PDE 14

We are going to let

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)\,u(x) = f(x) \quad \text{in } \Omega$$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \qquad 0<\lambda,\Lambda, \quad \text{all } \xi, x \in \mathbb{R}^n$$

$$a_{ij}, b_i, c, f \in C^{\alpha}(\Omega), \qquad g(x) \in C^{2,\alpha}(\partial\Omega)$$

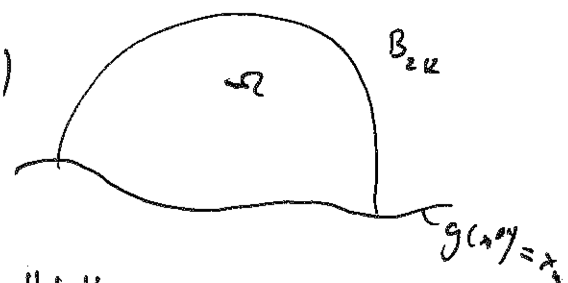
Theorem 1. If $u \in C^{2,\alpha}_{int}(\Omega)$ then

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \le C\left(\|f\|_{C^{\alpha}_{int}(\Omega)} + \|u\|_{C(\Omega)}\right)$$

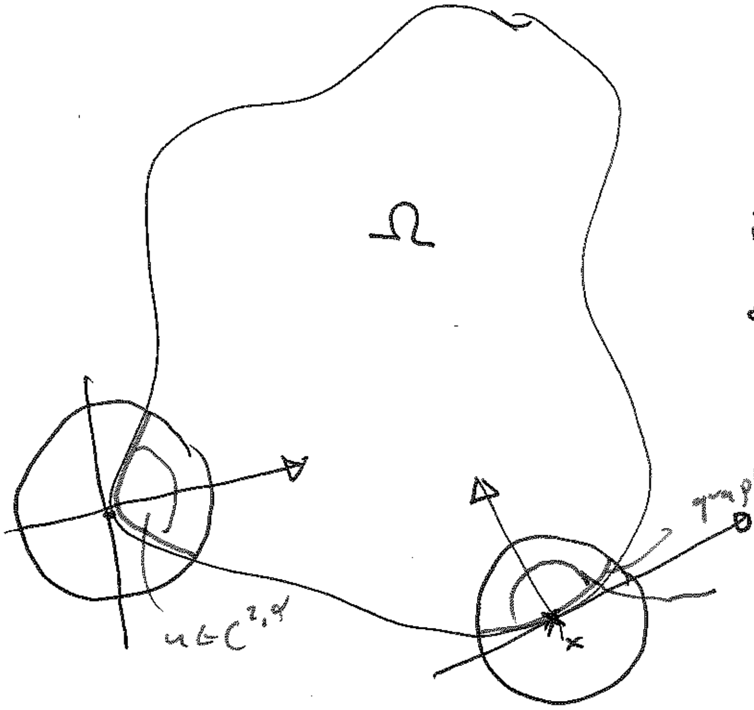
Theorem 2 If $u \in C^{2,\alpha}(\Omega)$

$u(x) = h(x) \in C^{2,\alpha}(\partial\Omega)$

Then



$$\|u\|_{C^{2,\alpha}(B_R\cap\Omega)} \le C\left(\|f\|_{C^{\alpha}(\Omega)} + \|h\|_{C^{2,\alpha}(\Omega)} + \|u\|_{C(B_{2R})}\right)$$



Definition We say that Ω is a $C^{2,\alpha}$-domain if $\partial\Omega$ can be written, for some r in $B_r(x)$ as the graph of a $C^{2,\alpha}$ function $g(x)$, for every $x \in \partial\Omega$

Theorem 3 $u \in C^{2,\alpha}(\Omega)$, Ω bdd $C^{2,\alpha}$ domain.

$$Lu = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Then $\exists C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$ s.t.

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)} + \|u\|_{C(\Omega)}\right)$$

Proof: Step 1 $\|u\|_{C^2(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)} + \|u\|_{C(\Omega)}\right)$

We only need to bound

$$|Du(x)| + |D^2u(x)| \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)} + \|u\|_{C(\Omega)}\right) \quad (*)$$

for each x. But either $x \in B_{r/4}(\xi)$ for some $\xi \in \partial\Omega$ and $(*)$ follows from Thm 7.

Or $x \in K = \underbrace{\overline{\Omega \setminus \{x;\ \mathrm{dist}(x, \partial\Omega) > r/4\}}}_{\text{compact.}}$

and $(*)$ follows from Thm 1.

<u>Step 2</u>: Estimate $\dfrac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \le C\left(\|f\|_{C^\alpha} + \|g\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right)$ $(**)$

If $|x-y| \geq r/4$ then

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \le \frac{2 \sup |D^2u|}{(r/4)^\alpha} \le C_r \sup |D^2u| \le C\|u\|_{C^2(\Omega)}$$

so it is enough to show $(**)$ for $|x-y| < r/4$.

