Instructions for the Oral Exam,
Selected topics in PDE, Fall 2014

The scope of the exam: The course includes all the course notes. Noting is excluded from the notes.

However, the notes are very technical so I figured that it might be good to get some guidance on the oral exam. This document contains 20 questions that you should know how to answer and you will be asked *at least* two questions from this document during the exam.

Format of the exam: The oral exam will be scheduled for 20-25 minutes. You will first randomly pick two questions from these instructions, one random question from part one and one random question from part two. You will have 5-10 minutes to answer the question in front of a blackboard. Most of the questions involve qualifying phrases like *"briefly proof"* or *"sketch a proof of"* this means that you are not expected to know all the details of the proofs but you should be able to show that you have understood the proof.

In case you pick a question that you can not answer you are allowed to take an alternative question. However, this will affect your grade.

After the two mandatory questions I might ask you to pick a third question or ask you other questions if I feel that I need more information to decide on your grade.

Grades: Masters students will get **F (fail), E , D ,C ,B** or **A**.

PhD students will get **pass** or **fail**. Roughly the requirement for a PhD student to pass will be at the B level on the masters students grade scale.

Questions on Part 1 - the Laplace equation.

Question 1: Briefly present the proof that if

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi, \qquad \text{for } f \in C_c^\alpha(\mathbb{R}^n)$$

then $\Delta u(x) = f(x)$. In particular stress how the assumption that $f \in C_c^\alpha(\mathbb{R}^n)$ (as opposed to $f \in C_c(\mathbb{R}^n)$) is used in the proof.

Question 2: What is a Green's function for a domain Ω and what is the idea behind the concept of Green's functions.

Question 3: Give an outline of the proof that if

$$u(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n g(\xi')}{\omega_n |x-\xi'|^n} d\xi',$$

then $\Delta u(x) = 0$ in $\mathbb{R}^n_+$ and $u(x', 0) = g(x')$ on $\mathbb{R}^{n-1}$. Comment on the corresponding theorem in a ball $B_r(0)$.

Question 4: Sketch the proof of the mean value property for harmonic functions and how the mean value property is used to prove the strong maximum principle.

Question 5: Sketch the proof of the fact that is $u(x) \in C(\Omega)$ is harmonic in Ω then $u(x) \in C^\infty(K)$ for any compact subset $K \subset\subset \Omega$.

Question 6: Sketch a proof of the Harnack inequality for harmonic functions.

Question 7: Sketch a proof of the fact that if $\{u^k\}_{k=1}^\infty$ is a uniformly bounded sequence of harmonic functions in a domain Ω. Then there exists a harmonic function u^0 and a subsequence u^{k_j} such that $u^{k_j} \to u^0$ uniformly on compact sets. Your answer should mention, very briefly, how to prove the Arzela-Ascoli theorem.

Question 8: Give the definition of a sub-harmonic function and prove that if $u(x)$ and $v(x)$ are sub-harmonic in Ω then $w(x) = \max(u(x), v(x))$ is sub-harmonic.

Question 9: Give a brief proof of the existence of solutions, using the Perron method, for the Dirichlet problem $\Delta u(x) = 0$ in Ω and $u(x) = g(x) \in C(\partial\Omega)$ on $\partial\Omega$. You do not have to show that $u(x)$ defined by the Perron method attains the boundary data.

Question 10: Explain the concept of a barrier and briefly explain how the existence of a barrier at $\xi \in \partial\Omega$ implies that a harmonic function defined via the Perron method attains it boundary data continuously at ξ.

Question 11: Give an example of a domain Ω such that the Perron solution does not necessarily obtain boundary data given by a continuous function at every point of $\partial\Omega$.

Questions on Part 2 - variable coefficient equations.

Question 12: Sketch a proof of the comparison principle for a second order variable coefficient PDE. Also explain how the comparison principle implies uniqueness of the solution.

Question 13: In the proof of Theorem 14.1 we split the integral representing

$$\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right|$$

into several parts $\int_{B_{2r}} \cdots$, $\int_{B_{2R}\setminus B_{2r}} \cdots$ etc. Explain why, or the idea, behind splitting the integral in this way.

Question 14: Explain the proof of Proposition 14.1. In particular, focus on why we define $v(x)$ the way we do and why we write $v = w + h$ and why define w and h the way we do.

Question 15: Explain the idea behind the *"Freezing of the Coefficients"* argument and how that is used to prove that

$$\|u\|_{C^{2,\alpha}_{\mathrm{int}}(\Omega)} \leq C \left(\|f\|_{C^{\alpha}_{\mathrm{int},(2)}(\Omega)} + \|u\|_{C(\Omega)} \right)$$

for solutions to variable coefficient PDE.

Question 16: What is the idea behind the proof of the interpolation inequalities.

Question 17: Give a brief sketch of how boundary estimates are proved - from the straight boundary for the laplacian to the general case for variable coefficients in $B_2(0) \cap \{x;\ x_n \geq h(x')\}$ for some $h(x') \in C^{2,\alpha}$.

Question 18: Assuming that interior and boundary regularity have been proved. Prove global estimates for second order elliptic equations with variable coefficients.

Question 19: Assuming that we have proved global apriori $C^{2,\alpha}(\Omega)$ estimates for solutions to linear PDE. Explain the method of continuity and how it is used to prove existence of solutions to second order linear elliptic PDE.

Question 20: Explain the Perron method for second order elliptic variable coefficient PDE.