Last week we proved that if

(1)
$$f \in C^\alpha(B_{2R}^+(0)) \quad \text{and}$$

$$u(x) = \int_{B_{2R}^+(0)} N(x-\xi) f(\xi) d\xi \quad \text{then}$$

$$[D^2 u]_{C^\alpha(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}^+)} + \frac{\|f\|_{C(B_R)}}{R^\alpha} \right)$$

We also remarked that we need to show some boundary estimates, that if

$$\sum a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} + c u = f(x) \quad \text{in } \Omega$$

then
$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C_1 \left(\|f\|_{C^\alpha} + \|u\|_{C^0} \right) \quad (2)$$

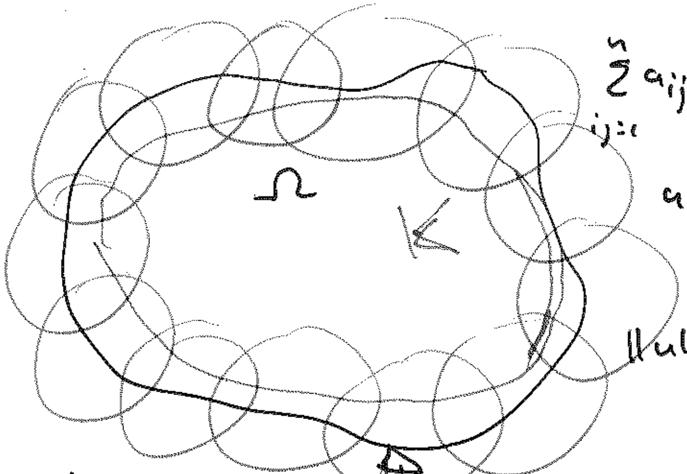
boundary data.

and that (1) is the first step in that proof.

Today we will prove something like (2)

We will do this, by a long and complicated construction...
(are you surprized?)

General problem

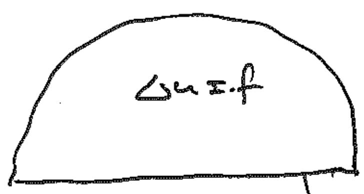


$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + c(x)u = f(x)$$

$u(x) = h(x)$ on $\partial\Omega$

$\|u\|_{C^{2,\alpha}(\Omega)} \leq C \quad]$

1) Simplify to the simplest case



flat boundary

$\|u\|_{C^{2,\alpha}(B_R^+)} \leq C$

$\Downarrow$

2) Simplify further

$$w = \int_{B_{2R}} N(x-\xi) f(\xi)\, d\xi$$

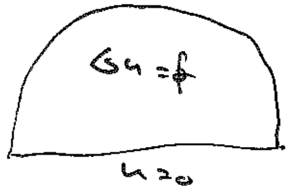
Explicit calculations last time

$[D^2 w]_{C^\alpha}$

$\Downarrow$

Interpolation $\|u\|_{C^{2,\alpha}} \leq C\left(\|f\|_{C^\alpha} + \|u\|_{C^0}\right)$

$\Downarrow$



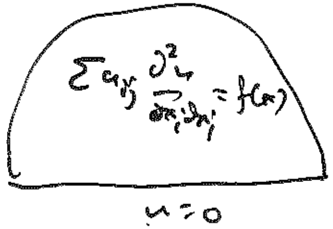
Write $u = v + w$

where $\Delta v = 0$

$\Delta w = f$.

Since the boundary is flat we can use a reflection of 2) to get $w(x',0)=0$ and a reflection of v