If both $x, y \in K$ then $(**)$ follows from Thm 1.

If $x \notin K$ then $x \in B_{r/4}(\xi)$ for some $\xi \in \partial\Omega$.

so $x, y \in B_{r/2}(\xi)$ and

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \le [D^2u]_{C^\alpha(B_{r/2} \cap \Omega)} \le C\left(\|f\| + \|g\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right)$$

by Thm 2.

It is rather unpleasing to have the $\|u\|_{C(\Omega)}$ term in the right hand side of all the estimates. We would therefore like to estimate $\|u\|_{C(\Omega)}$ by the known quantities. We can do that if $c(x) \leq 0$ since then we have a comparison principle.

Remember that if N is large then we proved earlier that

$$w(x) = e^{-Nr^2} - e^{-N|x|^2} \quad \text{satisfies}$$

$$Lw(x) \leq -1 \qquad \text{in } \mathbb{R}^n \setminus B_r(0)$$

$$w = 0 \qquad \text{on } B_r(0).$$

Theorem: Let $u \in C^{2,\alpha}(\Omega)$, Ω bdd $C^{2,\alpha}$ domain

$$Lu = f \quad \text{in } \Omega$$

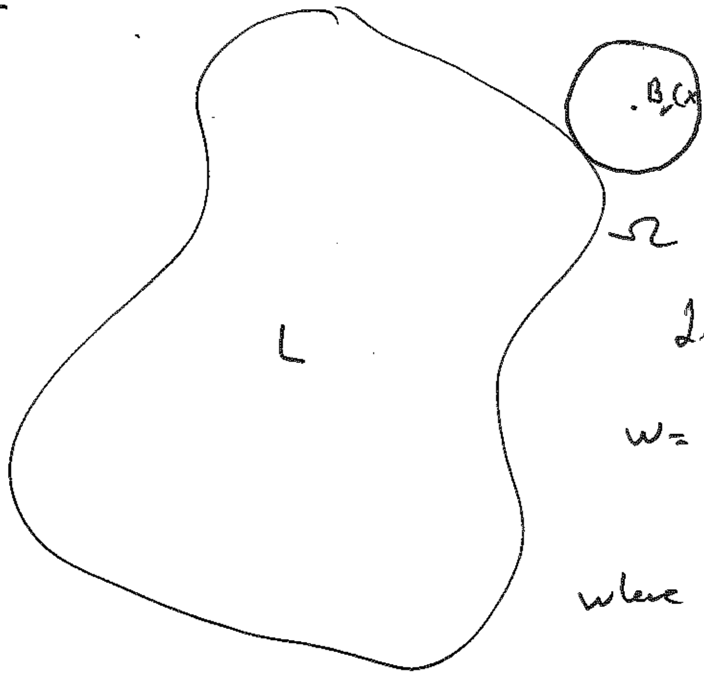
$$u = g \quad \text{on } \partial\Omega$$

AND $c(x) \leq 0$.

then $\exists C = C(n, \alpha, d, \Lambda, a_{ij}, b_i, c, \Omega)$ s.t.

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^{\alpha}(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)}\right)$$

Proof:



Let

$$w = \|f\|_{C(\Omega)}\left(e^{-Nr^2} - e^{-N|x-x_0|^2}\right)$$

where N is so large that $Lw \le -\|f\|_{C(\Omega)}$.

Then $w^+ = w + \|g\|_{C(\partial\Omega)}$ satisfies

$$Lw^+ = Lw + L\|g\|_{C(\partial\Omega)} \le -\|f\|_{C(\Omega)} + \underbrace{c(x)\|g\|_{C(\partial\Omega)}}_{\le 0} \le -\|f\|_{C(\Omega)}.$$

By the comparison principle

$$u(x) \le w^+(x) \le C\left(\|f\| + \|g\|\right).$$

Similarly $u(x) \ge -C\left(\|f\|_C + \|g\|_C\right)$

so $\|u\|_{C(\Omega)} \le C\left(\|f\|_{C(\Omega)} + \|g\|_{C(\partial\Omega)}\right)$.

By Thm

$$\|u\|_{C^{2,\alpha}(\Omega)} \le C\left(\|f\|_{C^{\alpha}(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)} + \cancel{\|u\|_{C(\Omega)}}^{\le C(\|f\|_{C(\Omega)} + \|g\|_{C(\partial\Omega)})}\right)$$

Contraction mapping principle.

