Lecture 8

Two weeks ago we saw that

$$\left.\begin{array}{l} f \in C^\alpha(\Omega) \\ \Delta u = f \quad \text{in } \Omega \end{array}\right\} \Longrightarrow \left\{ u \in C^2(\Omega) \right.$$

Is not enough to show existence, we would need something like

$$\left.\begin{array}{l} f \in C^\alpha(\Omega) \\ \Delta u = f \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} u \in C^{2,\alpha}(\Omega) \quad \text{and} \\ \sup_\Omega |D^2 u| + \sup_{x,y \in \Omega} \dfrac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \leqslant C_{f,\alpha} \end{array}\right.$$

Is this reasonable?

1) What we have $f \in C^\alpha \Longrightarrow u \in C^2$ doesn't really take the value of $\alpha\,(q_\alpha)$ into consideration. $f \in C^{10^{-10}}$ or $f \in C^{0,9999}$

gives the same regularity for u (that is $u \in C^2$).

So we ~~give up some~~ loose some information in the previous estimates.

Since we have derived an explicit estimate for $D^2 u$ in terms of f we should be able to estimate $|D^2 u(x) - D^2 u(y)|$

Theorem: Let $f \in C_c^\alpha(B_{2R}(0))$, $\alpha \in (0,1)$, and define

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi .$$

Then there exist a constant $C(n,\alpha)$ s.t.

$$\frac{\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right|}{|x-y|^\alpha} \le C_{\alpha,n} \left([f]_{C^\alpha(B_{2R})} + \frac{|x-y|^{1-\alpha} \sup_{B_R} |f(x)|}{R^2} \right)$$

for all $x, y \in B_R(0)$.

Proof: We have the representation

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_{2R}(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} (f(\xi) - f(x)) d\xi - f(x) \int_{\partial B_{2R}(0)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j \, dA(\xi).$$

Therefore $\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right| =$

constants

$$\Bigg| \int_{B_{2R}} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} (f(\xi) - f(x)) d\xi - f(x) \int_{\partial B_{2R}} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j \, dA(\xi)$$

Almost same

$$- \int_{B_{2R}} \frac{\partial^2 N(y-\xi)}{\partial x_i \partial x_j} (f(\xi) - f(y)) + f(y) \int_{\partial B_{2R}} \frac{\partial N(y-\xi)}{\partial x_i} \nu_j \, dA(\xi) \Bigg|$$

Almost same

In order to estimate the above we will use several of our old as well as some new ideas

1) We need to split the integral

$$\int_{B_{2R}(0)} = \int_{B_{2R}\setminus B_{2r}(x)} + \int_{B_{2r}(x)}$$ in order

to handle the singularity at x. easy since $|f(\xi)-f(x)| \le C|\xi-x|^{\alpha}$

and $r=|x-y|$.

2) ~~We also see that the boundary terms and~~

We also see that $\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} - \frac{\partial^2 N(y-\xi)}{\partial x_i \partial x_j}$

appears, and we need to estimate this term in $\int_{B_{2R}\setminus B_{2r}}$, $\int_{\partial B_{2r}}$.

3) $f(x)$ and $f(y)$ are constants so we may move them out from the integrals.

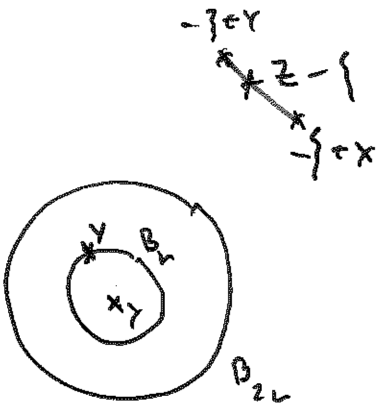
We start to address 2).

Lemma: Let $x, y \in \mathbb{R}^n$, $|x-y| = r$ and
$N_{ij}(x) = \dfrac{\partial^2 N(x)}{\partial x_i \partial x_j}$. Then

$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| \leq \frac{C\,|x-y|}{|x-\xi|^{n+1}}$$

for any $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$.