$\Leftarrow$



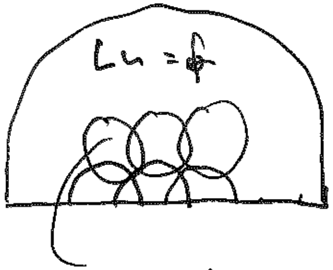
Constant coefficients

Make a linear transformation

$v(x) = u(Px) \implies \Delta v = f(Px)$

$v = 0$ on

$\Leftarrow$



Freezing

$$\sum a_{ij}(x^k) \frac{\partial^2 u}{\partial x_i \partial x_j} = $$

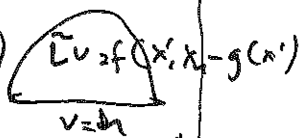
$$= f + \left(\sum (a_{ij}(x^k) - a_{ij}(x))\right) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

in each ball as last lecture

$\Uparrow$

$\partial\Omega = \{g(x') = x_n\}$

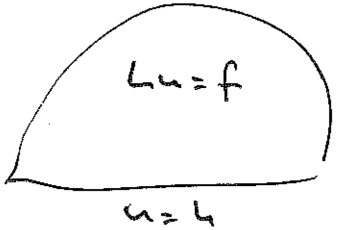
$\implies$



$v(x) = u(x', x_n + g'(x'))$

$\Uparrow$

$\Leftarrow$



Consider

$v = u(x) - h(x')$

$Lv = f - Lh = \tilde{f}$

Proposition Assume that $u \in C^2(B_{2R}^+)$ solves

$$\Delta u(x) = f(x) \quad \text{in } B_{2R}^+$$

$$u(x) = 0 \quad \text{on } B_{2R} \cap \{x_n = 0\}.$$

$$f \in C^\alpha(B_{2R}), \quad \alpha \in (0,1)$$

Then

$$\|u\|_{C^{2,\alpha}(B_R^+)} \leq C\left([f]_{C^\alpha(B_{2R})} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B_{2R})} + \frac{1}{R^{2+\alpha}}\|u\|\right)$$

Proof: We will write $u = v + h$

where
$$\left.\begin{aligned} \Delta v(x) &= f(x) \\ v(x', 0) &= 0 \end{aligned}\right\} \quad \text{Potential and reflection}$$

and
$$\left.\begin{aligned} \Delta h &= 0 \\ h(x', 0) &= 0. \end{aligned}\right| \quad \text{reflection and interior est. for harmonic functions}$$

Step 1 Construction of v

Reflect f in $x_n = 0$.

$$\hat{f}(x) = \begin{cases} f(x) & x_n \geq 0 \\ f(x', -x_n) & x_n < 0 \end{cases}$$

then $\hat{f} \in C^\alpha(B_{2R})$.

so

$$\hat{v}(x) = \int_{B_{2R}(0)} N(x-\xi)\,\hat{f}(\xi)\,d\xi \qquad \text{will satisfy}$$

$$\|\hat{v}\|_{C^{2,\alpha}(B_R(0))} \le \underbrace{C_{n,\alpha}\left([\hat{f}]_{C^\alpha} + \left(R^2 + R\tfrac{1}{R^\alpha}\right)\|\hat{f}\|_{C(B^+_{2R})}\right)}$$

also

$$\check{v} = \int_{B^+_{2R}} N(x-\xi)\,f(\xi)\,d\xi \qquad \text{satisfies}$$

$$\|\check{v}\|_{C^{2,\alpha}} \le \left(\qquad \right)$$

Observe that on $x_n = 0$ we have

$$\hat{v} = 2\check{v} \qquad \text{so}$$

$$v = 2\check{v} - \hat{v} = 0 \quad \text{on} \quad x_n = 0.$$

also

$$\Delta v = 2\Delta\check{v} - \Delta\hat{v} = 2f - \hat{f} = f \quad \text{in } B^+_{2R}(0)$$

so v satisfies the conditions and estimates

Step 2. Define $h = u - v$.

$$\left.\begin{aligned} \text{then} \quad \Delta h &= 0 \quad \text{in } B_{2R}^{+} \\ h(x', x_n) &= 0 \quad \text{on } \{x_n = 0\}. \end{aligned}\right\} \quad (1)$$

Consider the solution to

$$\Delta g = 0 \qquad \text{in } B_{2R}(0)$$

$$\left. g = \begin{cases} h(x) & x \in \partial B_{2R}(0), \; x_n > 0 \\ -h(x', -x_n) & x \in \partial B_{2R}, \; x_n < 0. \end{cases} \right\} \quad (2)$$

Then $g(x)$ is odd and thus

$g(x', 0) = 0$ on $x_n = 0$.

so g solves (1) as well

$\Rightarrow$ $g = h$ in $B_{2R}^{+}(0)$.

