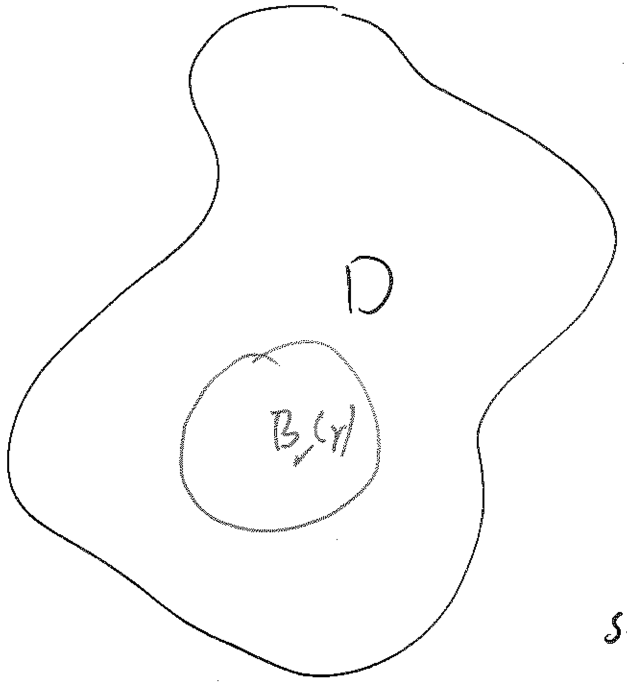
Lecture 5

Last week we set up a strategy for finding the solution to the general Dirichlet problem:

$$\Delta u(x) = 0 \quad \text{in } D$$
$$u(x) = g(x) \quad \text{on } \partial D.$$



By defining

$$u(x) = \overbrace{\sup_{v \in S_g} v(x)}^{\text{2) show that this operation is well behaved}}$$

where

$$S_g = \{ \text{subharmonic functions } v \text{ s.t. } v \le g \text{ on } \partial D \}$$

Sub-harmonic $\Rightarrow$ $\underbrace{\Delta v(x) \ge 0}$

1 $\left\{ \begin{array}{l} \text{Need to define} \\ \text{this without} \\ \text{reference to} \\ \text{2nd derivatives.} \end{array} \right.$

Since we plan to define

$$\tilde{v}(x) = \begin{cases} v(x) & x \notin B_r(y) \\ \displaystyle\int_{\partial B_r} \frac{r^2 - |x|^2}{r\omega_r} \frac{v(\xi)}{|x-\xi|^2} \, d\sigma_{\partial B_r}(\xi) & x \in B_r \end{cases}$$

Today we will focus on 1.

General strategy for extending a concept

$\Delta u \geq 0$ $\quad u \in C^2$ $\quad$??

If $A, B \Rightarrow C$, C is weaker then A

and $C, B \Rightarrow A$. We might use

C as a weaker definition for A.

So if we can find a property that characterizes harmonic functions without referring to second derivatives then we ~~are~~ ~~good~~ have a good chance to weaken the C^2 condition for harmonic functions and also for sub-harmonic functions.

Observation: If $u \in C^2(B_r(x^0)) \cap C(B_r(x^0))$

then

$$u(x_0) = \int_{\partial B_r(x^0)} K(x,y)\, u(y)\, dA_{\partial B_r(x^0)}(y) = \int_{\partial B_r} \frac{r}{\omega_n} \underbrace{\frac{1}{|y|^n}}_{= r^{n-1}} u(y)\, dA =$$

$$= \underbrace{\fint_{\partial B_r(x^0)}} u(y)\, dA(y) = \text{Average of } u \text{ on } \partial B_r(x^0).$$

$\fint_{\partial B_r}$ average $= \frac{1}{|\partial B_r|} \int_{\partial B_r}$

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Continuity

$$u(x^0) = \lim_{s \to 0} \fint_{\partial B_s(x^0)} u(y)\, dA$$

$$\frac{d}{ds} \fint_{\partial B_s(x^0)} u(y)\, dy = \frac{1}{\omega_n r^{n-1}} \int_{B_1} \Delta u(sy)\, dy$$