Proof:



$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| \leq \left\{\begin{matrix}\text{Mean}\\ \text{value}\\ \text{theorem}\end{matrix}\right\} \leq$$

$$\leq |x-y| \cdot |\nabla N_{ij}(z-\xi)|$$

But $N = -\dfrac{1}{(n-2)\omega_n} \dfrac{1}{|x|^{n-2}}$

so $D^3 N \approx C \dfrac{1}{|x|^{n+1}}$

so $|\nabla N_{ij}(z-\xi)| \leq \dfrac{C}{|z-\xi|^{n+1}} \leq \dfrac{C}{|x-\xi|^{n+1}}$

since $|x-\xi| \approx |z-\xi|$

so

$$|N_{ij}(x-\xi) - N_{ij}(y-\xi)| \leq C \frac{|x-y|}{|x-\xi|^{n+1}}.$$

∎

Now we may estimate $\left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j}\right| \leq$

$$\leq \left|\int_{B_{2r}(x)} \underbrace{\left|\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\right|}_{C\frac{1}{|x-\xi|^n}} \underbrace{|f(\xi)-f(x)|}_{\leq [f]_{C^\alpha}|\xi-x|^\alpha} d\xi\right| d\xi \quad + \left|\int_{B_{2r}(x)} \frac{\partial^2 N(y-\xi)}{\partial x_i \partial x_j}(f(\xi)-f(x))\right|$$

$$+ \left|\int_{B_{2R}\setminus B_{2r}(x)} \underbrace{\overbrace{\left|\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} - \frac{\partial^2 N(y-\xi)}{\partial x_i \partial x_j}\right|}^{C\frac{|x-y|}{|x-\xi|^{n+1}}} \overbrace{|f(\xi)-f(x)|}^{[f]_{C^\alpha}|\xi-x|^\alpha} d\xi}_{C[f]_{C^\alpha}|x-y|\frac{1}{|x-\xi|^{n+1-\alpha}}}\right| +$$

$$+ \left|\int_{B_{2R}\setminus B_{2r}(x)} N_{ij}(y-\xi)(f(y)-f(x)) - f(x)\int_{\partial B_{2R}} N_i(x-\xi)\nu_j \, dA\right.$$

$$\left. + f(y)\int_{\partial B_{2R}} N_i(y-\xi)\nu_j \, dA\right| \leq$$

$$\leq C_\alpha [f]_{C^\alpha} r^\alpha + C[f]_{C^\alpha} \overset{r^\alpha}{\cancel{|x-y|}}\left(\frac{1}{\cancel{r^{1-\alpha}}} - \cancel{\frac{1}{R^{1-\alpha}}}\right)$$

Put in C

$$+ \left|\int_{B_{2R}} \text{———} \quad \int_{\partial B_{2R}} dA\right|$$

We may estimate the last absolute value by an integration by parts

$$\left| \sim \right| = \Big| - \int_{\partial B_{2R}} N_i(y-\xi)\,(\cancel{f(y)} - f(x))\, v_j \, dA$$

$$- \int_{\partial B_{2r}} \underbrace{N_i(y-\xi)}_{\approx \frac{c}{r^{n-1}}} \underbrace{(f(y) - f(x))}_{|x-y|^\alpha [f]_{C^\alpha}} v_j \, dA \qquad \text{Cancel}$$

$$\pm f(x) \int_{\partial B_{2R}} N_i(x-\xi)\, v_j \, dA + \cancel{f(y) \int_{\partial B_{2R}} N_i(y-\xi)\, v_j \, dA} \Big| \lesssim$$

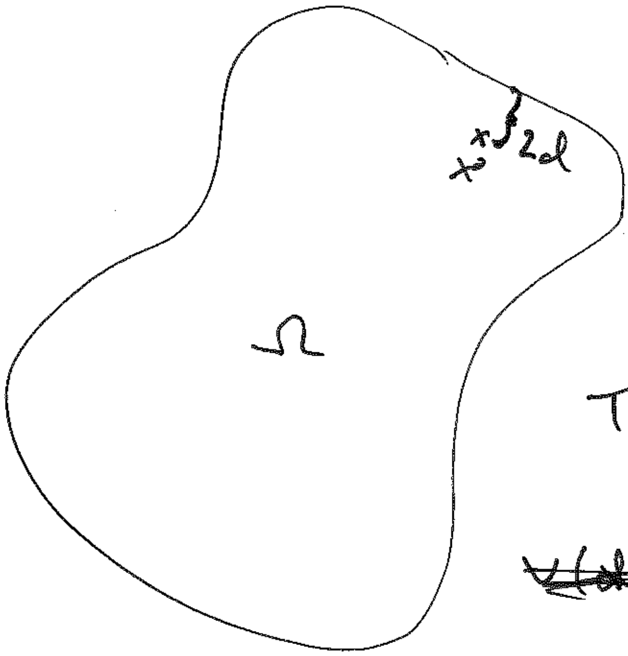
$$\lesssim C |x-y|^\alpha [f]_{C^\alpha} + \left| f(x) \int_{\partial B_{2R}} \underbrace{\big(N_i(y-\xi) - N_i(x-\xi)\big)}_{C\frac{|x-y|}{R^n}} v_j \, dA \right| \lesssim$$