But g is harmonic in B_{2R}, so we may estimate

$$\|D^3 g\|_{C(B_R)} \leq C \|g\|_{L^1} \;\; \text{———}$$

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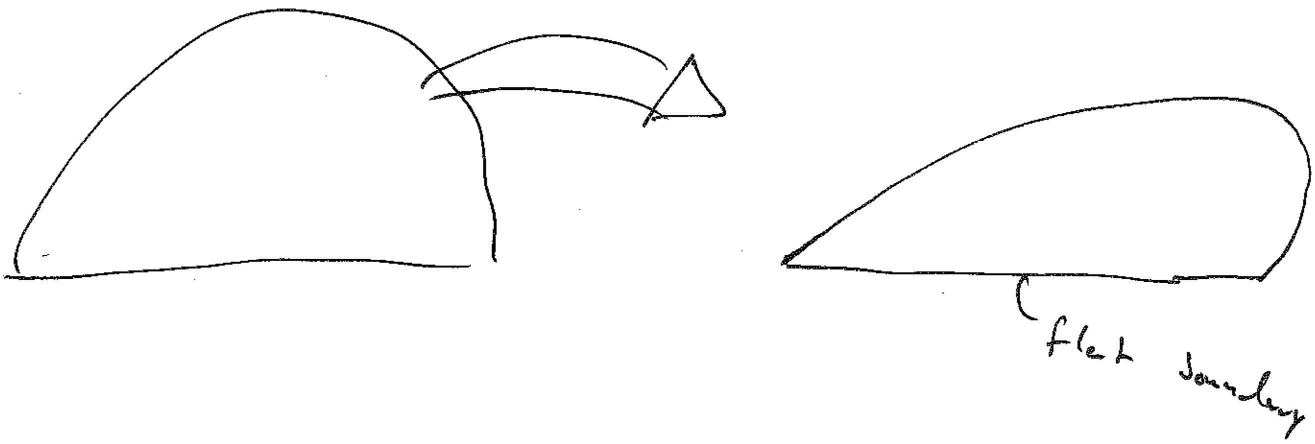
Prop. The same is true for constant coefficient PDE.

Proof: Same as in the interior.

We set $v(x) = u(Px)$

then $\Delta v(x) = f(Px)$ in $\{x \,;\, Px \in B_{2R}^+\}$.



Theorem: Let $u \in C^{2,\alpha}(B_{2R}^+)$ be a solution to

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x)$$

$$u(x) = h(x) \qquad \text{on} \qquad \{x_n = 0\} \cap B_{2R}(0).$$

$$\lambda|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for some } \Lambda, \lambda > 0$$

Then there exists a constant $C(\lambda, \Lambda, n, a_{ij}, b_i, c)$ s.t.

