Selected Topics in PDE.

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Contents

Preface to the course.

In this course we will try to understand the main aspects of the theory of partial differential equations (PDE). PDE theory is a vast subject with many different approaches and subfields. We can not cover everything in one course. Some selection has to be made. In this course we will try to achieve the following:

1: We will try to stress that the foundation of theory of PDE is basic real analysis.

2: We will try to motivate the increasing levels of abstraction in the theory. Our starting point will be a difficult problem, to find a solution $u(x)$ to the equation

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in some domain } D \\ u(x) = g(x) & \text{on the boundary } \partial D. \end{array} \tag{1}$$

We will use the tools we have from analysis to attack the problem. But with the tools at hand we will not be able to solve the problem in its full generality. Instead we are going to simplify the problem to something that we can solve and then add more and more (and more abstract) theory in order to solve the problem in its full generality. And at every step of the way we will try to motivate the theory and why we move into the more abstract areas of mathematics.

3: We will try to introduce and motivate a priori estimates in the theory of PDE. A priori estimates are one of the most important, most technical and most difficult to understand part of PDE theory. Often it is not mentioned on the masters level. But due to its importance we will introduce it and try to understand its significance.

4: We will try to show some of the different aspects of PDE theory. In particular, in the later parts of the course, we will discuss some functional analysis and viscosity solutions approaches in the course.

For various reasons this course will be based on lecture notes written by myself. I guess that only a madman would conceive to write a book in parallel with giving a course on that book. Writing the course material has some advantages. In particular for me! I get the course that I want and a course that covers the material that I think is important. The specific contents of the course will be whatever I put in these notes. It also have some disadvantages to write the course book in parallel to teaching it. Writing is time consuming and I am a terrible bad writer even under the best circumstances. Besides my lack of suitability as a course book author, the manuscript will inevitable contain many many typos.

I therefore feel that I should suggest some supporting literature that might be red in parallel to these notes.

One of the standard PDE texts today is Lawrence C. Evans *Partial Differential Equations* published by the American Mathematical society. Evans' book is an excellent introduction to PDE theory and it covers much material that we will not have the chance to discuss. In particular Evans' book covers elliptic, parabolic and hyperbolic equations of any order as well as variational calculus and Sobolev spaces. We will focus on second order elliptic equations - but we will go further than Evans' book in some respects. Evans' book could be seen as a complement to my notes.

The next book that can be seen as a complement to this course is D. Gilbarg and N.S. Trudingers *Elliptic Partial Differential Equations of Second Order* published by Springer. Gilbarg and Trudinger's book is an excellent PDE book that covers much regularity theory. However, Gilbarg-Trudinger's treatment of the topic is very terse and I don't think that it is suitable for a masters course. As a complement to this course it is however a great book. It also covers much more material than we will be able to cover in a term. One of my hopes is that you will be able to easily understand the first six chapters of Gilbarg-Trudinger after finishing this course.

Since the course will assume that you have a good understanding of basic analysis I would also recommend that you have an analysis book at hand. Something at the level of Walter Rudin's *Principles of Mathematical Analysis* published by McGraw-Hill Higher Education.

The course will be defined by my notes and no other course literature is necessary.

Chapter 1

The Laplace Equation, some Heuristics.

One of the most fundamental partial differential equations, and also one of the must studied object in mathematics is the Laplace equation:

Solving the laplace equation means to find a function $u(x)$ such that for every $x \in D$, where D is a given open set,

$$\Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = 0. \tag{1.1}$$

Equation (1.1) appears in many applications. For instance (1.1) models the steady state heat distribution in the set D. In applications it is often necessary not only to find just any solution to $\Delta u(x) = 0$ but a specific solution that attains certain values on ∂D (say the temperature on the boundary of of the domain).

Notation: *We will denote open sets in $\mathbb{R}^n$ by D. By ∂D we mean the boundary of D, that is $\partial D = \overline{D} \setminus D$. An open connected set will be called a domain.*

Since we will always consider domains in $\mathbb{R}^n$ we will use $x = (x_1, x_2, ..., x_n)$ to denote a vector in $\mathbb{R}^n$ with coordinates $(x_1, x_2, ..., x_n)$.

