Theory of PDE Lecture 4

Last week we stated the following Theorem

> **Theorem:** Let $g \in C_c(\mathbb{R}^n_+)$ and define
>
> $$u(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi')}{|x-\xi'|^n} \, d\xi' \qquad \text{for } x_n > 0$$
>
> then $\Delta u(x) = 0$ in $\mathbb{R}^n_+$
>
> (*) $\lim_{x_n \to 0} u(x', x_n) = g(x')$ uniformly on compact sets.

~~End of~~ Start of the proof last lecture.

We "showed" that $\Delta u(x) = 0$ and we started to prove (*). To that end we fixed a compact set $K \subset\subset \mathbb{R}^{n-1}$ and let

$$K^1 = \overline{\bigcup_{x \in K} B_1(x)},$$

then K^1 is closed and bounded and thus compact so $g(x')$ is uniformly continuous on K^1. That is for each $\varepsilon > 0$ ~~$\xi \in K$~~ there exists a $\delta_\varepsilon > 0$ such that

$$|g(x') - g(\xi')| < \tfrac{\varepsilon}{2} \quad \text{for} \quad |x' - \xi'| < \delta_\varepsilon \text{ and } x', \xi' \in K^1.$$

We may assume that $\delta_\varepsilon < 1$ and conclude that $|g(x') - g(\xi')| < \frac{\varepsilon}{2}$ for all $x' \in K$ and $|x' - \xi'| < \delta_\varepsilon$.

What we need to show is that for each ~~$x' \in K$~~ $\varepsilon > 0$ there exists a $\tilde{\delta}_\varepsilon > 0$ s.t.

$$|u(x', x_n) - g(x')| < \varepsilon \quad \text{for all} \quad x' \in K \text{ and } |x_n| < \tilde{\delta}_\varepsilon .$$

since $\int K(x,\xi')\,d\xi' = 1$

$$= \left| \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi') - g(x')}{|x-\xi'|^n}\, d\xi' \cancel{- g(x')} \right| \leq$$

$$\leq \left| \int_{B_{\delta_\varepsilon}(x')} \frac{x_n}{\omega_n} \frac{\overbrace{|g(\xi') - g(x')|}^{< \varepsilon/2}}{|x-\xi'|^n}\, d\xi' \right| + \left| \int_{\mathbb{R}^n \setminus B_{\delta_\varepsilon}(x')} \frac{x_n}{\omega_n} \frac{g(\xi') - g(x')}{|x-\xi'|^n}\, d\xi' \right| \leq$$

$$\leq \frac{\varepsilon}{2} \underbrace{\left| \int_{B_{\delta_\varepsilon}(x')} \frac{x_n}{\omega_n} \frac{1}{|x-\xi'|^n}\, d\xi' \right|}_{\leq 1} + 2\sup_{\mathbb{R}^{n-1}}|g| \, \frac{x_n}{\omega_n} \int_{\mathbb{R}^n \setminus B_{\delta_\varepsilon}(x')} \frac{1}{|x-\xi'|^n}\, d\xi' \leq$$

$= \frac{1}{r^n}$ in polar

$$\leq \frac{\varepsilon}{2} + \left(\frac{2\sup_{\mathbb{R}^{n-1}}|g|}{\cancel{\delta}} \int_{\delta_\varepsilon}^{\infty} \frac{1}{r^2}\, dr \right) x_n = \frac{\varepsilon}{2} + \frac{2\sup_{\mathbb{R}^{n-1}}|g|}{\delta_\varepsilon}\, x_n < \varepsilon$$

$$\text{if} \quad x_n < \frac{\delta_\varepsilon \varepsilon}{4 \sup_{\mathbb{R}^{n-1}} |g|} \equiv \tilde{\delta}_\varepsilon .$$

$\blacksquare$

Remark: The proof works just as well for $g \in C(\mathbb{R}^{n-1})$ and g bounded.

