Lecture 7

Theorem [Perron's Method]. Suppose that Ω is a bounded domain and $g \in C(\partial\Omega)$.

Define

$$u(x) = \sup_{v \in S_g(\Omega)} v(x)$$

where

$$S_g(\Omega) = \{ v \in C(\bar{\Omega}) \,;\, v \text{ is sub-harmonic in } \Omega, \; v \leq g \text{ on } \partial\Omega \}.$$

Then $\Delta u(x) = 0$ in Ω.

STEP 1 u is well defined.

Proof: Since g is continuous on $\partial\Omega$ it is bounded

so $v \in S_g(\Omega) \Rightarrow v \leq \sup_{\partial\Omega} g \leq C$ on $\partial\Omega$

so $v(x) \leq C$ in Ω by the maximum principle.

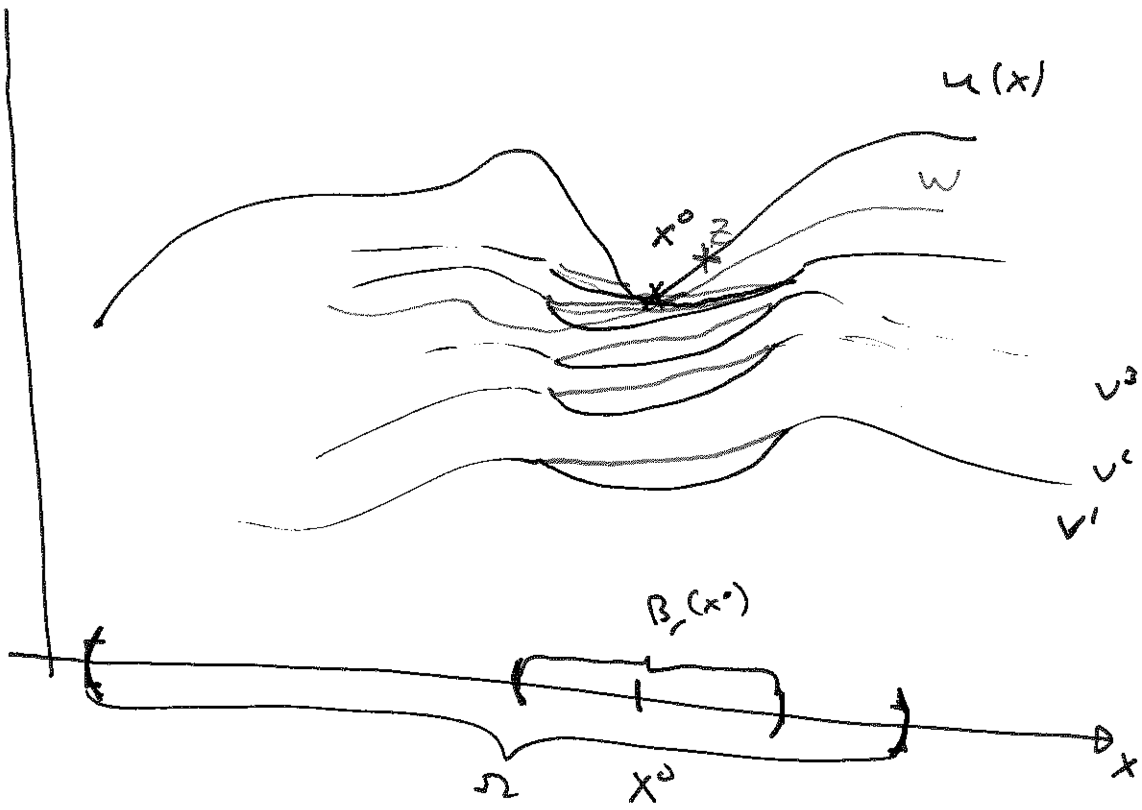
Furthermore $\inf_{\partial\Omega} g \geq -C$ (maybe different C)

and $-C$ is sub-harmonic (every constant is)

so $-C \in S_g(\Omega) \neq \phi$.

For each $x \in \Omega$ $u(x)$ is therefore the supremum of a non-empty set, bounded from above and is therefore well defined.

The ~~idea of~~ the rest of the proof



We want to show that for each $x^0 \in \Omega$ $\Delta u(x^0) = 0$. Fix an $x^0 \in \Omega$, and let $B_r(x^0) \subset \Omega$ (exsists since domains are open).

By definition there exsists $v^j \in S_g(\Omega)$ s.t $v^j(x^0) \to u(x^0)$.