It is also of interest to solve the problem $\Delta u(x) = f(x)$ for a given function $f(x)$. We therefore formulate the Dirichlet problem:

$$\begin{aligned} &\Delta u(x) \equiv \textstyle\sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = f(x) && \text{in } D \subset \mathbb{R}^n \\ &u(x) = g(x) && \text{on } \partial D. \end{aligned} \tag{1.2}$$

D is a given domain (open set in $\mathbb{R}^n$ that might equal $\mathbb{R}^n$) and $f(x)$ a given function defined in D and g a given function defined on ∂D. Later on we will have to make some assumptions on f, g and D.

Let us fix some notation before we begin to describe these equations.

Definition 1.1. *We say that a function $u(x)$ is harmonic in an open set D if*

$$\Delta u(x) = 0 \text{ in } D.$$

We call the operator $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ the Laplace operator or the laplacian.

We will call the problem of finding a solution to (1.2) the Dirichlet problem or at times the boundary value problem.

Our first goal will be to solve the equations (1.2).

1.1 A Naive Approach - and a motivation for the theory ahead.

Warning: *This section is an informal discussion to motivate the theory that we will develop later. Reading this should give you a feeling that you could, if given some time, come up with the main ideas yourself. There is nothing miraculous in mathematics - just ordinary humans carefully following their intuition and the mathematical method. Later we will give stringent arguments for the intuitive ideas presented in this section.*

We stand in front of a new, interesting and very difficult problem: given a domain D and two functions f (defined on D) and g (defined on ∂D) we want to find a function $u(x)$ defined on D such that

$$\begin{array}{ll} \Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = f(x) & \text{in } D \subset \mathbb{R}^n \\ u(x) = g(x) & \text{on } \partial D. \end{array} \tag{1.3}$$

One should remark that the problem is extremely difficult. We may, at least apriori, choose f (and g) in any way we want which means that for any of the infinitively many points $x^0 \in D$ we want to prescribe the value of $\Delta u(x^0)$ so we have infinitely many equations that we want to solve simultaneously at the same time as we want the solution to satisfy $u(x) = g(x)$ for every $x \in \partial D$.

How do we start? How do we approach a new problem? We need to play with it. Try something and see where it leads.

The easiest way to attack a new problem is to make it simpler! So let us consider the simpler problem[1]

$$\Delta u(x) = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.4}$$

This problem is still quite difficult. So let us simplify (1.4) further and look for solutions $u(x)$ that are radial, that is functions that only depend on $|x|$. This should make the problem solvable - since we know how to solve differential equations depending only on one variable.

[1] Remember that we are just playing with the problem now. At the end of this section we will see that this approach leads to a theory for the equation $\Delta u = f$ in $\mathbb{R}^n$. My point is that whenever we encounter a new problem in mathematics we need to find an "in". A way to approach the problem and tie it into formulas. The we need to see where those formulas take us. If we pay attention to the formulas, and if we are lucky, we will gain some understanding of the problem. Not always the understanding we set out to find.

Lemma 1.1. *If $\Delta u(x) = 0$ and $u(x)$ is radial: $u(x) = h(|x|)$. Then*

$$\frac{\partial h(r)}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial h(r)}{\partial r} = 0.$$

In particular

$$u(x) = \begin{cases} \frac{a}{|x|^{n-2}} + b & \text{if } n \geq 3 \\ a \ln(|x|) + b & \text{if } n = 2 \end{cases} \tag{1.5}$$

for some $a, b \in \mathbb{R}$.

Proof: If we set $r = |x|$ then we see that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \Rightarrow \frac{\partial}{\partial x_i} = \frac{x_i}{r}\frac{\partial}{\partial r} + \text{angular derivatives}$$

and

$$\frac{\partial^2}{\partial x_i^2} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{x_i^2}{r^2}\frac{\partial}{\partial r} - \frac{x_i^2}{r^3}\frac{\partial^2}{\partial r^2} + \text{ angular part.}$$

In particular for a radial function $u(x) = h(r)$ we have

$$\frac{\partial h(r)}{\partial x_i} = h'(r)\frac{\partial r}{\partial x_i} = h'(r)\frac{x_i}{r}$$

and

$$\frac{\partial^2 h(r)}{\partial x_i^2} = h''(r)\frac{x_i^2}{r^2} + h'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right),$$

where we have used that $\sum_{i=1}^n x_i^2 = r^2$.