We can in particular solve

$$\Delta u(x) = f(x) \quad \text{in} \quad \mathbb{R}^n_+$$

$$u(x',0) = g(x') \quad \text{on} \quad \partial\mathbb{R}^n_+$$

for any $f \in C^\alpha_c(\mathbb{R}^n)$ and $\underbrace{g \in C(\mathbb{R}^{n-1})}_{\text{bounded}}$.

In particular, let $v = \int_{\mathbb{R}^n} N(x-y) f(y) dy$

and

$$w(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi') - v(\xi',0)}{|x-\xi'|^n} d\xi'$$

then $u(x) = v(x) + w(x)$ solves

$$\Delta u(x) = \left\{\begin{matrix}\text{use} \\ \text{linearity}\end{matrix}\right\} = \underbrace{\Delta v(x)}_{= f(x)} + \underbrace{\Delta w(x)}_{=0} = f(x)$$

& $u(x,0) = \underbrace{g(x') - v(x',0)}_{w(x',0)} + \underbrace{v(x',0)}_{v(x',0)} = g(x')$.

Remark: We use linearity here. But our entire approach is based on linearity since the integral is linear, that is the methods we use would be quite useless to solve, say the porous medium equation $\operatorname{div}(u^m(x) \nabla u(x)) = f(x)$.

Since, if we could write

$$u = \int \text{kernel} \cdot f(\xi)\, d\xi \qquad \cancel{\text{then}}$$

~~if v solves div(v~~

and $$v = \int \text{kernel}\ g(\xi)\, d\xi$$

then the solution to

$$\operatorname{div}(w^m(x)\, \nabla w(x)) = f(x) + g(x)$$

would be $$w(x) = \int \text{kernel}\ (f(\xi) + g(\xi))\, d\xi = u(x) + v(x)$$

but $$\operatorname{div}((u+v)^m\, \nabla (u+v)) \neq \underbrace{\operatorname{div}(u^m \nabla u)}_{f} + \underbrace{\operatorname{div}(v^m \nabla v)}_{g}$$

in general since the operator is not linear, unless $m = 0$ and we are back at $\operatorname{div}(u^0 \nabla u) = \operatorname{div}(1 \nabla u) = \Delta u$.

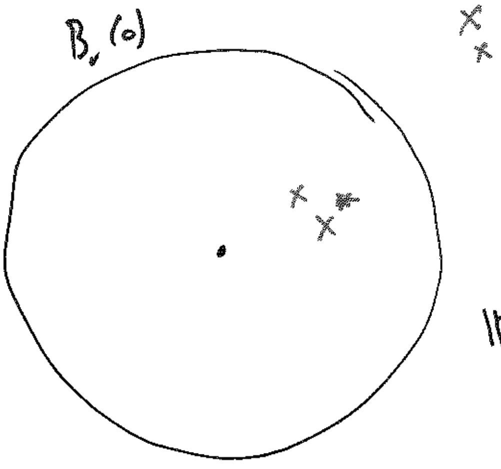
The Green's function in $B_r(0)$.

In $\mathbb{R}^n_+$ we could find a Green's function by reflecting $\mathbb{R}^n_-$ to $\mathbb{R}^n_+$ by $x \to \tilde{x} = (x', -x_n)$

We might try the same in $B_r(0)$.

In particular, to find a Green's function in $B_r(0)$ we need to solve, for any $x \in B_r(0)$,

$$\Delta \phi^x(\xi) = 0 \quad \text{in } B_r(0)$$
$$\phi^x(\xi) = N(\xi - x) \quad \text{on } \partial B_r(0).$$



How do we reflect?

We want to map

$$x \to x^* \quad \text{s.t.}$$

$$\mathbb{R}^n \setminus B_r(0) \longrightarrow \overline{B_r(0)}$$

and for $x \in \partial B_r(0)$ $\quad x = x^*$.