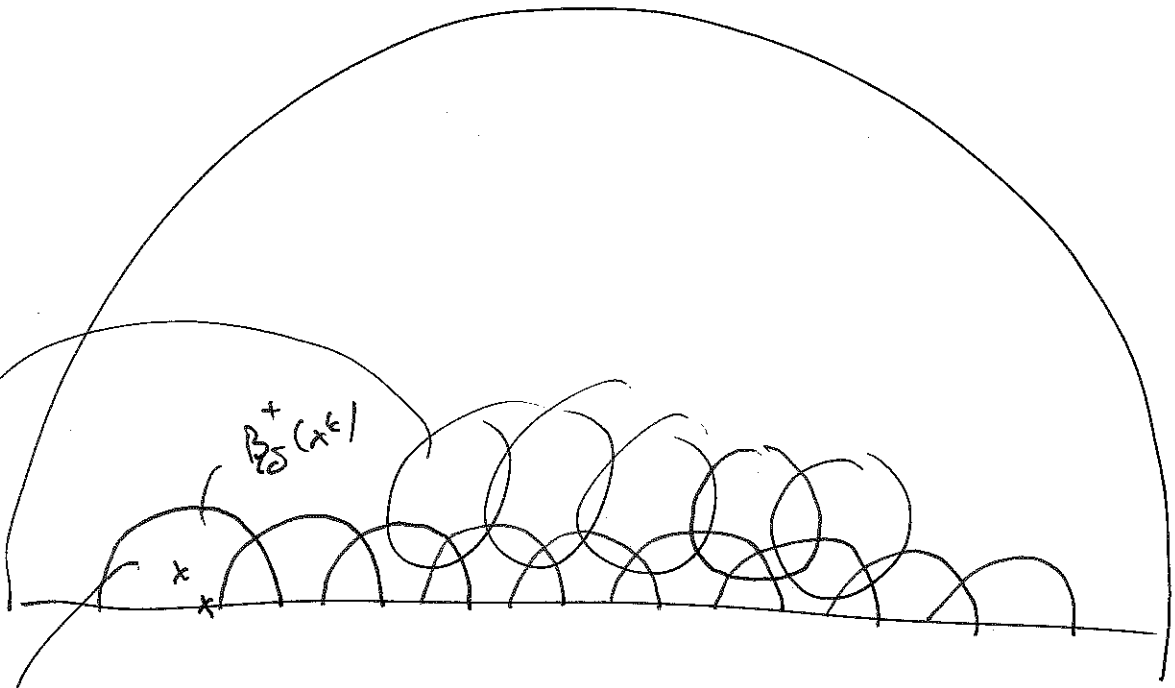
$$\|u\|_{C^{2,\alpha}(B_R^+)} \leq C\left([f]_{C^\alpha} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B_{2R}^+)} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B_{2R}^+(0))}\right.$$
$$\left. + \left(\frac{1}{R} + R^{3}\right)\|h\|_{C^{2,\alpha}(B_{2R}')}\right)$$

Proof: Notice that we may reduce to the case when $h = 0$ by considering $u(x) - h(x') = \tilde{u}$

then $L\tilde{u} = f - \underbrace{Lh}_{\in C^\alpha} = \hat{f}(x)$.

So w.l.o.g $h = 0$.