We may thus calculate

$$0 = \Delta u(x) = \Delta h(r) = \sum_{i=1}^n \frac{\partial^2 h(r)}{\partial x_i^2} = h''(r)\sum_{i=1}^n \frac{x_i^2}{r^2} + h'(r)\sum_{i=1}^n \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) =$$

$$= h''(r) + \frac{n-1}{r}h'(r).$$

We have thus shown that $h(r)$ satisfies the ordinary differential equation

$$0 = h''(r) + \frac{n-1}{r}h'(r) = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial h(r)}{\partial r}\right). \tag{1.6}$$

Multiplying (1.6) by r^{n-1} and integrating twice gives the desired result. $\square$

The solutions in (1.5) have a singularity, and are not differentiable, in $x = 0$. So Our radial solutions only solve $\Delta u(x) = 0$ in $\mathbb{R}^n \setminus \{x = 0\}$.

We would want to analyze the singularity at the origin. If you know anything about the theory of distributions you could consider $\Delta u(x)$ to be a distribution. Here we will use more elementary methods to analyze the singularity. The main difficulty with $u(x) = h(|x|)$ as defined in (1.5) is that u isn't differentiable at the origin. So let us approximate u by a two times differentiable function.

We may then analyze the approximation and use the information about the approximated function to say something about u.

In order to simplify things somewhat we will assume that $n = 3$ and we will define an approximation to u according to

$$u_\delta(x) = \begin{cases} -\left(\frac{3}{8\pi\delta} - \frac{1}{8\pi\delta^3}|x|^2\right) & \text{if } |x| \leq \delta \\ -\left(\frac{1}{4\pi}\frac{1}{|x|}\right) & \text{if } |x| > \delta. \end{cases} \tag{1.7}$$

Here we have chosen $a = \frac{1}{4\pi}$ and $b = 0$, the particular choice of a will be explained later. Moreover, we have chosen the coefficients u_δ so that u_δ is continuously differentiable.

Then

$$\Delta u_\delta(x) = \begin{cases} \frac{3}{4\pi\delta^3} & \text{if } |x| < \delta \\ 0 & \text{if } |x| > \delta. \end{cases}$$

Since we are only trying to gain an understanding of the problem we don't care so much about the value of Δu_δ on the set $\{|x| = \delta\}$[2] - the set where the second derivatives are not defined.

In order to simplify notation somewhat we will define the characteristic function of a set A according to

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{1.8}$$

Then

$$\Delta u_\delta = \frac{3}{4\pi\delta^3}\chi_{B_\delta(0)}(x).$$

We may also translate the function and solve, for an $x^0 \in \mathbb{R}^3$,

$$\Delta u_\delta(x - x^0) = \frac{3}{4\pi\delta^3}\chi_{B_\delta(x^0)}(x). \tag{1.9}$$

Notice that equation (1.3), with $D = \mathbb{R}^3$, is to find for each $x^0 \in \mathbb{R}^3$ a function $u(x)$ such that $\Delta u(x^0) = f(x^0)$. But $u_\delta(x - x^0)$ accomplishes almost that when δ is small.

In particular, given N points $x^1, x^2, ..., x^N \in \mathbb{R}^3$ and values $f(x^1), f(x^2), ..., f(x^N)$ such that $|x^i - x^j| > \delta$ for $i \neq j$ then the function

$$u(x) = \sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}u_\delta(x - x^j) \tag{1.10}$$

[2]If you are really worried about it we could define u_δ to be equal to $\frac{n^2+2n-8}{32\pi\delta^{n-2}} - \frac{n^2-4}{16\pi\delta^n}|x|^2 + \frac{n^2-2n}{32\pi\delta^{n+2}}|x|^4$ for $|x| \leq \delta$. With that definition u_δ becomes two times continuously differentiable and all the analysis in the rest of this section would follow with minor changes. Also, if you know anything about weakly differentiable functions, you will realize that u_δ has weak second derivatives in L^∞ which justifies the following calculations.

will solve

$$\Delta u(x) = \Delta\left(\sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}u_\delta(x-x^j)\right) =^3 \tag{1.11}$$

$$= \sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}\delta u_\delta(x-x^j) = \left\{\begin{array}{c}\text{if}\\ x = x^i\end{array}\right\} = f(x^i).$$

That is, given any finite set of points $x^1, x^2, ..., x^N$ and values $f(x^1), f(x^2)..., f(x^N)$, we may find a function u, defined according to (1.10), such that $\Delta u(x^i) = f(x^i)$.