So $|x| > r \Rightarrow |x^*| < r$

$|x| = r \Rightarrow |x^*| = r$

The simplest way is to define

$$x^* = \frac{r^2 x}{|x|^2}.$$

Definition: For any $x \in \mathbb{R}^n \setminus \{0\}$ we say that

$$x^* = \frac{r^2 x}{|x|^2}$$

is the reflection of x in $\partial B_r(0)$.

And if u is defined in Ω we say that

$$u^*(x) = \frac{r^{n-2}}{|x|^{n-2}} u(x^*) \quad \text{for } x^* \in \Omega$$

is the Kelvin transform of u in $\partial B_r(0)$.

Notice that if $u(x)$ is defined at $x \in \partial B_r(0)$ then $x^* = x$ and $|x| = r$ so $u^*(x) = u(x)$. More amazingly

Lemma: If u is harmonic in Ω then u^* is harmonic in $\Omega^* = \{x ; x^* \in \Omega\}$.

Proof:

$$\Delta u^*(x) = \Delta \left(\frac{r^{n-2}}{|x|^{n-2}} u\left(\frac{r^2 x}{|x|^2}\right)\right) = \left\{\begin{matrix}\text{Tedious} \\ \text{calculations,} \\ \text{use } \Delta u = 0\end{matrix}\right\} = 0$$

Corollary: Let $x \in B_r(0)$ then

$$(\phi^x(\xi) =)\ N^*(\xi - x) = \begin{cases} -\frac{1}{(n-2)\omega_n} \dfrac{r^{n-2}}{(|x|\,|x^* - \xi|)^{n-2}} & x \neq 0 \\ -\frac{1}{(n-2)\omega_n} & x = 0 \end{cases}$$

solves

$$\Delta_\xi N^*(\xi - x) = 0 \quad \text{in } B_r(0)$$

$$N^*(\xi - x) = N(\xi - x) \quad \text{on } \partial B_r(0).$$

We may thus define a Green's function

$$G(x,\xi) = N(\xi - x) - N^*(\xi - x) \qquad \text{in } B_r(0)$$

and the poisson Kernel.

$$\frac{\partial G(x,\xi)}{\partial \nu(\xi)} = \xi \cdot \nabla_\xi G(x,\xi) = K(x,\xi) = \frac{r^2 - |x|^2}{\omega_n r} \frac{1}{|x-\xi|^n}$$

Analogously with the $\mathbb{R}^n_+$ case we get

Thm: Let $g \in C(\partial B_r)$ and define

$$u(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{\omega_n r} \frac{g(y)}{|x-y|^n} \, dy$$

then $\Delta u = 0$ in $B_r(0)$

$u = g$ on $\partial B_r(0)$

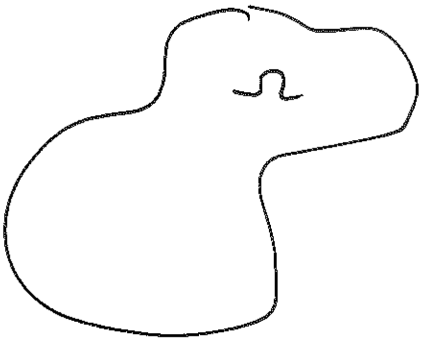
This is it, end of the ~~game~~.

The reflections $x \to \tilde{x}$ in $\partial \mathbb{R}^n_+$

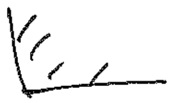
and $x \to x^*$ in $\partial B_r(0)$

is more or less as far as we get with this method.

Can you imagine creating a "reflection" of

 or 

or a general theory for ~~co~~ explicity writing down the solutions ϕ^x?

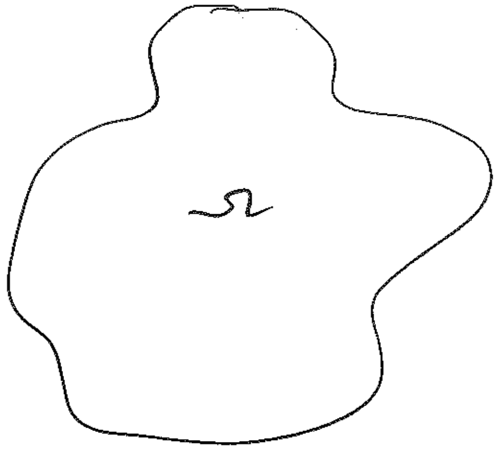
Maybe we can handle special cases like  B_1^+ or  $x_1, x_2 > 0$.

