Lecture 6

Last week we defined sub-harmonic as a function satisfying the sub-meanvalue property

$$(1) \qquad u(x) \leq ⨍_{\partial B_r(x^0)} u(y)\, dy \qquad \text{for any } B_r(x^0) \subset D.$$

Today we will investigate the convergence properties of harmonic functions and also verify that the definition (1) behaves well w.r.t. the harmonic replacement

$$\tilde{u}(x) = \begin{cases} u(x) & x \notin B_r(x^0) \\ \displaystyle\int_{\partial B_r(x^0)} \frac{r^2 - |x-x^0|^2}{\omega_n r} \, \frac{1}{|(x-x^0)-y|^n}\, u(y)\, dA(y) & x \in B_r(x^0) \end{cases}$$

that we need in our strategy to show that

$$u(x) = \sup_{v \in S_g} v(x)$$

$$S_g = \{ v \text{ sub-harmonic in } D,\ v \leq g \text{ on } \partial D \}.$$

We know that we can approximate any function $u(x)$ by a C^∞ function

$$u_\varepsilon(x) = \int_\Omega u(y)\,\phi_\varepsilon(x-y)\,dy \in C^\infty$$

where

$$\phi_\varepsilon = \begin{cases} \frac{c_0}{\varepsilon^n} e^{-\frac{1}{1-|\frac{x}{\varepsilon}|^2}} & x \in B_\varepsilon(0) \\ 0 & x \notin B_\varepsilon(0) \end{cases}$$

where c_0 is chosen such that $\int_{B_1(0)} \phi_\varepsilon = 1$

But

$$u_\varepsilon(x) = \int_{B_\varepsilon(x)} u(y) \underbrace{\phi_\varepsilon(x-y)}_{\substack{=0 \text{ outside} \\ B_\varepsilon(x)}} dy = \left\{ \begin{array}{c} \text{polar} \\ \text{coordinates} \end{array} \right\} =$$

$$= \int_0^\varepsilon \phi_\varepsilon(s)\, s^{n-1}\omega_n \underbrace{\left[⨍_{\partial B_s(x)} u(y)\,dA_y \right]}_{\substack{=u(x) \\ \text{by the MVP}}} ds = u(x) \underbrace{\int_0^\varepsilon \phi_\varepsilon(s)\, s^{n-1} \omega_n\, ds}_{=\int_{B_\varepsilon(x)} \phi_\varepsilon(x-y)dy = 1}$$

$= 1$. So $u(x) = u_\varepsilon(x) \in C^\infty$.

Theorem: If u satisfies the mean value property in D (say $\Delta u = 0$) then $u \in C^\infty(D)$.

So $u \in C^\infty$ - can it be better?

Yes, since we will be interested in ~~compact~~ the convergence of harmonic functions it is more important to have estimates of the solution, estimates that assures compactness.

We know from the Arzela-Ascoli theorem that equicontinuous functions ~~converge~~ sequences of functions have convergent subsequences.

Say if u^j is a ~~sequence~~, bdd sequence of functions on D s.t $\underbrace{|\nabla u^j| \leq C}_{\text{equicontinuity}}$ on D

then u^j has a ~~conver~~ pointwise convergent sub-sequence.

So what we need is some estimate on the derivative - not just a C^∞ regularity.

Theorem Suppose that $u \in C^2(D)$ and $\Delta u = 0$ (it is enough that u satisfies the MVP). Then for each ball $B_r(x^0) \subset \Omega$ we have the following estimate

$$\left|\frac{\partial u(x^0)}{\partial x_i}\right| \leq \frac{n(2^{n+1} n)^n}{\omega\, r^{n+1}} \int_{B_r(x^0)} |u(y)|\,dy$$

Remarks:

1) Why is it important?

If we have a sequence u^j s.t.

$$\underbrace{\int |u^j| \leq C}_{\text{say } |u^j| \leq C} \quad \text{then } u^j \text{ is equicontinuous}$$

2) Notice that if $|u| \leq C$ then

$$\int_{B_r(x^0)} |u|\,dy \leq r^n C \implies \left|\frac{\partial u(x^0)}{\partial x_i}\right| \leq \frac{C_0}{r}$$

so the estimate breaks down when x^0 is close to the boundary, since if $\operatorname{dist}(x^0, \partial D) = s$ then $r \leq s \implies \frac{C}{r} \geq \frac{C}{s}$

So s small gives bad estimates.

This is what we call an interior estimate - since we don't get good bounds at the boundary.

Proof: We know that $u \in C^\infty$ (all harmonic functions are)

so
$$\Delta \frac{\partial u}{\partial x_i} = \frac{\partial \Delta u}{\partial x_i} = \frac{\partial 0}{\partial x_i} = 0$$

so $u_i = \frac{\partial u}{\partial x_i}$ is harmonic.

Therefore

$$\left| u_i(x^0) \right| = \left| \fint_{B_{r/2}(x_0)} \frac{\partial u(x)}{\partial x_i}\, dx \right| = \left\{ \begin{array}{l} \text{integration} \\ \text{by parts} \end{array} \right\} =$$

(2)

$$= \left| \frac{n}{\omega_n \left(\frac{r}{2}\right)^{n}} \int_{\partial B_{r/2}} \underbrace{u(x)}_{\leq \sup_{\partial B_{r/2}} |u(x)|} \underbrace{e_i \cdot \nu}_{|\ | \leq 1} \, dA(x) \right| \leq \frac{2n}{r} \sup_{\partial B_{r/2}} |u|$$

But for any $y \in \partial B_{r/2}(x^0)$ we have $B_{r/2}(y) \subset B_r(x^0)$ and thus

$$|u(y)| = \left| \fint_{B_{r/2}(y)} u(x)\, dx \right| \leq \fint_{B_{r/2}(y)} |u(x)|\, dx$$

(3)

$$= \frac{n}{\omega_n \left(\frac{r}{2}\right)^n} \int_{B_{r/2}(y)} |u(x)|\, dx \leq \frac{n}{\omega_n \left(\frac{r}{2}\right)^n} \int_{B_r(x^0)} |u(x)|\, dx$$

put (2) and (3) together

$$\left| \frac{\partial u}{\partial x_i}(x^0) \right| \leq \frac{2n}{r} \sup_{y \in \partial B_r(x_0)} |u| \leq \frac{2^{n+1} n^2}{r^{n+1} \omega_n} \int_{B_r(x^0)} |u(y)|\, dy$$

Corollary: If $\Delta u = 0$ in D then

$$\left| \frac{\partial^{|\alpha|} u(x^0)}{\partial x^\alpha} \right| \leq \frac{n\,(2^{n+1}\, n k)^k}{\omega_n\, r^{n+k}} \iint_{B_r(x^0)} |u|$$

for any multiindex α with length k

and ball $B_r(x^0) \subset D$.

Proof: Induction on k. Just apply the theorem

on u_i which is harmonic and bdd....

∎

An application

Theorem [The Liouville theorem] Suppose that

$u \in C(\mathbb{R}^n)$ is harmonic and

$|u(x)| \leq C$ for some constant C (indep of x)

Then $u(x)$ is constant in $\mathbb{R}^n$.

Proof:

$$\left| \frac{\partial u(x^0)}{\partial x_i} \right| \leq \frac{2^{n+1}\, n^2}{r^{n+1}\, \omega_n} \int_{B_r(x^0)} \underbrace{|u(y)|}_{C}\, dy \leq \frac{2^{n+1}\, n^2}{r}\, C$$

for any $r > 0$. Let $r \to \infty$ and deduce

$$\left| \frac{\partial u}{\partial x_i}(x^0) \right| = 0 \quad \text{for any } x^0, \text{ and } i.$$

∎

The Arzela-Ascoli thm

Def 1: Let $\mathcal{F}$ be a set of functions defined on D. We say that $\mathcal{F}$ is equicontinuous on D if ~~the~~ for any $\varepsilon > 0$ and $x \in D$ exists a $\delta_{x,\varepsilon}$ s.t

$$|f(x) - f(y)| < \varepsilon \qquad \text{for all } |x-y| < \delta_{x,\varepsilon}$$
$$\text{and } f \in \mathcal{F}.$$

Arzela-Ascoli Theorem. Let $\{f_j\}$ be a uniformly bounded set of equicontinuous functions on D. Then there exists a sub-sequence

$f_{k_j} \longrightarrow f_0$ uniformly on compact subsets $K \subset D$

Theorem Let u^j be a uniformly bounded sequence of harmonic functions on D. Then $\exists$ sub-sequence $u^{k_j} \rightarrow u^0$ uniformly on compact sets of D.

Furthermore $\Delta u^0 = 0$ on D.

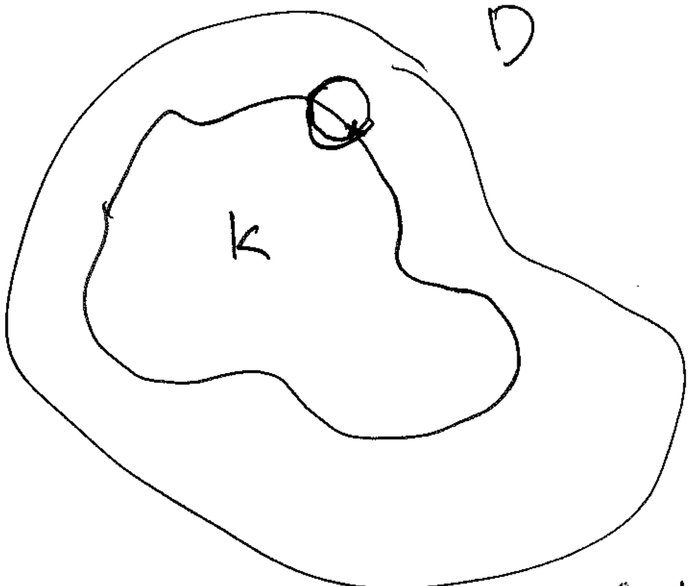
Proof: We want to use the Arzela-Ascoli Thm and to that end we need to prove equicontinuity.

It is enough to show that $|\nabla u^j| \leq C$ uniformly on compact sets. Right?

Let us fix a compact set $K \subset \bigcup_{x \in K} B_{r_x}(x)$ where $r_x = \operatorname{dist}(x, \partial D) \underbrace{> 0}_{\substack{\text{since} \\ D \text{ is open.}}}$ Since K is compact

There is a finite subcover $K \subset \bigcup_{j=1}^{N} B_{r_{x^j}}(x^j)$

in particular for any $x \in K$ $\quad r_x > \frac{1}{2} \inf_j r_{x^j} = r > 0.$



so $B_r(x) \in D$ for any $x \in K$.

Now

$$\left|\frac{\partial u^j(x)}{\partial x_i}\right| \leq \frac{C}{r} \sup |u^j| \leq C_K$$

So u^j is equicontinuous and we may find a uniformly convergent sub-sequence by the Arzela-Ascoli theorem.

To show that $u^{k_j} \to u_0$ satisfying $\Delta u_0 = 0$ we use the MVP. For k_j large enough

$$2\varepsilon > |u^0(x) - u^{k_j}(x)| = \left| u^0(x) - \fint_{B_r(x)} u^{k_j}(y)\,dy \right| =$$

$$= \left| u^0(x) - \fint_{B_r} u^0(y)\,dy + \fint_{B_r} u^0(y) - u^{k_j}(y)\,dy \right| \cancel{\geq}$$

$$\cancel{\geq} \left| u^0 - \fint_{B_r} u^0(y)\,dy \right| - \left| \fint \underbrace{|u^0(y) - u^{k_j}(y)|}_{\leq \varepsilon} dy \right| \cancel{\geq}$$

$$\geq \left| u^0(x) - \fint_{B_r(x)} u^0(y)\,dy \right| \cancel{-\varepsilon}$$

for any $r \leq \dfrac{\mathrm{dist}(x, \partial D)}{2}$.

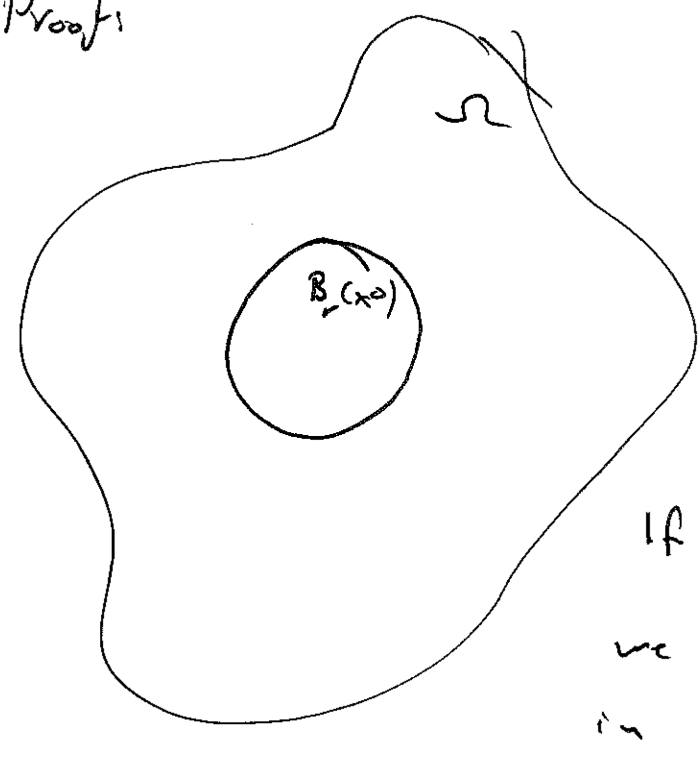
$\blacksquare$

Lemma: Suppose that v is sub-harmonic in D. Moreover assume that $B_r(x^0) \subset D$. If we define the harmonic replacement of v in $B_r(x^0)$ by

$$\tilde{v}(x) = \begin{cases} v(x) & x \in D \setminus B_r(x^0) \\ \displaystyle\int_{\partial B_r(x^0)} \frac{r^2 - |x-x^0|^2}{\omega_n r} \frac{1}{|(x-x^0)-y|^n} \, dA_{\partial B_r(x^0)}(y) & x \in B_r(x^0) \end{cases}$$

then $\tilde{v}$ is sub-harmonic in D.

Proof:



We need to show that

$$\tilde{v}(y) \leq \fint_{\partial B_\varsigma(y)} \tilde{v}(z) \, dz$$

for any $B_\varsigma(y) \subset D$.

If $B_\varsigma(y) \subset B_r(x^0)$ then we are done since $\Delta \tilde{v}(y) = 0$ in $B_r(x^0)$. Similarly, if $B_\varsigma(y) \subset \Omega \setminus B_r(x^0)$ then we are done since $\tilde{v} = v$ which is subharmonic.

Therefore we only need to consider

$B_s(y) \cap \partial B_r(x^0) \neq \phi$.

Step 1. Let $\tilde{h}$ solve

$$\begin{aligned} \Delta \tilde{h} &= 0 && \text{in } B_s(y) \\ \tilde{h} &= \tilde{v} && \text{on } \partial B_s(y) \end{aligned}$$

then $\tilde{h} \geq \tilde{v}$ in $B_s(y) \setminus B_r(x^0)$

~~Proof~~: Proof step 1: Since v is subharmonic

$v \leq h$ ~~for~~ where

$$\begin{aligned} \Delta h &= 0 && \text{in } B_s(y) \\ h &= v && \text{on } \partial B_s(y). \end{aligned}$$

Also, by the comparison principle

$v - \tilde{v} \leq 0$ in $B_r(x^0)$ so

$\underbrace{\tilde{h} \geq h}$ on $\partial B_s(y)$

equal on $\partial B_s(y) \setminus B_r(x^0)$
$\leq$ on $\partial B_s(y) \cap B_r(x^0)$

so by the maximum principle

$v \leq h \leq \tilde{h}$ in $B_s(y) \setminus B_r(x^0)$

but $\tilde{v} = v$ — (——

Step 2 $\quad \tilde{v} \le \tilde{h} \quad$ in $\quad B_s(y) \cap B_r(x^0) = \Omega$

$$\Delta \tilde{v} = \Delta \tilde{h} = 0 \quad \text{in} \quad \text{— '' —}$$

so
$$\tilde{v} - \tilde{h} \le \sup_{\partial\Omega} (\tilde{v} - \tilde{h}) = \max\Big(\underbrace{\sup_{\partial B_s \cap B_r} \tilde{v} - \tilde{h}}_{=0},\ \sup_{B_s \cap \partial B_r} \tilde{v} - \tilde{h} \Big)$$

$$= \left\{ \begin{array}{l} \text{since} \\ \tilde{v} = v \text{ on } \partial B_r \\ \text{and } \tilde{v} \le \tilde{h} \end{array} \right\} = \max\Big(0, \overbrace{\sup_{B_s \cap \partial B_r} (\tilde{v} - \tilde{h})}^{\le 0} \Big) = 0.$$

in Ω. It follows that $\tilde{v} \le \tilde{h}$.

Step 3. $\tilde{v}$ satisfies the sub-meanvalue property.

$$\tilde{v}(x^0) \le \tilde{h}(x^0) = \fint_{\partial B_r(x^0)} \overbrace{\tilde{h}(y)}^{= \tilde{v}(x^0)} dy = \fint_{\partial B_r(x^0)} \tilde{v}(x^0)\, dy.$$

Now everything is set up for a proof by the Perron method