This opens up for many possibilities. Could we consider a dense set of points $\{x^j\}_{j=1}^{\infty}$ and find a solution u^N to (1.11) for the points $\{x^j\}_{j=1}^{N}$ and values $\{f(x^j)\}_{j=1}^{N}$? Then let $N \to \infty$ and hope that $u = \lim_{N\to\infty} u^N$ solves $\Delta u(x) = f(x)$ for any $x \in \mathbb{R}^3$? This might work, but will use a different approach.

For that we need to notice that

$$\int_D g(x)dx \approx \sum_j \left(\text{volume}(\Omega_j)g(x^j)\right),$$

if $g(x)$ is continuous and Ω_j are a collection of disjoint sets such that $D \subset \cup_j \overline{\Omega}_j$ and the diameter of Ω_j is small. If we compare that to (1.10) we see that

$$u(x) = \sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}u_\delta(x-x^j) = \sum_{j=1}^{N} f(x^j)\text{volume}(B_\delta(x^j))u_\delta(x-x^j) \approx$$

$$\approx \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy,$$

we are very informal here and we can not claim that we have proved anything. But let us, still very informally, make the following conjecture:

An informal conjecture: *Let $f(x)$ be a continuous function and $\delta > 0$ a small real number. Then*

$$u^\delta(x) = \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy$$

should be an approximate solution to

$$\Delta u^\delta(x) = f(x).$$

Let us try to, still very informally, see if the conjecture makes sense. We make the following calculation

$$|\Delta u^\delta(x) - f(x)| = \left|\Delta \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy - f(x)\right| =$$

[3] Here we use an important property for the laplace equation, that it is linear. We will say more about the linearity later.

$$= \left\{ \begin{array}{l} \text{differentiation under} \\ \text{the integral} \end{array} \right\} =$$

$$= \left| \int_{\mathbb{R}^3} f(y)\Delta u_\delta(x-y)dy - f(x) \right| = \left| \frac{3}{4\pi\delta^3} \int_{\mathbb{R}^3} f(y)\chi_{B_\delta(x)}(y)dy - f(x) \right| =$$

$$= \left| \frac{3}{4\pi\delta^3} \int_{B_\delta(x)} f(y)dy - f(x) \right| \leq \frac{3}{4\pi\delta^3} \int_{B_\delta(x)} |f(y)-f(x)|\, dy \leq$$

$$\leq \sup_{y\in B_\delta(x)} |f(x)-f(y)|,$$

where we used that $\int_{B_\delta(x)} dy = \frac{4\pi\delta^3}{3}$ in the last step and that $\int_{\mathbb{R}^3} \chi_A g(x)dx = \int_A g(x)dx$ at the end of the third line of the calculation.

If f is uniformly continuous[4] then for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that

$$\sup_{y\in B_\delta(x)} |f(x)-f(y)| \leq \epsilon \qquad \text{for all } x \in \mathbb{R}^3.$$

That is if we choose $\delta = \delta_\epsilon$ then we have, at least informally, shown that

$$|\Delta u^\delta(x) - f(x)| < \epsilon.$$

It appears that

$$u(x) = \lim_{\delta\to 0} u^\delta(x) = \lim_{\delta\to 0} \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy = \left\{ \text{ informally } \right\} =$$

$$= \int_{\mathbb{R}^3} f(y)\frac{1}{4\pi|x-y|}dy$$

solves $\Delta u(x) = f(x)$ in $\mathbb{R}^3$.

We may thus make the following new conjecture

Another informal conjecture: *Let $f(x)$ be a uniformly continuous function defined in $\mathbb{R}^3$. Then*

$$u(x) = \int_{\mathbb{R}^3} f(y)\frac{1}{4\pi|x-y|}dy \tag{1.12}$$

solves $\Delta u(x) = f(x)$.

We will consider the informal conjecture as a working hypothesis to motivate the formal theory we develop later. One can already see that we need more assumptions on $f(x)$ in order for the conjecture to make sense. For instance, we need some assumption on $f(x)$ to assure that the integral in (1.12) is well defined.

[4]In our informal conjecture we just assumed continuity, but after some calculations we see that uniform continuity is a more natural assumption. The point in playing with mathematics is that we have the opportunity to see what assumptions we will need in the theorems we prove. At this point we have not proved anything. But we need to understand the problem before we can get down to the work of writing a proof.

Moreover, in proving the conjecture (with whatever extra assumptions we need) we need to be much more formal than we have been so far. In doing mathematical research one needs to be able to take a leap in the dark and argue informally to set up a reasonable hypothesis. And then have the technical skill to turn that hypothesis into a stringent proof. So far we have, what I feel to be, a reasonable hypothesis for how a solution to $\Delta u(x) = f(x)$, for uniformly continuous $f(x)$, should look. In the next section we will prove this.

