \leq$$

$$\leq C\left(\sup_{K\subset\subset\Omega}\left(\operatorname{dist}(\bar{K},\partial\Omega)^2 \sup_K |f| + \operatorname{dist}(K,\partial\Omega)^{2+\alpha} \sup_{x,y\in\bar{K}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}\right.\right.$$

$$\left.\left. + \sup_\Omega |u|\right)\right)$$

interpol inequality

$$\leq C_\varepsilon \|u\|_{C(\Omega)} + \varepsilon [D^2u]_{C^\alpha_{int,(2)}(\Omega)}$$

So it is enough to show

$$[D^2u]_{C^\alpha_{int,(2)}} \leq C\left(\|f\|_{C^\alpha_{int,(2)}} + \|u\|_{C(\Omega)}\right)$$

since then

$$+ [D^2u]_{C^\alpha_{int,(2)}} \leq C_\varepsilon \|u\|_{C(\Omega)} + (1+\varepsilon)[D^2u]_{C^2_{int,(2)}}$$

$$\leq \cancel{C_\varepsilon} C(1+\varepsilon)\left(\|f\|_{C^\alpha_{int,(2)}} + \|u\|_{C(\Omega)}\right) + \cancel{C_\varepsilon \|u\|_{C(\Omega)}}$$

~~PDE 15~~

So far we have shown

1) $\Delta u(x) = f(x) \implies \|u\|_{C^{2,\alpha}_{int}} \le C\left(\|f\|_{C^{\alpha}_{int,(2)}} + \|u\|_{C}\right)$

Constant coeff. $\sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \implies$ 〃

2) If $a_{ij}, b_i, c \in C^{\alpha}$ then and $Lu = f(x)$

$$\underbrace{\sum a_{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}}_{\text{constant coeff.}} = \overbrace{\underbrace{f(x)}_{C^{\alpha}} - \left(\underbrace{\sum (a_{ij}(x) - a_{ij}(x^0))}_{} \frac{\partial^2 u}{\partial x_i \partial x_j} + \underbrace{\sum b_i \frac{\partial u}{\partial x_i} + c(x) u}_{}\right)}^{F(x)}$$

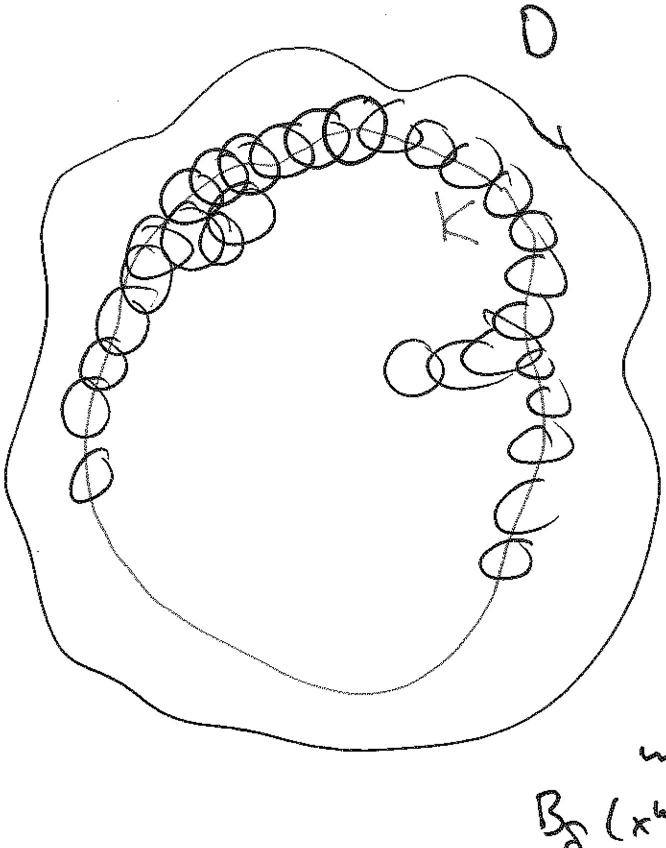
$(1-\varepsilon)\, \|D^2 u\|_{C^{\alpha}_{int,(2)}}$

If $\|a_{ij}(x) - a_{ij}(x^0)\| \le \varepsilon$ (C) then this should behave like $\varepsilon \|D^2 u\|_{C^{\alpha}_{int,(2)}}$

Controll by interpolation ineq.

$\|u\|_C + \varepsilon [D^2 u]_{C^{\alpha}_{int,(2)}}$

If $\|a_{ij}(x) - a_{ij}(x^0)\|_{C^{\alpha}} < \varepsilon$.