$$\lesssim C \, \overset{r^\alpha}{\cancel{|x-y|^\alpha}} \, [f]_{C^\alpha} + C \sup|f(x)| \, \frac{\overset{r}{\cancel{|x-y|}}}{R} \, ,$$

Put all estimates together gives the Thm.

One obvious shortcoming in the Theorem is that the estimates only work in $B_R(0)$. What do we get ~~at~~ close to the boundary $\partial B_{2R}(0)$?

Let Ω be a general domain.



If we want to estimate $D^2u(x^0)$ for an x^0 close to the boundary.

Then we may consider

~~$v(dx+x^0)$~~ $v(x) = u(dx+x^0)$

Then

$$\Delta v(x) = d^2 \Delta u(dx+x^0) = d^2 f(dx+x^0) \quad \text{in } B_2(0)$$

and then apply the Theorem to v with $R=1$.

We may thus deduce that

$v(x) = w + h$

where $\displaystyle W = \int_{B_2} \mathcal{N}(x-\xi)\, d^2 f(d\xi)\, d\xi$

$\Delta h = 0.$

Since $\Delta h = 0$ we know that

$$\sup_{B_1} |D^3 h| \leq C \sup_{B_2} |h| \Rightarrow \left| \frac{\partial^2 h(x)}{\partial x_i \partial x_j} - \frac{\partial^2 h(y)}{\partial x_i \partial x_j} \right| \leq C |x-y|^{\alpha} \sup h$$

And by the theorem

$$\frac{\left| \frac{\partial^2 w(x)}{\partial x_i \partial x_j} - \frac{\partial^2 w(y)}{\partial x_i \partial x_j} \right|}{|x-y|^{\alpha}} \leq C_{\alpha,n} \left(\left[d^2 f(d\cdot) \right]_{C^{\alpha}(B_2)} + d^2 \sup |f| \right)$$

So
$$\underbrace{\left| \frac{\partial^2 v(x)}{\partial x_i \partial x_j} - \frac{\partial^2 v(y)}{\partial x_i \partial x_j} \right|}_{d^2 \left| \frac{\partial^2 u(dx+x^0)}{\partial x_i \partial x_j} - \frac{\partial^2 u(dy+x^0)}{\partial x_i \partial x_j} \right|} \leq \left| \frac{\partial^2 w(x)}{\partial x_i \partial x_j} - \frac{\partial^2 w(y)}{\partial x_i \partial x_j} \right| + \left| \frac{\partial^2 h}{\partial x_i \partial x_j} - \partial^2 h \right| \leq$$

$$\leq \left(C \left([d f(d\cdot)]_{C^{\alpha}} + d^2 \sup |f| + d^2 \sup |f| + \sup |u| \right) \right) |x-y|^{\alpha}$$

We can thus conclude

Proposition: Let Ω be a domain

$$\Delta u(x) = f(x) \quad \text{in } \Omega$$

$$|u| \leq M \quad \text{in } \Omega$$

$$\sup_{x \in K} |f(x)| \leq \frac{C_{0,f}}{\operatorname{dist}(K, \partial\Omega)^2} \quad \text{for all compact } K \subset \Omega$$

$$|f(x) - f(y)| \leq \frac{C_{\alpha,f} |x-y|^\alpha}{\operatorname{dist}(K, \partial\Omega)^{2+\alpha}} \quad \text{— " —}$$

Then

$$\sup_{x,y \in K} \frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|}{\operatorname{dist}(K, \partial\Omega)^{2+\alpha}}.$$