Observe that this is just a first step in the development of the theory for the laplace equation. Later on we will need to find methods to handle the boundary conditions, that is to find solutions in a domain $D \subset \mathbb{R}^n$ for which the boundary condition $u(x) = g(x)$ on ∂D is satisfied.

Chapter 2

The Laplace Equation in $\mathbb{R}^n$.

2.1 The fundamental solution.

In this chapter we will be much more stringent than in the previous chapter. In particular we will take care to prove every statement that we make and to clearly define our terms. Our goal is to prove that if $f(x)$ is an appropriate function in $\mathbb{R}^n$ then

$$u(x) = -\frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy \quad \text{if } n \geq 3 \tag{2.1}$$

or

$$u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy \quad \text{if } n = 2 \tag{2.2}$$

solves $\Delta u(x) = f(x)$. Here ω_n is the area of the unit sphere in $\mathbb{R}^n$.

It is clear that we need to choose $f(x)$ in a class of functions such that the integrals (2.1) and (2.2) are well defined. To that end we make the following definition.

Definition 2.1. *For a continuous function f we call the closure of all the points where f is not equal to zero the support of f. We denote the support of f by $spt(f) = \overline{\{x;\ f(x) \neq 0\}}$.*

We will denote by $C^k_{loc}(D)$ the set of all functions $f(x)$ defined on D that are two times continuously differentiable on every compact set $K \subset D$.

We denote by $C^k_c(D)$ the set of all functions in $C^k_{loc}(D)$ that has compact support. That is:

$$C^k_c(D) =$$
$$= \left\{f \in C^k_{loc}(D);\ \textit{there exists a compact set } K \subset D s.t. f(x) = 0 \textit{ for all } x \notin K\right\}$$

In what follows we will often use the functions $-\frac{1}{(n-2)\omega_n}\frac{1}{|x|^{n-2}}$ and $-\frac{1}{2\pi}\ln(|x|)$ appearing in (2.1) and (2.2) so it is convenient to make the following definition.

Definition 2.2. *The function defined in $\mathbb{R}^n \setminus \{0\}$ defined by*

$$N(x) = \begin{cases} -\frac{1}{2\pi}\ln(|x|) & \text{for } n = 2 \\ -\frac{1}{(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{for } n \neq 2, \end{cases}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, will be called either the Newtonian potential or the fundamental solution of Laplace's equation.

With this definition we see that (2.1) and (2.2) reduces to

$$u(x) = \int_{\mathbb{R}^n} N(x-y)f(y)dy. \tag{2.3}$$

Notice that $N(x)$ is a radial function. That means that $N(x)$ depends only on $|x|$. The name fundamental solution is somewhat justified by the following Lemma (which is essentially covered in Lemma 1.1).

Lemma 2.1. *Let $N(x)$ be the fundamental solution to Laplace's equation then*

$$\Delta N(x) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Proof: This follows from the calculations of Lemma 1.1. □

In order to prove that $\Delta u(x) = f(x)$, where u is defined in (2.3), we need to show that:

1. The function $u(x)$ is well defined. That is that the integral in (2.3) is convergent for each x.
2. That the second derivatives of $u(x)$ are well defined. This in order to make sense of $\Delta u(x) = \sum_{i=1}^n \frac{\partial^2 u(x)}{\partial x_i^2}$.
3. Show that $\Delta u(x) = f(x)$.

Let us briefly reflect on these steps in turn.

1: Since the singularity of $N(x-y)$ is integrable it should be enough to assume that $f(x)$ has compact support in order to assure that the integral in (2.3) is convergent.

2: This is a more subtle point. If we naively[1] differentiate twice under the integral sign in (2.3) we see that, if we for definiteness assume that $n \geq 3$,

$$\frac{\partial^2 u(x)}{\partial x_i^2} = \int_{\mathbb{R}^n} \left(\frac{1}{\omega_n}\frac{1}{|x-y|^n} - \frac{n}{\omega_n}\frac{|x_i - y_i|^2}{|x-y|^{n+2}} \right) f(y)dy.$$

We see that the formal expression of $\frac{\partial^2 u(x)}{\partial x_i^2}$ involves integration of the function $\frac{f(y)}{|x-y|^n}$ which is not locally integrable in $\mathbb{R}^n$ since

$$\int_{B_1(0)} \frac{1}{|y|^n}dy = \left\{ \begin{array}{l} \text{polar} \\ \text{coordinates} \end{array} \right\} = \omega_n \int_0^1 \frac{1}{r}dr = \infty.$$

[1]By naively I mean that we do not care to verify that it is justified at this point.

It is therefore far from certain that $\frac{\partial^2 u(x)}{\partial x_i^2}$ exists. As a matter of fact, we will have to add new assumption in order to assure that $u(x)$ is two times differentiable. (See also Exercise 3 at the end of the chapter.)

3: We already have good intuition that this should be true.

In view of the second point above it seems that we need to make some extra assumption of $f(x)$ in order to prove that $u(x)$ has second derivatives. Therefore we define the following class of functions.

Definition 2.3. *Let $u \in C(D)$ and $\alpha > 0$ then we say that $u \in C^\alpha(D)$ if*

$$\sup_{x,y\in D, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

We define the norm on $C^\alpha(D)$ to be

$$\|u\|_{C^\alpha(D)} = \sup_{x\in D}|u(x)| + \sup_{x,y\in D, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

If $u \in C^\alpha(D)$ then we say that u is Hölder continuous in D.

We say that that $u \in C^k(D)$ if u is $k-$times continuously differentiable on D and

$$\|u\|_{C^k(D)} = \sum_{j=0}^{k} \sup_{x\in D} |D^j u(x)| < \infty.$$

Moreover we say that that $u \in C^{k,\alpha}(D)$ if $u \in C^k(D)$ and for every multiindex[2] *β of length $|\beta| = k$*

$$\sup_{x,y\in D, x\neq y} \frac{|u_\beta(x)-u_\beta(y)|}{|x-y|^\alpha} < \infty,$$

where $u_\beta = \frac{\partial^{|\beta|}u}{\partial x^\beta}$.

We define the norm on $C^{k,\alpha}(D)$ according to

$$\|u\|_{C^{k,\alpha}(D)} = \|u\|_{C^k(D)} + \max_{|\beta|=k}\left(\sup_{x,y\in D, x\neq y} \frac{|u_\beta(x)-u_\beta(y)|}{|x-y|^\alpha}\right),$$

where the max *is taken over all multiindexes β of length $|\beta| = k$.*

We are now ready to formulate our main theorem.

Theorem 2.1. *Let $f \in C_c^\alpha(\mathbb{R}^n)$ for some $\alpha > 0$ and define*

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi. \tag{2.4}$$

Then $u(x) \in C^2_{loc}$ and satisfies

$$\Delta u(x) = f(x).$$

[2] See the appendix for an explanation of this notation.

Proof: We will only prove the Theorem for $n \geq 3$, the proof when $n = 2$ is the same except for very small changes. The proof is rather long so we will split it up into several smaller steps.

Step 1: *The function $u(x)$ in (2.4) is well defined.*

Proof of step 1: We need to show that the integral $\int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi$ is convergent for every x. Notice that the integral is generalized in two ways. First the integrand have a singularity at $x = \xi$, and secondly the domain of integration is not bounded. Therefore we need to show that

$$\lim_{\epsilon \to 0, R \to \infty} \int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \tag{2.5}$$

exists.

Since $f(\xi)$ is continuous by assumption and $\frac{1}{|x-\xi|^{n-2}}$ is continuous for $\xi \neq x$ it is clear that

$$\int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi$$

is well defined for each $R, \epsilon > 0$. Moreover, since $f(\xi)$ has compact support there exists an R_0 such that $f(\xi) = 0$ for every $\xi \notin B_{R_0}(x)$. This means that

$$\lim_{R \to \infty} \int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi = \int_{B_{R_0}(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi,$$

so the limit as $R \to \infty$ causes no difficulty.

So we only need to consider the limit as $\epsilon \to 0$. To that end we show that $\int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi$ is Cauchy in ϵ. That is, for each $\mu > 0$ there exists a $\delta_\mu > 0$ such that

$$\left| \int_{B_R(x) \setminus B_{\epsilon_2}(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi - \int_{B_R(x) \setminus B_{\epsilon_1}(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \right| < \mu \tag{2.6}$$

for every $0 < \epsilon_1 \leq \epsilon_2 < \delta_\mu$.

We may rewrite the left hand side in (2.6) as

$$\left| \int_{B_{\epsilon_2}(x) \setminus B_{\epsilon_1}(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \right| \leq$$

$$\leq \sup_{B_{\epsilon_2}(x)} |f(\xi)| \int_{B_{\epsilon_2}(x) \setminus B_{\epsilon_1}(x)} \frac{1}{|x-\xi|^{n-2}} d\xi =$$

$$= \left\{ \begin{array}{l} \text{polar} \\ \text{coordinates} \end{array} \right\} = \omega_n \sup_{B_{\epsilon_2}(x)} |f(\xi)| \int_{\epsilon_1}^{\epsilon_2} r dr \leq \frac{\omega_n \epsilon_2^2}{2} \sup_{B_{\epsilon_2}(x)} |f(\xi)|.$$

Clearly (2.6) follows with $\delta_\mu = \frac{1}{2}\sqrt{\frac{2\mu}{\omega_n}}$. It follows that $u(x)$ is well defined.

Step 2: *The function $u(x)$ is $C^1_{\text{loc}}(\mathbb{R}^n)$ and*

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi.$$

Proof of step 2: First we notice, using a similar argument as in step 1, that

$$w_i(x) = \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \tag{2.7}$$

is well defined for every $i = 1, 2, ..., n$. We aim to show that $w_i(x) = \frac{\partial u(x)}{\partial x_i}$ (which is what we would expect by differentiating under the integral sign). To prove this we define $u_\epsilon(x)$ according to

$$u_\epsilon(x) = -\frac{1}{\omega_n(n-2)} \int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-2}} \eta_\epsilon(|x-\xi|) d\xi = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) \eta_\epsilon(|x-\xi|) d\xi$$

where $\eta_\epsilon(|x|) \in C^\infty(\mathbb{R}^n)$ is an increasing function such that $\eta'_\epsilon(|x|) \leq C/\epsilon$ and satisfies[3]

$$\eta_\epsilon(|x|) = \left\{ \begin{array}{ll} 0 & \text{if } |x| < \epsilon \\ 1 & \text{if } |x| > 2\epsilon. \end{array} \right\}$$

The reason we define u_ϵ in this way is that we have no singularity in the integral in the definition of u_ϵ. This means that we may manipulate u_ϵ with more ease than u. In particular, since the integrand in the definition of u_ϵ is in $C^1_c(\mathbb{R}^n)$ and we integrate over a compact set (since $f = 0$ outside a compact set) with respect to x we may use Theorem 2.3 in the appendix and differentiate under the integral sign and conclude that $u_\epsilon \in C^1_{\text{loc}}(\mathbb{R}^n)$.

Clearly $u_\epsilon \to u$ uniformly since

$$|u(x) - u_\epsilon(x)| = \left| \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \right| \leq$$

$$\leq \frac{\sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)|}{(n-2)\omega_n} \left| \int_{B_{2\epsilon}(x)} \frac{1}{|x-\xi|^{n-2}} d\xi = \right| \leq$$

$$\frac{\sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)|}{(n-2)} \epsilon^2.$$

If we can show that $\frac{\partial u_\epsilon(x)}{\partial x_i} \to w_i(x)$ uniformly it follows that u is the uniform limit of a sequence of C^1 functions whose derivatives converge uniformly to $w_i(x)$. It follows, from Theorem 2.2 in the appendix, that $u(x) \in C^1_{\text{loc}}(\mathbb{R}^n)$ and that $\frac{\partial u(x)}{\partial x_i} = w_i(x)$ and the proof is done.

It remains to show that $\frac{\partial u_\epsilon(x)}{\partial x_i} \to w_i(x)$ uniformly. To that end we estimate

$$\left| \frac{\partial u_\epsilon(x)}{\partial x_i} - w_i(x) \right| =$$

---

[3] Such an η_ϵ exists. Take for instance η_ϵ, as in Lemma 2.2 in the appendix, where $\eta(x) = \chi_{\mathbb{R}^n \setminus B_{2\epsilon}}$ and χ is the characteristic function defined in (1.8).

$$= \left| \frac{\partial}{\partial x_i} \int_{\mathbb{R}^n} N(x-\xi) f(\xi) \eta_\epsilon(|x-\xi|) d\xi - \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \right| \leq$$

$$= \left\{ \begin{array}{l} \text{diff. under} \\ \text{integral} \end{array} \right\} =$$

$$= \left| \int_{\mathbb{R}^n} \left(N(x-\xi) f(\xi) \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i} + \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) \eta_\epsilon(|x-\xi|) \right) d\xi - \right.$$

$$\left. - \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \right| \leq$$

$$\leq \left| \int_{\mathbb{R}^n} N(x-\xi) f(\xi) \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i} d\xi \right| + \tag{2.8}$$

$$+ \left| \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) \left(\eta_\epsilon(|x-\xi|) - 1 \right) d\xi \right|$$

where we used the triangle inequality in the last step and differentiation under the integral is justified by Theorem 2.3.