We will prove this in a different way that explicitly uses $\Delta u = 0$

> Theorem [The mean value property] Suppose that $u \in C^2(D) \cap C(\bar{D})$ ~~$u \in C^2(B_r(x^0)) \cap C$~~ and that $\Delta u = 0$ then for any ball $B_r(x^0) \subset D$
>
> $$u(x^0) = \fint_{\partial B_r(x^0)} u(y)\, dA_{\partial B_r(x^0)}(y) = \fint_{B_r(x^0)} u(y)\, dy.$$

Is it true (Yes it is, ofcourse we just proved it.) (clearly, since u is continuous

$$u(x^0) = \lim_{s \to 0} \fint_{\partial B_s(x^0)} u(y)\, dA = \lim_{s \to 0} \fint_{\partial B_1(0)} u(sy + x^0)\, dA_{\partial B_1}(y) =$$

$$\left\{ \text{Continuity} \right\} = \fint_{\partial B_1} u(x^0)\, dA_{\partial B_1} = u(x^0).$$

So it is enough to show that $\frac{d}{ds} \fint_{\partial B_1(0)} u(x^0 + sy)\, dA = 0$

We may clearly assume that $x^0 = 0$ (or consider $v(x) = u(x^0 + x)$) and try to prove

$$\frac{d}{ds} \fint_{\partial B_1(0)} u(sy)\, dA = \fint_{\partial B_1(0)} \overbrace{y}^{= \text{normal of } \partial B_1} \cdot \nabla u(sy)\, dA_{\partial B_1} = \fint_{\partial B_1} \frac{\partial u}{\partial \nu}(sy)\, dA_{\partial B_1} =$$

$$= \left\{ \begin{array}{c} \text{divergence} \\ \text{theorem} \end{array} \right\} = \frac{1}{s^{n-1}\omega_n} \int_{B_1} \operatorname{div}(\nabla u(sy))\, dy = \frac{1}{r^{n-1}\omega_{n-1}} \int_{B_1} \overbrace{\Delta u}^{=0}\, dy = 0$$

To prove

$$u(x^0) = \fint_{B_r(x)} u(y)\,dy = \left\{ \begin{array}{l} \text{polar} \\ \text{coordinates} \end{array} \right\} =$$

$$= \frac{1}{n\omega_n r^n} \int_0^r \underbrace{\int_{\partial B_s(x^0)} u(y)\, dA_{\partial B_s}(y)}_{= s^{n-1}\omega_n u(x^0)}\, ds = \frac{1}{n\omega_n r^n} \int_0^r s^{n-1}\omega_n\, ds\, u(x^0) = u(x^0).$$

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The maximum principle.

Theorem [The strong maximum principle]

Weak MP: Suppose that $\Delta u = 0$ in D and (D bdd) $u \in C^2(D) \cap C(\bar{D})$. Then

$$\sup_{x \in D} u(x) = \sup_{x \in \partial D} u(x).$$

Strong part: Furthermore if D is connected and there exsists a point $x^0 \in D$ s.t.

$$u(x^0) = \sup_D u$$

then u is constant.

Proof: The nice thing with the ~~maximum principle~~ mean value property is that we may control the solution in a ball $B_r(x^0)$ if we just know the value at one point x^0. Let us use this and ~~argue by contradiction~~ assume that x^0 is an interior maximum and see what happens.

Let
$$M = \underbrace{\sup_{x\in\bar{D}} u(x)}_{\substack{\text{exsists since} \\ u \text{ is continuous} \\ \bar{D} \text{ is compact.}}} = u(x^0) \quad \text{where } x^0 \in D.$$

Now let $B_r(x^0) \subset D$, then

$$M = u(x^0) = \fint_{\partial B_r} u(y)\,dy \overset{1}{\le} \left\{ u \le M \right\} \le \fint_{B_r(x^0)} M\,dy = M$$

with equality in 1 if and only if $u = M$ for all $x \in B_r(x^0)$. So if $u(x) = M$ then there exsists some ball $B_r(x)$ s.t. $u = M$ in $B_r(x)$.