Define $\tilde{v}^j = \begin{cases} v^j & x \notin B_r(x^0) \\ \text{harmonic} & \text{else.} \end{cases}$

Now $\tilde{v}^j \geq v^j$ and $\tilde{v}^j \in S_g(\Omega)$ so

$\lim_{j\to\infty} \tilde{v}^j(x^0) = u(x^0)$.

Moreover, by compactness of harmonic functions,

$\tilde{v}^j \to v^0$ where $\Delta v^0 = 0$ in $B_{r/2}(x^0)$

(at least for a sub-sequence which we may assume to be the entire sequence.)

$$\boxed{\begin{array}{l}\text{Claim: } v^0 = u \text{ in } B_{r/2}(x^0), \text{ in particular} \\ \Delta u(x) = 0 \text{ in } B_{r/2}(x^0).\end{array}}$$

If not then there exsists a $z \in B_{r/2}(x^0)$ s.t. $u(z) > v^0(z)$. And thus a $w(x) \in S_g(\Omega)$ s.t. $u(z) \geq w(z) > v^0(z)$. This since $u(z) = \sup_{w \in S_g(\Omega)} w(z)$

Let $w^j(x) = \max(\tilde{v}^j(x), w(x))$ then $w^j \in S_g(\Omega)$ since the maximum of two sub-harmonic functions is sub-harmonic.

Moreover

$$\tilde{w}^j = \begin{cases} w^j & x \notin B_r(x^0) \\ \text{harmonic} & x \in B_r(x^0) \end{cases}$$

will also be in $S_g(\Omega)$.

Since $\cancel{\tilde{w}^j} \leq w^j \leq \tilde{w}^j \leq u$

It follows that $\tilde{w}^j(x^0) \to u(x^0) = \cancel{\ } v^0(x^0)$

and $\lim \tilde{w}^j(\cancel{x}) \geq \cancel{V}^0(\cancel{x})$.

Furthermore by compactness $\tilde{w}^j \to w^0$ in $B_{r/2}(x^0)$

where
$$\left.\begin{aligned} &\Delta w^0(x) = 0 \\ &w^0(x) \geq v^0(x) \\ &w^0(x^0) = v^0(x^0) \\ \text{and}\quad &w^0(z) > \cancel{w}^0(z) \end{aligned}\right\} \text{ in } B_{r/2}.$$

That is $h = w^0 - v^0$ is a non-negative harmonic function in $B_{r/2}(x^0)$ s.t.

$h(x^0) = 0 \Rightarrow h(x) = 0$ by the strong max principle

$\underline{h(z) > 0}$ Contradiction.

We may thus conclude that
$$u(z) = v^0(z) \quad \text{for all } z \in B_{r/2}(x^0)$$

It follows that $\Delta u(x^0) = 0$ for any $x^0 \in \Omega$.

∎

Remarks

1) We only claim that $\Delta u = 0$. in Ω
But we do not claim, nor prove, that
$u = g$ on $\partial\Omega$. (Next hour)

2) However, if there exist a solution

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases}$$

then $w \in S_g(\Omega)$ and $w \geq v$

for any $v \in S_g(\Omega)$ by the maximum principle.
Thus $u = \sup_{v \in S_g(\Omega)} v = w$

so we find the ~~#~~ unique harmonic function.

3) One would be tempted to say that we prove that

$$u \leq g \quad \text{on } \partial\Omega.$$

This is indeed true, but we need to be very carefull since u might not be continuous up to $\partial\Omega$ so for $x^0 \in \partial\Omega$ we may have

$$\lim_{\substack{x \to x^0 \\ x \in \Omega}} u(x) > u(x^0), \qquad \text{so } u(x^0) \leq g(x^0)$$

doesn't mean that ~~the~~ u takes the bdry values.

Attaining the boundary data.

Question: Let $u = \sup_{v \in S_g(\Omega)} v(x)$ and $x^0 \in \partial\Omega$

will $\lim_{\substack{x \to x^0 \\ x \in \Omega}} u(x) = g(x^0)$? (1)

The easiest way to assure (1) is to construct a sub-harmonic function v s.t

$$\lim_{x \to x^0} v(x) = g(x^0) \quad , \quad v(x) \leq g(x) \quad \text{on } \partial\Omega$$

and a superharmonic function w s.t

$$\lim_{\substack{x \to x^0 \\ x \in \Omega}} w(x) = g(x^0) , \quad w(x) \geq g(x) \quad \text{on } \partial\Omega.$$

Then

$$\underbrace{v(x) \leq u(x)}_{\text{by construction}} \underbrace{\leq w(x)}_{\text{by comparison}}$$

So $\lim_{x \to x^0} u(x) = g(x^0)$ by the sandwich theorem.

We will be a little smarter and define one function that will work for all g.

Definition. Let Ω be a domain and $\xi \in \partial\Omega$. We say that w is a barrier (~~[illegible]~~ at ξ relative to Ω) if

1) $w \in C(\bar{\Omega})$
2) $w > 0$ in $\bar{\Omega} \setminus \{\xi\}$ and $w(\xi) = 0$
3) w is super-harmonic in Ω.

If Ω is a domain and there exsists a barrier at $\xi \in \partial\Omega$ then we say that Ω is regular at ξ.

Theorem: Let Ω be a *bounded* domain, $g \in C(\partial\Omega)$,

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

and finally we assume that $\xi \in \partial\Omega$ is a regular point of Ω. Then

$$\lim_{\substack{x \to \xi \\ x \in \Omega}} u(x) = g(\xi).$$

After we prove this theorem we will give rather general examples of domains that have barriers at every point. And also of domains with points that doesn't have a barrier.

Proof: We need to find, for each $\varepsilon > 0$, a $\delta_\varepsilon > 0$ s.t.

$$\sup_{x \in B_{\delta_\varepsilon}(\xi)} |u(x) - g(\xi)| < \varepsilon . \qquad (1)$$

Note that if $g(x) = 0$ in a neighbourhood of ξ then $-kw(x) \le u(x) \le kw(x)$ by the comparison principle so (1) follows since $w(x)$ is continuous at ξ.

When g is non-zero we have to work a little harder.

Since g is continuous $\exists\, \delta_{g,\varepsilon/2} > 0$ s.t.

$$\sup_{x \in B_{\delta_{g,\varepsilon/2}}(\xi)} |g(x) - g(\xi)| < \varepsilon/2 .$$

Let w be a barrier at ξ and let

$$\varkappa = \inf_{x \in \partial\Omega \setminus B_{\delta_{g,\varepsilon/2}}(\xi)} w(x) > 0$$

since $w > 0$ on the compact set $\partial\Omega \setminus B_{\delta_{g,\varepsilon/2}}(\xi)$.

If we define

$$k = \frac{2}{\varkappa} \sup_{x \in \partial\Omega} |g(x) - g(\xi)|$$

Then

$$-\frac{\varepsilon}{2} - k\,w(x) \le g(x) - g(\xi) \le \frac{\varepsilon}{2} + k\,w(x) \quad \text{on } \partial\Omega \qquad (2)$$

(In particular $g(x) \le g(\xi) + \frac{\varepsilon}{2} + k\,w(x)$, $g(x) \ge g(\xi) - \frac{\varepsilon}{2} - k\,w(x)$)

since $|g(x) - g(\xi)| < \frac{\varepsilon}{2}$ on $B_{\delta_{\varepsilon/2},g}(\xi) \cap \partial\Omega$

and $k\,w(x) \ge k\kappa = \frac{\sup |g(x) - g(\xi)|}{\cancel{\kappa}}\cancel{\kappa}$ on $\partial\Omega \setminus B_{\delta_{\varepsilon/2},g}(\xi)$.

Now since $w(\xi) = 0$ and $w \in C(\Omega)$ there ~~exist~~.

a δ_w s.t.

$$\sup_{x \in B_{\delta_w}(\xi)} |w(x)| \le \frac{\varepsilon}{2k}.$$

Since w is superharmonic it follows that

$g(\xi) + \frac{\varepsilon}{2} + w(x)$ ~~$w(x)$~~ $\ge v(x)$ for ~~all~~ $v \in S_g(\Omega)$

So ~~$w(x) \ge u(x)$~~. $g(\xi) + \frac{\varepsilon}{2} + w(x) \ge u(x)$. (3)

Also since $g(\xi) - \frac{\varepsilon}{2} - k\,w(x)$ is sub-harmonic

and $\le g(x)$ on $\partial\Omega$ it follows that

$$g(\xi) - \frac{\varepsilon}{2} - k\,w(x) \le u(x). \qquad (4)$$

③ & ④ implies that

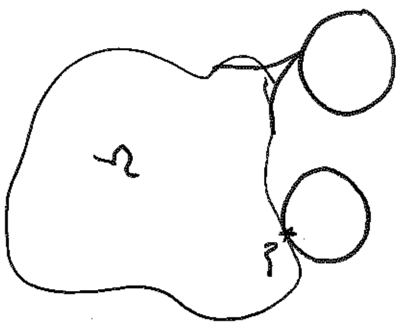
$$|u(x) - g(\xi)| \leq \frac{\varepsilon}{2} + k\, w(x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

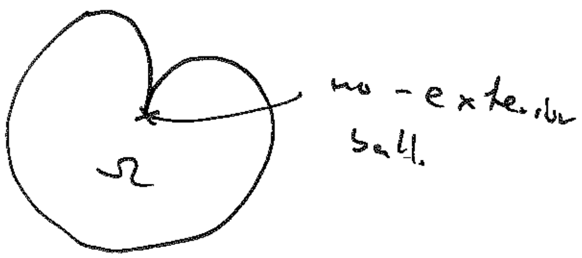
for any $x \in B_{\delta_w}(\xi)$.

$\boxed{30}$

~~Proposition:~~

Definition: We say that a domain Ω satisfies the exterior ball condition at $\xi \in \partial\Omega$ if there exists a ball $B_r(y)$ s.t. $B_r(y) \cap \Omega \neq \emptyset$ and $\xi \in \partial B_r(y) \cap \partial\Omega$.

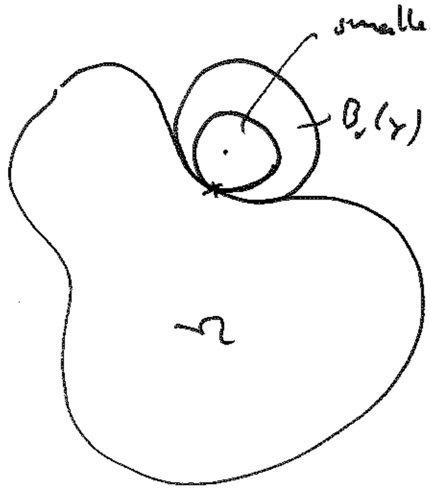




Proposition: Assume that Ω satisfies the exterior ball condition at $\xi \in \partial\Omega$.
Then ξ is a regular point w.r.t. Ω

Proof: We need to show that there exists a barrier w at ξ w.r.t. Ω.

There is no loss of generality to assume that $\partial B_r(y) \cap \partial\Omega = \{\xi\}$.



Now

$$w(x) = \begin{cases} \ln(|x-y|) - \ln(r) & n=2 \\ \dfrac{1}{r^{n-2}} - \dfrac{1}{|x-y|^{n-2}} & n \geq 3. \end{cases}$$

Then clearly $w(x)=0$ on $\partial B_r(y)$ since $|x-y|=r$ on $\partial B_r(y)$. Also $w(x)>0$ in $\mathbb{R}^n \setminus \overline{B_r(y)}$ and $\Delta w(x) = 0$ in $\mathbb{R}^n \setminus B_r(y)$ and w is therefore super-harmonic. (Even harmonic).

∎

Thm: If $\overbrace{\Omega}^{\text{a bounded domain}}$ satisfies the exterior ball condition at ~~all~~ $\xi \in \partial\Omega$,
And if $g \in C(\partial\Omega)$ and $f \in C^{\alpha}(\Omega)$.
Then there exsists a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to

$$\Delta u(x) = f(x) \quad \text{in } \Omega$$
$$u(x) = g(x) \quad \text{on } \partial\Omega.$$

Proof: Exercise. (After 7 weeks of hard work I have decided to leave the theorem that we have been trying to prove as an exercise!)

Example: Let $\Omega = B_1(0) \setminus \{0\}$ in $\mathbb{R}^2$ (or $\mathbb{R}^n$, $n \geq 3$)
Then $0 \in \partial\Omega$ is not a regular point.

Proof. Let $g = 0$ on ∂B_1
$g(0) = -1$.

Then $v_k(x) = \max\left(\frac{\ln(|x|)}{k}, -1\right) \in S_g(\Omega)$

so $u(x) = \sup_{v \in S_g(\Omega)} v(x) \geq \lim_{k\to\infty} v_k(x) = 0$ for $x \in \Omega$

But $u(x) \leq 0$ by the maximum principle.
So $u = 0$ in Ω. So there isn't any meaningfull way that $u(x) = g(x)$ at ~~x~~ $= 0$.