In order to make $\|a_{ij}(x) - a_{ij}(x^0)\|$ small we ~~will cover D~~ will fix a compact set K. Since K is compact we may cover K by finitely (not needed) many balls $B_{\delta_k}(x^k)$ such that

$$|a_{ij}(x) - a_{ij}(x^k)| < \varepsilon \quad \text{in} \quad B_{2\delta_k}(x^k)$$

and $$4\delta_k \le \mathrm{dist}(x^k, \partial\Omega)$$

<u>Claim</u>: ~~For any x, x~~ The following estimate holds

$$\delta_k^2 \|D^2 u\|_{C^\alpha_{int,}(B_{2\delta_k})} \le C_L\left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{int,(2)}(\Omega)}\right)$$

where C_L depends only on L, α and the dimension

In $B_{2\delta_k}(x^k)$ u solves

$$\sum a_{ij}(x^k)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) - \left(\sum (a_{ij}(x) - a_{ij}(x^k))\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} + cu\right) = F(x)$$

We may thus use the estimates for constant coefficient equations to estimate

$$\delta^2\|D^2 u\|_{C} + \delta^{2+\alpha}\|D^2 u\|_{C^\alpha} \le [\![D^2 u]\!]_{C^\alpha_{int}(B_{2\delta_k})} \le C\left(\|u\|_{C(\Omega)} + \|F\|_{C^\alpha_{int,(2)}(B_{2\delta})}\right)$$

$$\le C\Big(\overbrace{\|u\|_{C(\Omega)}}^{\text{ok!}} + \overbrace{\Big\|\sum (a_{ij}(x) - a_{ij}(x^k))\frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_{C^\alpha_{int,(2)}(B_{2\delta}(x^k))}}^{\mathrm{I}} + \overbrace{\|f\|_{C^\alpha_{int,(2)}(\Omega)}}^{\text{ok!}}$$

$$+ \underbrace{\Big\|\sum_i b_i(x)\frac{\partial u}{\partial x_i}\Big\|_{C^\alpha_{int,(2)}(\Omega)}}_{\mathrm{II}} + \underbrace{\|c\,u\|_{C^\alpha_{int,(2)}}}_{\mathrm{III}}\Big).$$

We need to estimate I, II and III

In order to estimate these terms we remind ourselves of the multiplicative estimate for Hölder norms

$$[\phi e]_{C^\alpha_{int,(2)}} \le [\phi]_{C^\alpha_{int,(2)}} \|e\|_{C(\Omega)} + [e]_{C^\alpha_{int,(2)}} \|\phi\|_{C_{int,(2)}(\Omega)}$$

$$+ \|e\|_{C(\Omega)} \|\phi\|_{C_{int,(2)}(\Omega)}$$

Thus

$$I \le [D^2 u]_{C^\alpha_{int,(2)}} \overbrace{\|a_{ij}(x) - a_{ij}(x^0)\|_{C(\Omega)}}^{\le \varepsilon} + \overbrace{[a_{ij}(x) - a_{ij}(x^0)]_{C^\alpha}}^{C \text{ only dep on } \varepsilon} \underbrace{\|D^2 u\|_{C_{int,(2)}}}_{\text{interpolation}}$$

$$+ \|a_{ij}(x) - a_{ij}(x^0)\|_{C(\Omega)} \|D^2 u\|_{C_{int,(2)}(\Omega)} \le$$

$$\le C\varepsilon [D^2 u]_{C^\alpha_{int,(2)}} + C\left(C_\varepsilon \|u\|_{C_\Omega} + \cancel{\varepsilon [D^2 u]_{C^\alpha_{int,(2)}}} \right)$$

$$\le C\varepsilon [D^2 u]_{C^\alpha_{int,(2)}} + C\|u\|_{C(\Omega)}$$

To estimate II, III works similarly

We can thus estimate

$$[D^2 u]_{C^\alpha_{int,(2)}(B_{2\delta})} \le 2C\|f\|_{C^\alpha_{int,(2)}(B_{2\delta})} + 2C\|u\|_{C(B_{2\delta})}$$

$$+ \cancel{2C\varepsilon [D^2 u]_{C^\alpha_{int,(2)}}} .$$

Now we may assume that $2C_\varepsilon < \frac{1}{2}$ by choosing ε small

Now since $[D^2u]_{C^\alpha_{int,(2)}(B_{2\delta}(x^k))} \le C(\|f\| + \|u\|)$

it follows that

$$\underbrace{\delta^{2+\alpha}[D^2u]_{C^\alpha(B_\delta(x^k))}}_{} \le C(\|f\| + \|u\|)$$

$$\frac{dist(K,\partial\Omega)^{2+\alpha}}{\cancel{(4)^{2+\alpha}}}[D^2u]_{C^\alpha(B_\delta(x^k))} \le 4^{2+\alpha} C(\qquad)$$