Thus the set

$\{x;\ u(x) = M\}$ is relatively open in $\bar{D}$

but u is continuous so

$\{x;\ u(x) = M\}$ is ~~relatively~~ closed in $\bar{D}$.

So $\{x;\ u(x) = M\}$ is open and closed in $\bar{D} \Rightarrow u(x) = M$ in D.

Corollary: If u satisfies the MVP, $u \in C(D)$ then u satisfies the strong maximum principle in D.

Proof: We only used the MVP in the previous proof.

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Theorem: Assume that $u, v \in C^2(D) \cap C(\bar{D})$ and

$$\Delta u = \Delta v = f(x) \quad \text{in } D$$

$$u = v \quad \text{on } \partial D$$

then $u = v$ on D

Proof: Let $w^{\pm} = \pm(u-v)$ then

$$\Delta w^{\pm} = \pm(\Delta u - \Delta v) = \pm(f - f) = 0 \quad \text{in } D$$

$$\pm w = 0 \quad \text{on } \partial D$$

so by the strong maximum principle

$$\sup_D w^+ \leq \sup_{\partial D} w = 0$$

and

$$\sup_D w^- \leq \sup_{\partial D} w^- = 0$$

so $0 \leq u - v \leq 0$.

Corollary: If $u, v \in C(\bar{D})$ and

$u = v = g$ on ∂D

and ~~[illegible]~~, u, v satisfies

the MVP then $u = v$.

Proof: Same as before.

Observation. If $\Delta u = 0$ in D D Solid.

$u = g$ on ∂D

then u is the only continuous function satisfying the mean-value property in D.

Lemma: If $u \in C^2(D)$ Then the following are equivalent

1) $u(x^0) \le \fint_{\partial B_r(x^0)} u(y)\, dA_{\partial B_r}(y)$ for all $B_r(x^0) \subset D$

2) ~~then~~ $\Delta u(x) \geq 0$ in D.

Proof:

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Observe that we have

$$\left.\begin{array}{l} A)\ \Delta u \geq 0 \\ B)\ u \in C^2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} C)\ \text{Sub mean value property} \\ B)\ u \in C^2 \end{array}\right.$$

So we can hope that the sub mean value property is enough to characterize $\Delta u \geq 0$.

Definition: We say that $u \in C(D)$ is sub-harmonic in D if

$$u(x^0) \leq \fint_{\partial B_r(x^0)} u(y)\, dA_{\partial B_r}(y)$$

for all balls $B_r(x^0) \subset D$.

If $-u$ is sub-harmonic then we say that u is super-harmonic.

Thm: The strong maximum principle holds for sub-harmonic functions. That is if $u \in C(\bar{D})$ and u is sub-harmonic then

$$\sup_D u = \sup_{\partial D} u \quad \text{with equality}$$

only if u is constant.

Theorem: Let $u, v \in C(D)$ be sub-harmonic.

Then $w(x) = \sup(u(x), v(x))$

is sub-harmonic.

Proof: Since the supremum of two continuous functions are continuous it follows that $w \in C(D)$.

Moreover let $x^0 \in D$ and $B_r(x^0) \subset D$. Then either $w(x^0) = u(x^0)$ or $w(x^0) = v(x^0)$, say $w(x^0) = u(x^0)$ for definiteness.

Then

$$w(x^0) = u(x^0) \underset{\substack{\text{Since} \\ u \text{ is} \\ \text{sub-harmonic}}}{\leq} \fint_{\partial B_r(x^0)} u(y)\, dA_{\partial B_r} \leq \left\{ u \leq w \right\} \leq \fint_{\partial B_r(x^0)} w(y)\, dA_{\partial B_r}$$

So w satisfies the sub-mean value property and is thus sub-harmonic. ∎