We want to solve

$$Lu = f(x) \quad \text{in } \Omega \qquad\qquad \Omega \text{ is a } C^{2,\alpha}\text{-domain}$$
$$u = g \quad \text{on } \partial\Omega \qquad\qquad g \in C^{2,\alpha}(\partial\Omega). \qquad (*)$$

But we can not. However, we can solve

$$\left.\begin{aligned} \Delta u_1 &= f(x) \quad \text{in } \Omega \\ u_1 &= g \quad \text{on } \partial\Omega. \end{aligned}\right\} \quad (1)$$

~~which "differs~~

~~from~~ ~~So it is enough to solve~~

$\cancel{Lu = f(x)}$

$$Lu_1 = \Delta u_1 + (Lu_1 - \Delta u_1) = f(x) + \underbrace{(Lu_1 - \Delta u_1)}_{\text{Error!}}$$
$$u_1 = g$$

So if we can solve

$$Lu = \underbrace{-Lu_1 + \Delta u_1}_{f_1} \qquad (2)$$
$$u = 0$$

then $u_1 + u$ solves $(*)$. We can not solve (2) but we can solve

$$\Delta u_2 = f_1 \qquad (3)$$
$$u = 0$$

And then

$$Lu_2 = \overset{f_1}{\cancel{\Delta u_2}} + \underbrace{(Lu_2 - \Delta u_2)}_{\text{error.}}$$

Let us try to formalize this. We view L, Δ as maps between $\underbrace{C^{2,\alpha}(\Omega)}_{B}$ and $\underbrace{C^{\alpha}(\Omega)}_{V}$.

$$u_1 = \Delta^{-1}(f(x)) \qquad \text{that is} \qquad \Delta u_1 = f, \ \boxed{u_1 = 0}$$

$$u_2 = \Delta^{-1}(f_1) = \Delta^{-1}(-Lu_1 + \Delta u_1)$$

$$u_3 = \Delta^{-1}(-Lu_1 + \Delta u_2)$$

$$\vdots$$

$$u_{k+1} = \Delta^{-1}(-Lu_k + \Delta u_k) = \Delta^{-1}\left[(-L+\Delta)u_k\right]$$

So, by our appriori estimates,

$$\|u_{k+1}\|_{C^{2,\alpha}(\Omega)} \leq C\|(-L+\Delta)u_k\|_{C^{\alpha}(\Omega)} \leq$$

$$\leq C\Big(\underbrace{\sum \|a_{ij} - \delta_{ij}\|_{C^{\alpha}(\Omega)} + \sum \|b_i\|_{C^{\alpha}} + \|c\|_{C^{\alpha}}}_{\text{if this is } \leq \frac{\lambda}{2C},\ \lambda < 1}\Big)\|u_k\|_{C^{2,\alpha}(\Omega)}$$

$$\leq \frac{\lambda}{2}\|u_k\|_{C^{2,\alpha}(\Omega)}$$

$$\Rightarrow \|u_{k+1}\|_{C^{2,\alpha}} \leq \lambda^k \|u_1\|_{C^{2,\alpha}(\Omega)}.$$

But $u = \sum_{k=1}^{\infty} u_k$ so $\|u\|_{C^{2,\alpha}(\Omega)} \leq \sum_k \|u^k\|_{C^{2,\alpha}} < \infty$

So u exists.

Definition: We say that $F : B \to B$ is a contraction if

$$\|Fu - Fv\|_B \leq \lambda \|u-v\|_B \quad \text{for some } \lambda < 1.$$

So $Fu = \Delta^{-1}\left(f(x) + \Delta u(x) - Lu(x)\right)$

That is

$$\begin{cases} \Delta Fu(x) = f(x) + \Delta u - Lu & \text{in } \Omega \\ Fu(x) = g(x) & \text{on } \partial\Omega \end{cases}$$

Well defined since Δ has solutions

Is a contraction if L and Δ are close enough

Since

$$\|F(u) - F(v)\|_{C^{2,\alpha}} \leq \overbrace{C\left\|\left(\cancel{f(x)} + \Delta u - Lu\right) - \left(\cancel{f(x)} + \Delta v - Lv\right)\right\|_{C^\alpha}}^{\text{since } \Delta \text{ has estimates.}}$$

$$= C\|\underbrace{(\Delta - L)}_{\|\cdot\|_{C^\alpha} < \varepsilon}(u-v)\|_{C^\alpha} \leq C\varepsilon \|u-v\|_{C^{2,\alpha}}.$$

Proposition: [contracting mapping principle] Let F be a contracting map on a Banach space B. Then there exists a $u \in B$ s.t. $Fu = u$.

Example with F as above

$$\Delta Fu = f + \Delta u - Lu \quad \text{in } \Omega$$

$$\cancel{\Delta u} = f + \cancel{\Delta u} - Lu \quad \text{in } \Omega$$

$$\Rightarrow \quad Lu = f \quad \text{in } \Omega$$

So contraction mapping principle $\Rightarrow$ existence.