But how do we create a general theory applicable on all domains Ω?

This is a very challenging problem and the solution will take considerable time to develop (culminating in Perron's method and a homework assignment in maybe 3-4 weeks).

Let us stop for a second and look ahead.

Take a general domain

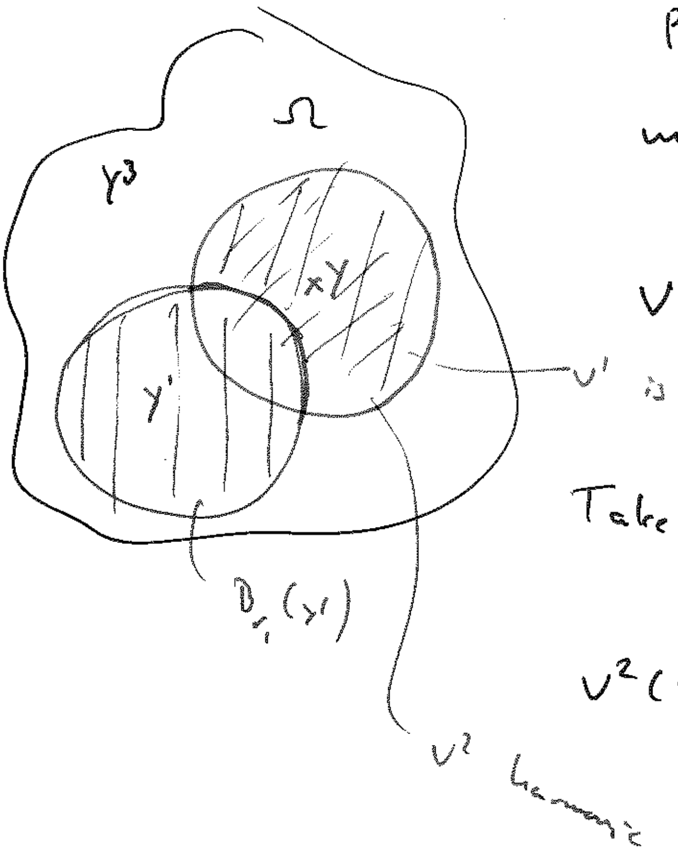


and assume that $g \in C(\bar{\Omega})$. We want to solve

$$\Delta u = \cancel{f}\,0 \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega.$$

The only thing we can do is to solve the Dirichlet problem in $B_r(x')$ (or $\mathbb{R}^n_+$)

So given $g(x)$, if you give me a point $y \in \Omega$ I can modify g and define



$$v^1(x) = \begin{cases} \text{harmonic} = \displaystyle\int_{\partial B_r(y)} \frac{r^2-|x|^2}{r\omega_n} \frac{g(\xi)}{|x-\xi|^n}\, d\xi & \text{in } B_r(y) \\ g & \text{in } \Omega \setminus B_r(y). \end{cases}$$

Take a new y, say y'

$$v^2(x) = \begin{cases} \displaystyle\int_{\partial B_{r_1}(y')} \frac{r^2-|x|^2}{r\omega_n} \frac{v^1(x)}{|x-\xi|^n}\, d\xi & \text{in } B_{r_1}(y') \\ v^1(x) & \text{else} \end{cases}$$

We can continue to define $v^3, v^4 \ldots$.
The problem is that we might never get a harmonic function. But what if $\lim_{k\to\infty} v^k(x) = u(x)$

then $u(x)$ would be a good guess ~~of a~~ for a solution. But choosing y^k and r_k are arbitrary. ~~way~~ We need to find a different way of doing things — a non-arbitrary.