Now we may use the Freezing of coefficients argument. That is cover

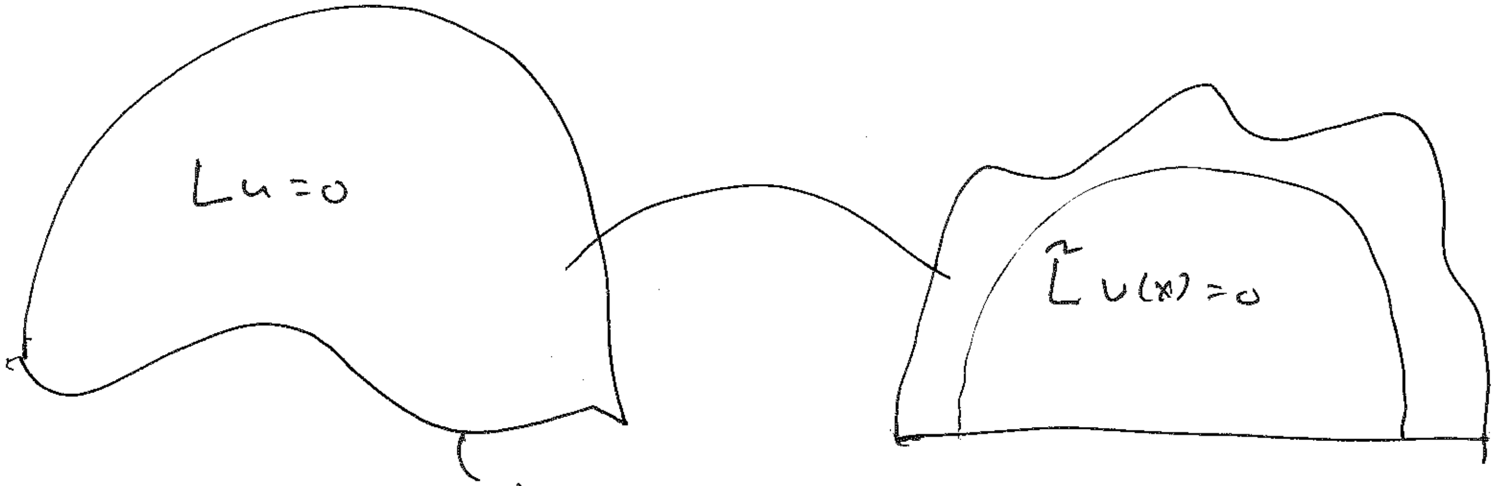


Here we write $Lu=f$ as

$$\sum a_{ij}(x^k)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) - \Big(\underbrace{\sum \underbrace{(a_{ij}(x) - a_{ij}(x^k))}_{\le \varepsilon} \frac{\partial^2 u}{\partial x_i \partial x_j} + \underbrace{\sum b_i \frac{\partial u}{\partial x_i} + cu}_{\text{interpolation}}}_{\text{interpolation}} \Big)$$

and show that $\|u\|_{C^{2,\alpha}(B_\delta^+(0))} \le C\Big(\qquad \Big)$ --

Here we use interior estimates as before.



$x_n = g(x') \in C^{2,\alpha}$

Lemma: Let $g(x') \in C^{2,\alpha}(B'_{2R}(0))$, $g(0) = |\nabla' g(x')| = 0$

~~Ω is~~ $\Omega = B_{2R}(0) \cap \{x_n > g(x')\}$.

Assume furthermore that u is a solution to

$$Lu(x) = f(x) \quad \text{in } \Omega$$

$$u(x) = 0 \quad \text{on } \partial\Omega \cap B_{2R}.$$

Then there is a constant c s.t. if $|\nabla g| \le c$. Then

$$v(x) = u(x', x_n - g(x'))$$

solves an elliptic PDE

$$\tilde{L}v(x) = 0 \quad \text{in } \{x;\ (x', x_n - g(x')) \in \Omega\}$$

$$v(x) = 0 \quad \text{on } x_n = 0.$$

Proof: We calculate

$$i,j \neq n$$

$$\frac{\partial u(x', x_n)}{\partial x_i} = \frac{\partial v(x', x_n + g(x'))}{\partial x_i} + \frac{\partial g}{\partial x_i} \frac{\partial v(x', x_n + g(x'))}{\partial x_n}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 v}{\partial x_i \partial x_j} + \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_n} + \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial v}{\partial x_n^2} + \frac{\partial g}{\partial x_i} \frac{\partial^2 v}{\partial x_j \partial x_n}$$

$\vdots$

etc...

If we, for simplicity assume that $b_i, c = 0$, then we get

$$f(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n} \tilde{a}_{ij}(x) \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_n}$$

~~where $\tilde{a}_{ij}$ is the matrix~~

$\Rightarrow$

$$f(x', x_n - g(x')) = \sum_{i,j=1}^{n} \underbrace{\tilde{a}_{ij}(x', x_n - g(x'))}_{\hat{a}_{ij}} \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial v(x)}{\partial x_n}$$

where $\hat{a}_{ij}$ is given in matrix representation

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & c \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ & & & \\ g_1 & g_2 & \cdots & 1+g_n \end{bmatrix}}_{G^T} \underbrace{\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & g_1 \\ 0 & 1 & 0 & & g_2 \\ 0 & 0 & 1 & & g_3 \\ & & & & \\ c & 0 & \cdots & & 1+g_n \end{bmatrix}}_{G}$$

so

$$\frac{\Lambda}{2}|\xi|^2 \geq \sum \hat{a}_{ij}(x)\xi_i\xi_j = (G\xi)^T \cdot A \cdot (G\xi) \geq \frac{\lambda}{2}|\xi|^2$$

But if $|\nabla g| < c$ then $\frac{1}{\sqrt{2}}|\xi| < |G\xi| \leq \sqrt{2}|\xi|$

so, since A is elliptic

$$2\Lambda|\xi|^2 \geq (G\xi)^T A (G\xi) \geq \lambda|G\xi|^2 \geq \frac{\lambda}{2}|\xi|^2$$

Thus $\hat{A}$ is elliptic.