In particular it follows that if

$x, y \in K$ and $|x-y| < \delta$ then

$$dist(K,\partial\Omega)^{2+\alpha}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \le (\|f\|_{C^\alpha_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)})$$

and if $|x-y| \ge \delta$ we get

$$(4)^\alpha\, dist(K,\partial\Omega)^{2+\cancel{\alpha}}\frac{|D^2u(x) - D^2u(y)|}{\cancel{|x-y|^\alpha}} \le 2(4^\alpha)\, dist(K,\partial\Omega) \sup_K |D^2u(x)| \le$$

$$\le \ldots \le 2C\left(\|f\|_{C^\alpha_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)}\right).$$

Thus

$$[D^2u]_{C^\alpha_{int,(2)}(\Omega)} \le C\left(\|f\|_{C^\alpha_{int,(1)}(\Omega)} + \|u\|_{C(\Omega)}\right)$$

The general case follows by interpolation. $\square$

So far we have been able to show that if $u \in C^{2,\alpha}_{int,\alpha}$ solves $Lu = f(x)$ then

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \le C\left(\|f\|_{C^{\alpha}_{int,(\alpha)}} + \|u\|_{C(\Omega)}\right).$$

The "int" indicates that the norm is interior and the second derivatives may go to ∞ as fast as $\mathrm{dist}(x, \partial\Omega)^{-2}$. This leads to a serious problem.

If we don't know that u exists and wants to solve

$$\begin{aligned} Lu &= f \\ u &= g \text{ on } \partial\Omega \end{aligned} \implies \sum a_{ij}(0)\frac{\partial^2 u}{\partial x_i \partial x_j} = \overbrace{f(x) - \underbrace{\left(\sum (a_{ij}(x) - a_{ij}(0))\frac{\partial^2 u}{\partial x_i \partial x_j} - \right.}_{\text{may go to } \to \infty}}^{F}$$

Then we would need to find a barrier $b(x)$ at every bdry point. That is a <u>super solution</u> to

$$\sum_{i,j=1}^{n} a_{ij}(0)\frac{\partial^2 u}{\partial x_i \partial x_j} = F(x) \approx \frac{1}{\mathrm{dist}(x, \partial\Omega)^2} \quad \text{in } \Omega$$

Consider the simplest case. $a_{ij}(0) = \delta_{ij}$, $\Omega = B_1^+(0)$

Then to assume that $b(x)$ is a super solution we would need

$$\Delta b(x) \le -\frac{1}{x_n^2}$$

$$b(0) = 0$$

$$b(x) \ge 0 \quad \text{on } \{x_n = 0\} \cap B_1(0).$$

To simplify further let us consider $n = 1$

Then $\frac{\partial^2 b(x)}{\partial x^2} = -\frac{1}{x^2} \Rightarrow \frac{\partial b}{\partial x} = \frac{1}{x} + c \Rightarrow b = \ln(x) + cx + d$

But $\ln(x)$ is singular at $x=0$ so no matter how we choose c & d we can not get $b(0)=0$.

The estimate is truly "interior" in the sense that it allows the ~~bd~~ solution to break down so fast that we loose all control over the bdry data - hence the $\ln(x)$ term.

In order to construct a barrier, and thus a solution we will have to improve the interior estimates close to the bdry.

From the above calculations it would be enough to show that $|F(x)| \leq C \mathrm{dist}(x,\partial\Omega)^{-2+\varepsilon}$ for some $\varepsilon > 0$. We will however go the entire way and show that, if the bdrl and the bdry data are good, then $u \in C^{2,\alpha}(\Omega)$.

The proof is almost line for line the same as in the interior case. So if you understand that proof everything will be smooth sailing from now on.

We will start to do the boundary estimates for the Laplace eq.

<u>Lemma</u> Let $f(x) \in C^{\alpha}(B_{2R}^{+}(0))$ for some $0 < \alpha < 1$ and define

$$u(x) = \int_{B_{2R}^{+}} N(x-\xi) f(\xi)\, d\xi$$

then there exist a $C_{n,\alpha}$ s.t.

$$[D^2 u]_{C^{\alpha}(B_R^{+}(0))} \leq C_{n,\alpha}\left([f]_{C^{\alpha}(B_{2R}^{+}(0))} + \frac{\sup|f|}{R^{\alpha}} \right)$$

<u>Proof</u>: