# Lecture 8

Last week we proved existence for

$$\Delta u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega.$$

This week we will move on and start to consider general linear elliptic equations.

This is a type of PDE that generalizes the Laplacian:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

Now consider

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u(x)$$

where $a_{ij}$, $b_i$ and $c$ are given functions.

What assumptions should we make on the coefficients? Applications? Or theory?

Under what assumptions can we assure that a solution to $Lu(x) = f(x)$ is unique?

That is: When do we have a maximum principle?

Example: Lets assume that $Lu = \varepsilon$ in $\Omega$ and that $u(x^0)$ is an ~~an~~ interior max point. Assume furthermore that

i) $u(x^0) \geq 0$

ii) $c(x^0) \leq 0$

iii) All eigenvalues of $A(x^0)$ are $\geq 0$.

then $\varepsilon \leq 0$.

Solution: We may assume, by a rotation of the coordinate system that

$$a_{ij}(x^0) = \begin{cases} \lambda_i & i=j \\ 0 & i \neq j \end{cases}$$

Since $x^0$ is an interior maximum it follows that $\nabla u(x^0) = 0$ and

$$\frac{\partial^2 u(x^0)}{\partial x_i^2} \leq 0. \quad \text{Thus}$$

$$\varepsilon = \sum_i \underbrace{\underbrace{\lambda_i}_{\geq 0} \underbrace{\frac{\partial^2 u(x^0)}{\partial x_i^2}}_{\leq 0}}_{\leq 0} + \underbrace{\sum_i b_i(x^0) \frac{\partial u(x^0)}{\partial x_i}}_{=0} + \underbrace{\underbrace{c(x^0)}_{\leq 0} \underbrace{u(x^0)}_{\geq 0}}_{\leq 0} \leq 0.$$

**Definition:** We say that a ~~PDE~~ partial differential operator

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x) u(x)$$

is elliptic in $\Omega$ if there exists a constant $\lambda > 0$ s.t

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \qquad \text{for all } x \in \Omega \qquad (*)$$

and all vectors $\xi \in \mathbb{R}^n$ s.t $|\xi| \neq 0$.

We will only consider elliptic operators in the next couple of weeks.

Remark: Observe that if $\xi$ is an eigenvector to $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{21} & & & \\ & & & \\ & & & a_{nn} \end{bmatrix}$ ~~then~~ with eigenvalue $\mu$ then

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j = \xi^T A \xi = \xi^T \mu \xi = \mu |\xi|^2$$

so $(*)$ implies that all eigenvalues $\geq \mu$.

Example. $\Delta$ is elliptic. For $\Delta$ we have

$$a_{ij}(x) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else.} \end{cases}$$

So $\sum a_{ij} \xi_i \xi_j = \sum |\xi_i|^2 = |\xi|^2 \geq \lambda |\xi|^2$ for $\lambda = 1 > 0$.

Since we have defined elliptic to give us a maximum principle (for $c(x) \leq 0$) the following lemma should not come as a surprize.

**Lemma** [The weak maximum principle]. Suppose that $u \in C^2(\Omega)$ $\Omega$ added, $Lu(x) = f(x) \in C(\Omega)$.

Assume furthermore that $c(x) \leq 0$ and $f(x) > 0$ for all $x \in \Omega$.

enough with one strict.

Then $u(x)$ does not have any positive local maxima in $\Omega$.

In particular, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ then

$$\sup_{\Omega} u(x) \leq \sup_{\partial\Omega} u(x).$$

Proof: Notice that

$$\sum_{ij} a_{ij}(x^0) \frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} + \sum_i b_i(x^0) \frac{\partial u(x^0)}{\partial x_i} = \underbrace{f(x^0) - c(x^0)u(x^0)}_{>0} > 0$$

And since $x^0$ is a maxima $\nabla u(x^0) = 0$

$D^2 u(x^0)$ is non-positive so

$$0 < \sum_{ij} a_{ij}(x^0) \frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} = -\mathrm{trace}\left(A(x^0) \cdot (D^2 u(x^0))\right) \qquad (*)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & & a_{nn} \end{bmatrix}$

**Corollary** Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Omega$ bdd,

~~f~~ $c(x) \leq 0$ and

$$L u(x) \geq L v(x) \quad \text{in } \Omega$$

$$u(x) \leq v(x) \quad \text{on } \partial\Omega$$

then $u(x) \leq v(x)$ in $\Omega$.

Proof: Let $N$ be very large and $w$ as before. Then

$$h_\varepsilon(x) = u(x) - v(x) - \varepsilon w(x)$$

solves

$$L h_\varepsilon = \underbrace{Lu - L(v)}_{\geq 0} \underbrace{- \varepsilon L w}_{> 0} > 0$$

and $h_\varepsilon = u - v - \varepsilon w \leq -\varepsilon w$ on $\partial\Omega$

It follows that $h_\varepsilon$ doesn't have any interior maxima, so

$$h_\varepsilon \leq \sup_{\partial\Omega} -\varepsilon w \quad \text{in } \Omega.$$

Send $\varepsilon \to 0$ and we can deduce that $u - v = h_0 \leq 0$ in $\Omega$. $\blacksquare$

<u>Corollary 2</u>. Let $\Omega$ be bounded and

$u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$Lu = Lv \quad \text{in} \quad \Omega$$
$$u = v \quad \text{on} \quad \partial\Omega.$$

Then $u = v$.

Proof apply the previous corollary $\Rightarrow$ $u \le v$ in $\Omega$ and then again with the roles reversed $\Rightarrow$ $v \le u$.

$\boxtimes$

Now we know from linear algebra that trace $(A\cdot B) \geq 0$ if $A, B$ are non-negative. So $(*)$ implies that

$$0 < 0.$$

$\blacksquare$

For the next corollary we need the following barrier like function.

Let

$$w = e^{N r^2} - e^{N|x|^2}$$

Then $w \geq 0$ in $B_r(0)$

$$Dw(x) = -2N x\, e^{N|x|^2}$$

$$D^2 w(x) = -2N I e^{N|x|^2} - 4N^2 \begin{bmatrix} x_1^2 & x_1 x_2 & \cdots \\ & & \\ & & \end{bmatrix} e^{N|x|^2}$$

So the eigenvalues of $D^2 w(x)$ are $-2N e^{N|x|^2}$ with multiplicity $n-1$

$(-2N - 4N^2|x|^2)$ — " — $1$

We may conclude that if $c(x) \leq 0$

$$Lw \leq \left(-2N\left(1 + 2N|x|^2\right)\lambda + \sup|b|\, 2N|x|\right) e^{N|x|^2} < 0 \text{ if } N >> 1$$

# Exsistence theory - a plan.

We know that solutions are unique - if they exsist. But how do we prove exsistence.

Observe that the only reason we could solve $\Delta u = 0$ in $B_R(0)$ was because $\Delta$ is rotationally symmetric and we could thus explicitly write a ~~fu~~ fundamental solution by solving an ODE. In general, that is not possible.

Since we are working in analysis it is natural to begin by considering PDE $L$ that are very close to $\Delta$.

We will therefore start to consider

$$L_t u = \Delta u + \sum_{i,j=1}^{n} t \underbrace{(a_{ij}(x) - \delta_{ij})}_{\hat{a}_{ij}} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + t\, b_i(x) \frac{\partial u}{\partial x_i} + t c(x) u(x) = f(x)$$

that is

$$L_t u_t = \Delta u_t + t \overbrace{\left( \sum \hat{a}_{ij}(x) \frac{\partial^2 u_t(x)}{\partial x_i \partial x_j} + b_i \frac{\partial u_t}{\partial x_i} + c(x) u_t(x) \right)}^{\hat{L} u_t} = f(x)$$

when $t$ is small we have $L_t \approx \Delta$ (in some sense)

when $t = 1$ we have a general elliptic PDE

Now, if we can expand

$$u_t(x) = \sum_{k=0}^{\infty} t^k u^k(x).$$

Then putting $t=0$ we see that

$\Delta u_0(x) = f(x)$ in $\Omega$ which we can solve.
$u_0(x) = g(x)$ on $\partial\Omega$

For $t$ small we have

$$L_t u_t(x) = \Delta\left(\sum_{k=0}^{\infty} t^k u^k\right) + t\hat{L}\left(\sum_{k=0}^{\infty} t^k u^k\right) \approx \begin{Bmatrix}\text{disregard}\\ t^2 \text{ terms}\end{Bmatrix} =$$

$$\approx \underbrace{\Delta u^0}_{=f} + \underbrace{t\Delta u^1 + t\hat{L}u^0}_{=0}$$

So $L_t u_t \approx f$ if

$$\left.\begin{aligned} &u^1 = 0 \quad \text{on } \partial\Omega \\ &\underbrace{\Delta u^1 = -\hat{L}u^0}_{} \quad \text{in } \Omega. \quad \text{in } \Omega \end{aligned}\right\} \quad (**)$$

Can we solve this? Almost!

If $-\hat{L}u^0(x) \in C^\alpha(\Omega)$ then we can solve $(**)$

But if $f(x) \in C^\alpha(\Omega)$ then

$u^0 \in C^2(\Omega)$ so $\displaystyle\sum_{i,j=1}^{n} (a_{ij}(x) - \delta_{ij}) \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\Omega)$

and not better! We need $C^\alpha$!

Questions:

1) Does $\left.\begin{array}{l} f \in C^\alpha(\Omega) \\ \text{and } \Delta u = f \end{array}\right\} \Rightarrow D^2 u \in C^\alpha(\Omega)$ ?

Need to prove this!

Remark: Even if $D^2 u \in C^\alpha$ we can not conclude that

$$\sum_{i,j=1}^{n} \left(a_{ij}(x) - \delta_{ij}\right) \frac{\partial^2 u}{\partial x_i \partial x_j} \in C^\alpha$$

unless we also assume that $a_{ij} \in C^\alpha$.

Newton - Raphson

Observe that if we can solve $\Delta u^1 = -\hat{L} u^0$ then we can, in principle, solve

$L_t u_t = f(x)$ for $t$ small. Then

$$f(x) = L_t(u^0 + t u^1 + t^2 u^2) = \underbrace{\Delta u^0}_{=f} + \underbrace{t\hat{L} u^0 + t \Delta u^1}_{=0} + t^2 \Delta u^2 + t^2 \hat{L} u^1 + O(t^3)$$

So $\Delta u^2 = -\hat{L} u^1 \dots$

$\vdots$

$\Delta u^k = -\hat{L} u^{k-1}$

And we expect that $u_t = \sum t^k u^k$ should solve $L_t u_t(x) = f(x)$

But for what $t$ does $\sum_{k=0}^{\infty} t^k u^k$ converge? And in what sense?

Question: Assume that

$$\Delta u = f(x) \in C^\alpha \text{ in } \Omega \quad \Big(\text{say } \Delta u^k = \underbrace{-\hat{L} u^{k-1}}_{=f}\Big)$$

Can we say that

$$[D^2 u]_{C^\alpha(\Omega)} \leq C\,[f]_{C^\alpha} \quad ? \qquad (\ast\ast\ast)$$

Remember

$$[D^2 u]_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega} \frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha}.$$

If this where the case then

$$[D^2 u^0]_{C^\alpha} \leq C\,[f]_{C^\alpha}$$

$$\Rightarrow \quad [D^2 u^1]_{C^\alpha} \leq C\,[D^2 u^0]_{C^\alpha} \leq C^2 [f]_{C^\alpha}$$

$$\vdots$$

$$[D^2 u^k]_{C^\alpha} \leq \ldots \leq C^{k-1} [f]_{C^\alpha}$$

and $\left[\sum_{k=1}^{\infty} t^k u^k\right]_{C^\alpha(\Omega)} < \infty$ if $t < \frac{1}{C}$.

Conclusion. We need apriori estimates $[D^2 u]_{C^\alpha} \leq C\,[f]_{C^\alpha}$ or such!

Plan:

1) Derive some basic estimates of the form (***).

2) Show a very ~~simple~~ basic [but still complicated] existence theorem using the approach above.

3) Identify several gaps in our theory that we need to prove a more general existence theory.

   i) More general apriori estimates to handle larger $t$.

   ii) More vocabulary to express our thoughts (Banach spaces)

   iii) We need better understanding of the behavior of solutions near the boundary.

   This leads to horrendous technicalities.

4) Fill the gaps

We would like to show that if

$$\Delta u(x) = f(x) \in C^{\alpha}(B_{2R}) \quad \text{then}$$

$$[D^2 u]_{C^{\alpha}(B_R)} \leq C [f(x)]_{C^{\alpha}(B_{2R})} \quad \text{or such.}$$

That is we need to estimate

$$\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right| = \left| \int_{B_{2R}(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \big( f(\xi) - f(x) \big) d\xi \right.$$

$$- f(x) \int_{\partial B_{2R}} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j \, dA_{\partial B_{2R}} - \int_{B_{2R}(y)} \frac{\partial^2 N(y-\xi)}{\partial x_i \partial x_j} \big( f(\xi) - f(y) \big) d\xi$$

$$\left. + f(y) \int_{\partial B_{2R}} \frac{\partial N(y-\xi)}{\partial x_i} \nu_j \, dA_{\partial B_{2R}} \right|$$

Need to estimate this!

It will be convenient to have the following lemma

**Lemma:** Let $x, y \in \mathbb{R}^n$, $|x-y| = r$ and

$N_{ij}(x) := \dfrac{\partial^2 N(x)}{\partial x_i \partial x_j}$ be the derivatives of

the Newtonian kernel. Then

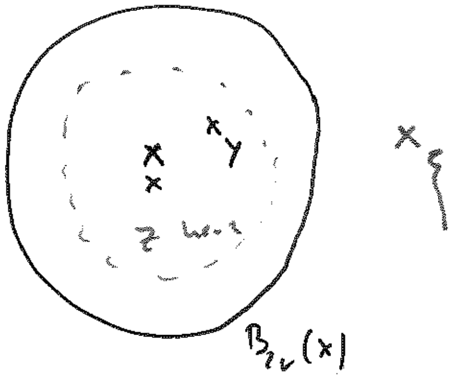
$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| \le \frac{C\,|x-y|}{|x-\xi|^{n+1}}$$

for any $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$.

**Proof:** Fix $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$ then

$N(z-\xi) \in C^\infty(\mathbb{R}^n \setminus \{z=\xi\}) \implies N(z-\xi) \in C^\infty(B_{3/2}(x))$.

Thus




$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| =$$

$$= \left| \int_0^1 (x-y)\, \nabla N_{ij}(sx+(1-s)y-\xi)\, ds \right| \le$$

$$\le \cancel{KA}\, |x-y| \sup_{z \in B_r(x)} |\nabla N_{ij}(\underbrace{z-\xi}_{|\,\cdot\,| \ge |x-\xi|-r \ge r})|$$

Now $|\nabla N_{ij}| \le |D^3 N| \approx \left| D^3 \dfrac{C}{|x|^{n-2}} \right| \approx \dfrac{C}{|x|^{n+1}}$

so $|\nabla N_{ij}(z-\xi)| \le \dfrac{C}{(\frac{1}{2}|x-\xi| \cancel{\ge r})^{n+1}} \le \dfrac{2^{n+1} C}{|x-\xi|^{n+1}}.$