Furthermore, we need to show that $v^k(x) \to u(x)$. How do we prove convergence?

The easiest way is by monotonicity?
If

$$v^1(x) \geq v^2(x) \geq \cdots \geq -C$$

then $v^k(x) \to u(x)$ since bounded decreasing sequences are convergent.

When is $v^k \geq v^{k+1}$, $\Leftrightarrow w := v^k(x) - v^{k+1}(x) \leq 0$.

$\Rightarrow$ Well if $\exists x^0 \in \Omega$ s.t

$$w^k(x) = v^k(x_0) - v^{k+1}(x_0) = \cancel{\sup_{x\in\Omega}(v^k - v^{k+1})} \quad \sup_{x\in\Omega} w^k(x) \geq 0$$

then $\nabla w^k(x) = 0$ and $\sum_{i=1}^n \frac{\partial^2 w^k(x^0)}{\partial x_i^2} \leq 0$

so if $\underbrace{\Delta w^k > 0}$ then $0 \geq w^k(x) = v^k(x) - v^{k+1}(x)$

$\Rightarrow \Delta v^k > \Delta v^{k+1}$!

Preliminary definition: If $v \in C^2(\Omega)$ and $\Delta v(x) \geq 0$ then we say that v is sub-harmonic.

The mean value property.

If $u \in C^2(\overline{B_r(0)})$ is given by the poisson integral. Then

$$u(0) = \int_{\partial B_r(0)} \frac{\cancel{r^2 - 0^2}}{\omega_n \cancel{r}} \underbrace{\frac{1}{|y|^n}}_{r^{n-1}} u(y)\, dA_{\partial B_r(0)}(y) = \fint_{\partial B_r(0)} u(y)\, dA_{\partial B_r}(y)$$

So $u(x^0) = \fint_{\partial B_r(x^0)} u(y)\, dA_{\partial B_r}(y)$ for any Ball $B_r \subset \Omega$.

Theorem: [The mean value theorem].

Utvärdering.

Teoriövningar är någon form av idioti som jag kom på inför den här terminen i hopp om att möjligheten att jobba med kursens svårare bevis och definitioner kommer att öka förståelsen av dessa. Det vore bra för mig att få veta om ni tycker att teoriövningen tjänar det syftet eller om teoriövningar är et totalt tidsslöseri. Ringa in ett svarsalternativ för varje fråga och lämna lappen till er övningsledare - era svar kommer att avgöra om vi har fler teoriövningar i framtiden.

Fråga 1) Tycker du att teoriövningen ökade din förståelse för kursens teoretiska moment?

1 (Inte alls) **2** **3** **4** **5** (Väldigt mycket.)

Fråga 2) Förstår du hur teoriövningen anknyter till teorin i boken.

1 (Inte alls) **2** **3** **4** **5** (Jag förstår hur uppgifterna är tänkta att belysa kursens teori.)

Fråga 3) Tycker du att teoriövningen var kul/intressant?

1 (Jämförbart med att titta på målarfärg som torkar) **2** **3** **4** **5** (Jätteintressant och kul.)

Fråga 4) Skulle du vilja ha fler teoriövningar.

1 (Nej, jag vill absolut inte ha fler teoriövningar, vanliga övningar är mycket bättre) **2**

3 (Det vore bra att ha ca 3-4 till under terminen) **4** **5** (Jag vill bara har teoriövningar i framtiden.)

Fråga 5) Vad tycker du om tempot på föreläsningarna?

1 (allt för långsamt) **2** **3** (ganska bra) **4** **5** (allt för snabbt).

Fråga 6) Hur skulle du vilja att fokus ligger på föreläsningarna.

1 (Bara räkna exempel) **2** **3** (50% exempel räkning 50% teori) **4** (som fokus ligger nu) **5** (Vill ha mer bevis, definitioner och teori.)

Kommentarer till kursen, övningar och eller teoriövningar: