Last week we ~~proved that~~ stated the informal conjecture

$$\boxed{\begin{array}{l}\textbf{Conjecture: } \text{Assume that } f(x) \text{ is uniformly continuous with compact support in } \mathbb{R}^n. \text{ Then} \\[1ex] \quad u(x) = \int_{\mathbb{R}^n} -\frac{1}{(n-2)\omega_n} \frac{1}{|x-y|^{n-2}} f(y)\,dy \qquad (n \geq 3) \\[1ex] \text{solves} \\ \quad \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n\end{array}}$$

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Today we will ~~prove something~~ try to prove it. We need to show

i) <u>$u(x)$ is well defined</u> (Last week we showed that $\int_{\mathbb{R}^n} g(x)\,dx$ is well defined for $g(x)$ that are continuous on $\mathbb{R}^n \setminus \{0\}$ has compact support and $|g(x)| \leq \frac{C}{|x|^{n-\alpha}}$ for any $\alpha > 0$.)

ii) We also made ~~the calculation~~ some informal calculations, that indicated that $\Delta u(x) = f(x)$

iii) We also need $u(x) \in C^2_{loc}(\mathbb{R}^n)$.

So let us informally calculate $\frac{\partial^2 u(x)}{\partial x_i \partial x_j} =$

$$= \left\{\begin{array}{l} \text{diff} \\ \text{under} \\ \text{the integral} \\ \text{Justified?} \end{array}\right\} = -\frac{n}{\omega_n}\int_{\mathbb{R}^n} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}} f(y)\,dy = \left\{\begin{array}{l}\text{put} \\ x=0\end{array}\right\} =$$

$$= -\frac{n}{\omega_n}\int_{\mathbb{R}^n} \underbrace{\frac{y_i y_j}{|y|^{n+2}}}_{\approx \frac{1}{|y|^n}} f(y)\,dy.$$

which is not integrable.

Can we ~~do~~ find an $f(x) \in C_c(\mathbb{R}^n)$ s.t.

$\int_{\mathbb{R}^n} \frac{y_i y_j}{|y|^{n+2}} f(y)\,dy$ is not well defined?

Example: $(\mathbb{R}^2)$: Let $f(y) = \frac{y_1 y_2}{|y|^2 \, |\ln |y||}$ (continuous)

In polar coordinates $f(r,\theta) = \frac{1}{2}\frac{\sin(2\theta)}{|\ln r|}$

and $\frac{y_1 y_2}{|y|^{n+2}} = \frac{1}{2}\frac{\sin(2\theta)}{r^2}$ so

$$\int_{B_{1/2}} \frac{y_1 y_2}{|y|^4} f(y)\,dy = \frac{1}{4}\int_0^{2\pi}\int_0^{1/2} \frac{\sin^2(2\theta)}{r^2 |\ln(r)|} r\,dr\,d\theta = c\int_0^{1/2} \frac{1}{r\,|\ln(r)|}\,dr$$

$$= \lim_{r\to 0}\Big[c \ln|\ln(r)|\Big]_r^{1/2} = \infty.$$

Notice that it would be enough for

$$\underbrace{|f(y)|}_{\text{No } x \text{ dependence}} \leq C|x-y|^\alpha \quad \text{in order to define}$$

$$\int_{\mathbb{R}^n} \frac{(y_i-x_i)(y_j-x_j)}{|x-y|^{n+2}} f(y)\,dy \quad \text{since then}$$

$$\left| - \ (\ ) - \right| \leq C \frac{|x-y|^2 \, |x-y|^\alpha}{|x-y|^{n+2}} = C \frac{1}{|x-y|^{n-\alpha}}$$

which is integrable.

Definition: We say that $f \in C_c^\alpha(\mathbb{R}^n)$ if f is continuous with compact support and

$$\sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq C \quad \text{for some}$$

constant C.

Then if $f \in C_c^\alpha(\mathbb{R}^n)$ we have, informally,

$$-\frac{n}{\omega_n} \int_{\mathbb{R}^n \setminus B_R} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}} f(y)\,dy \, \cancel{dy} = \frac{n}{\omega_n} \int_{\mathbb{R}^n \setminus B_R} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}} (f(y)-f(x))\,dy$$

$$-\frac{1}{\omega_n} \cancel{\tfrac{1}{\ldots}} \int_{\mathbb{R}^n \setminus B_R} \frac{\partial}{\partial y_j} \left(\frac{x_i-y_i}{|x-y|^{n}} \right) f(x) = \left\{ \begin{matrix} \text{int} \\ \text{by} \\ \text{parts} \end{matrix} \right\} = \int_{\mathbb{R}^n \setminus B_R} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}} (f(y)-f(x))\,dy$$

$$-\frac{f(x)}{\omega_n} \int_{\partial B_R} \frac{x_i-y_i}{|x-y|^n} \cdot \nu_j \, dA_{\partial B_r}(y)$$

Notice that the right hand side is well defined. ~~So in using the right hand side~~ We will use the right hand side as the expression for the second derivatives.

Theorem: Assume that $f \in C_c^\alpha(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^n} -\frac{1}{(n-2)\omega_n} \frac{1}{|x-y|^{n-2}} f(y)\,dy$$

is a $C^2_{loc}(\mathbb{R}^n)$ function s.t.

$$\Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n.$$

Proof: The proof uses the same idea several times so we will only do the last steps (which is the most difficult ones).

So assume that we have proved that $u \in C^1$ and

$$\frac{\partial u(x)}{\partial x_i} = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^n} f(y)\,dy$$

(The proof is similar to the proof of the existence of second derivatives and can be found in the lecture notes.)

Claim: $u \in C^2_{loc}(\mathbb{R}^n)$ and

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_R(x)} \frac{\partial^2 N(x-y)}{\partial x_i \partial x_j} \big(f(y) - f(x)\big)\, dy$$

$$+ f(x) \int_{\partial B_R(x)} \frac{\partial N(x-y)}{\partial x_i} \cdot \nu_j(y)\, dA_{\partial B_R}(y)$$

where $B_R(x)$ is so large that $\operatorname{spt}(f) \subset B_R(x)$ and $\nu_j(y)$ is the ~~outer~~ j:th component of the outer normal of $\partial B_R(x)$.

The difficulty in manipulating the integrals comes from the bad singularity at $x=y$. To get around that problem we define an approximation of $\frac{\partial u}{\partial x_i}$ that does not have a singularity and is thus easier to handle.

Define

$$V_\varepsilon(x) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^n}\, \eta_\varepsilon(|x-y|)\, f(y)\, dy$$

where
$$\eta_\varepsilon = \begin{cases} 1 & |x-y| \geq 2\varepsilon \\ 0 & |x-y| < \varepsilon \end{cases}$$ is increasing

with $|\nabla \eta_\varepsilon| \leq \frac{C}{\varepsilon}$ and $\eta_\varepsilon \in C^\infty$

Notice that the integrand has no singularity at $x=y$ since $\eta_\varepsilon(|x-y|)=0$ in a neighbourhood of $x=y$.

Next notice that

$$\left|\frac{\partial u(x)}{\partial x_i} - v_\varepsilon(x)\right| = \left|\frac{1}{\omega_n}\int_{\mathbb{R}^n} \frac{x_i-y_i}{|x-y|^n} f(y)\,dy - \frac{1}{\omega_n}\int_{\mathbb{R}^n} \frac{x_i-y_i}{|x-y|^n}\eta_\varepsilon(|x-y|)f(y)\,dy\right|$$

$$\leq \frac{1}{\omega_n}\int_{B_{2\varepsilon}} \left|\frac{x_i-y_i}{|x-y|^n}\right| |f(y)|\,dy \leq \frac{\sup|f|}{\omega_n}\int_{B_{2\varepsilon}(x)} \frac{1}{|x-y|^{n-1}}\,dy \leq$$

$$\leq C\sup|f|\,\varepsilon.$$

So $v_\varepsilon \to \frac{\partial u}{\partial x_i}$ uniformly as $\varepsilon \to 0$.

This means that whatever we prove for v_ε should imply something for $\frac{\partial u}{\partial x_i}$.

In particular, v_ε is continuously differentiable so if

$$\frac{\partial v_\varepsilon}{\partial x_j} \to w = \int_{B_R(x)} \frac{\partial N(x-y)}{\partial x_i\partial x_j}(f(y)-f(x))\,dy$$

$$+ f(x)\int_{\partial B_R(x)} \frac{\partial N(x-y)}{\partial x_i}\cdot \nu_j(y)\,dA_{\partial B_R}(y)$$

then $w \in C_{loc}(\mathbb{R}^n)$ and $\frac{\partial^2 u}{\partial x_i \partial x_j} = w$.

Notice that

$$\frac{\partial v_\varepsilon}{\partial x_j} = \int_{B_R^n} \frac{\partial}{\partial x_j}\left(\frac{\partial N(x-y)}{\partial x_i}\,\eta_\varepsilon(|x-y|)\right)\left(f(y)-f(x)\right)dy$$

$$+ \int_{B_R^n} \underbrace{\frac{\partial}{\partial x_j}}_{=-\frac{\partial}{\partial y_j}}\left(\frac{\partial N(x-y)}{\partial x_i}\,\eta_\varepsilon(|x-y|)\right) f(x)\,dy =$$

$$= \int_{B_R} \frac{\partial^2 N(x-y)}{\partial x_j \partial x_i}\,\eta_\varepsilon\,(f(y)-f(x))\,dy + \int_{B_R} \frac{\partial N(x-y)}{\partial x_i}\,\frac{\partial \eta_\varepsilon}{\partial x_j}\,(f(y)-f(x))\,dy$$

$$- f(x)\int_{\partial B_R} \frac{\partial N(x-y)}{\partial x_i}\,\underbrace{\eta_\varepsilon(|x-y|)}_{=1}\,\nu_j(y)\,dA(y)$$

So

$$\left|\frac{\partial v_\varepsilon}{\partial x_j} - w\right| = \Bigg| \int_{B_R} \frac{\partial^2 N(x-y)}{\partial x_j \partial x_i}\,\underbrace{(\eta_\varepsilon - 1)}_{\substack{=0 \text{ in } \mathbb{R}^n\setminus B_{2\varepsilon} \\ \leq 1 \text{ in } B_{2\varepsilon}}}\,(f(x)-f(y))\,dy$$

$$+ \int_{B_R} \frac{\partial N(x-y)}{\partial x_i}\,\underbrace{\frac{\partial \eta_\varepsilon}{\partial x_j}}_{\substack{=0 \text{ in } \mathbb{R}^n\setminus B_{2\varepsilon} \\ < C/\varepsilon \text{ in } B_{2\varepsilon}}}\,(f(y)-f(x))\,dy \Bigg| \leq$$

$$\leq \int_{B_{2\varepsilon}} \underbrace{\left|\frac{\partial^2 N(x-y)}{\partial x_j \partial x_i}\right|}_{< C\frac{1}{|x-y|^n}}\,\underbrace{|f(x)-f(y)|}_{< C|x-y|^\alpha} + \frac{C}{\varepsilon}\int_{B_{2\varepsilon}} \underbrace{\left|\frac{\partial N(x-y)}{\partial x_i}\right|}_{< \frac{C}{|x-y|^{n-1}}}\,\underbrace{|f(x)-f(y)|}_{\leq C|x-y|^\alpha}\,dy$$

$$\leq C\int_{B_{2\varepsilon}} \frac{1}{|x-y|^{n-\alpha}}\,dy + \frac{C}{\varepsilon}\int_{B_{2\varepsilon}} \frac{1}{|x-y|^{n-1-\alpha}}\,dy \leq C\varepsilon^\alpha$$

So $v_\varepsilon \to u$ uniformly

$\underbrace{\frac{\partial v_\varepsilon}{\partial x_j}}_{\text{continuous}} \to w$ uniformly so w is continuous.

It follows that $\frac{\partial u}{\partial x_j} = w$ which is continuous.

Claim $\Delta u(x) = f(x)$

Proof

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} \int_{B_R} \underbrace{\frac{\partial^2 N(x-y)}{\partial x_i^2}}_{=\Delta N = 0} \left(f(y) - f(x)\right) dy$$

$$\sum_{i=1}^{n} f(x) \int_{\partial B_R} \frac{\partial N(x-y)}{\partial x_i} \nu_i(y)\, dA(y) = f(x) \int_{\partial B_R} \nabla N(x-y) \cdot \nu =$$

$$= \left\{ \begin{array}{l} \nabla N = \frac{1}{\omega_n} \frac{x-y}{|x-y|^n} \\ \nu = \frac{y-x}{|x-y|} \end{array} \right\} = \frac{f(x)}{\omega_n} \int_{\partial B_R} \underbrace{\frac{1}{|x-y|^{n-1}}}_{=\frac{1}{R^{n-1}}} dA = f(x).$$