Proof ~~Let $u_0 = 0$ and define~~

Pick a random u_0 and define

$u_{k+1} = Lu_k$ then

$$\|u_{k+1} - u_k\| = \|Lu_k - Lu_{k-1}\| \leq \tau \|u_k - u_{k-1}\| \leq$$

$$\leq \tau \|Lu_{k-1} - Lu_{k-2}\| \leq \tau^2 \|u_{k-1} - u_{k-2}\| \leq .$$

$$\cdots \leq \tau^k \|u_1 - u_0\|$$

So u_k is a Cauchy sequence.

and thus $u_k \to u$.

Since F is a contraction it is continuous and thus

$$u = \lim_{k\to\infty} u_k = \lim_{k\to\infty} Lu_{k-1} = F\left(\lim_{k\to\infty} u_k\right) = Fu.$$

∎

Method of continuity.

Notice that $Lu = f$
$u = g$

has solutions follows from

1) Δ has solutions
2) Δ has estimates
3) $\|L - \Delta\|$ is small.

So if we define the operator, for a general L,

$$L_t = \cancel{\Delta}\,(1-t)\Delta + tL$$

then

1) $L_0 \cancel{= \Delta}$ has solutions
2) $L_0 \cancel{= \Delta}$ has estimates
3) L_t satisfies $\|L_t - \Delta_\varepsilon\|$ small if $|t| \leq \varepsilon$.

$\Rightarrow$ L_ε has solutions

<u>Proposition</u> [Method of continuity]: Let L_0 and L_1 be to bounded linear opperators from B to V (Banach spaces). Furthermore denote $L_t = (1-t)L_0 + tL_1$ and assume that there exists a constant C s.t.

$$\|u - v\|_B \leq C \|L_t(u-v)\|_V \quad \text{for all } t \in [0,1]$$

Then $L_t u = f$ has a solution for every $f \in V$ and every $t \in [0,1]$ if and only if $L_0 v = f$ has a solution for every $f \in V$.

Proof: It is enough to show

$L_0 u = f$ has solutions $\Rightarrow$ $L_t u = f$ has solutions for all $t \in [0,1]$

since the opposite implication is trivial.

Let $T = \{ t \in [0,1]$, s.t. $L_t u = f$ has sol solution for every $f \in V \}$.

Then $T \neq \phi$.

<u>Claim</u>: T is an open set

Proof: Let $t_0 \in T$, then we need to show that $(t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1] \subset T$.

To that end we define

$$F(u) = L_{t_0}^{-1}\left(f + L_{t_0} u - L_{t_0+\delta} u\right)$$

$$\Rightarrow \|F(u) - F(v)\|_B = \| L_{t_0}^{-1} (f + \underbrace{L_{t_0} u - L_{t_0+\delta} u}_{-\delta L_0 u + \delta L_1 u}) - L^{-1}(f + L_{t_0} v - L_{t_0+\delta} v)\|$$

$$= \underbrace{\delta \| L_{t_0}^{-1} \underbrace{\left(-L_0(u-v) + L_1(u-v)\right)}_{\substack{\leq C \|u-v\| \\ \text{since } L \text{ is } \text{sold}}} \|}_{\substack{\leq C \|u-v\| \text{ since} \\ L_{t_0}^{-1} \text{ has estimates}}} \leq C\delta \|u-v\| \leq \frac{1}{2}\|u-v\|$$

if $\delta < \frac{1}{2C} \equiv \varepsilon$.

So F is a contraction on $[t_0-\varepsilon, t_0+\varepsilon]$

So we may conclude that $\exists u$ s.t.

$$F(u) = u$$

$$L_{t_0}^{-1}\left(f + L_{t_0}u - L_{t_0+\delta}u\right) = u$$

$$\Rightarrow \quad f + \cancel{L_{t_0}u} - L_{t_0+\delta}u = \cancel{L_{t_0}u}$$

$$\Rightarrow \quad f = L_{t_0+\delta}u. \qquad \text{so} \qquad L_{t_0+\delta}u = f$$

has a solution for every $\delta \in (t_0-\varepsilon, t_0+\varepsilon)$

so T is open.

<u>Claim T is closed.</u>

Assume that $t_j \in T$ $\quad t_j \to t_\infty$

Then for any $f \in V$ $\quad \exists u_j$ s.t.

$$L_{t_j}u_j = f \quad \Rightarrow \quad L_{t_\infty}u_j = \underbrace{\left(L_{t_\infty} - L_{t_j}\right)}_{\to 0}u_j + f \to f.$$

so $L_{t_\infty}u = f$ also has a solution.

Thus T is open, closed and non-empty and thus $T = [0,1]$.

$\boxtimes$