Notice that $|\eta_\epsilon(|x-\xi|) - 1| \leq 1$ for $\xi \in B_{2\epsilon}(x)$ and $|\eta_\epsilon(|x-\xi|) - 1| = 0$ for $\xi \notin B_{2\epsilon}(x)$ and that $\left| \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i} \right| \leq \frac{C}{\epsilon}$ for $\xi \in B_{2\epsilon}(x) \setminus B_\epsilon(x)$ and $\left| \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i} \right| = 0$ else.

We may thus estimate (2.8) from above by

$$\frac{C}{\epsilon} \int_{B_{2\epsilon}(x) \setminus B_\epsilon(x)} |N(x-\xi) f(\xi)| \, d\xi + \int_{B_{2\epsilon}} \left| \frac{\partial N(x-\xi)}{\partial x_i} \right| |f(\xi)| \, d\xi \leq$$

$$\leq \sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)| \left(\frac{C}{\epsilon} \int_{B_{2\epsilon}} \frac{1}{|x-\xi|^{n-2}} d\xi + C \int_{B_{2\epsilon}} \frac{1}{|x-\xi|^{n-1}} d\xi \right) \leq$$

$$\leq C \sup_{\xi \in \mathbb{R}^n} |f(\xi)| \epsilon.^{4} \tag{2.9}$$

We may thus conclude that $\frac{\partial u_\epsilon}{\partial x_i} \to w_i$ uniformly and that $u_\epsilon \to u$ uniformly. It follows, from Theorem 2.2, that $\frac{\partial u(x)}{\partial x_i} = w_i \in C^0_{\text{loc}}(\mathbb{R}^n)$. Step 2 is thereby proved.

Step 3: *The function $u \in C^2_{loc}(\mathbb{R}^n)$ and*

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x) \right) d\xi - f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi)$$

where $B_R(x)$ is any ball such that $spt(f) \subset B_R(x)$. Here $\nu_j(\xi)$ is the j :th component of the exterior normal of $B_R(x)$ at the point $\xi \in \partial B_R(x)$ and $A(\xi)$ is the area measure with respect to ξ.

[4]Here, as is very common in PDE theory, we do not really distinguish between constants that only depend on the dimension. We will often denote them by C - but C will often mean different things within the same experssion. We may for instance write $2C \leq C$. By this we mean that for any constant C_0 there is another constant C_1 such that $2C_0 \leq C_1$. But we usually don't the indicate that we intend different constants C_0 and C_1 with an index.

Proof of step 3: Before we prove step 3, let us try to explain the idea. We will use the same method of proof as in step 2. However, since $\left|\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\right|$ grows like $\frac{1}{|x-\xi|^n}$ as $|x-\xi| \to 0$ which is not integrable in $\mathbb{R}^n$. We can not say that

$$\int_{R^n} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} f(\xi) d\xi \tag{2.10}$$

exists.

Since $f(\xi) \in C_c^\alpha(\mathbb{R}^n)$ we know that $|f(\xi) - f(x)| \leq C|x-\xi|^\alpha$ for some constant C and $\alpha > 0$. This implies that

$$\left|\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right)\right| \leq C \frac{1}{|x-\xi|^n} |x-\xi|^\alpha \leq \frac{C}{|x-\xi|^{n-\alpha}}$$

which is integrable close to the point $x = \xi$. We may thus integrate

$$\int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) d\xi \tag{2.11}$$

for any ball $B_R(x)$. The difference between what we want to integrate (2.10) and what we can integrate (2.11) is the term

$$-\int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} f(x) d\xi = -\int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial \xi_i \partial \xi_j} f(x) d\xi =$$

$$= \left\{ \begin{array}{l} \text{a very formal} \\ \text{integration by parts} \end{array} \right\} =$$

$$= \int_{B_R(x)} \frac{\partial N(x-\xi)}{\partial \xi_i} \frac{\partial f(x)}{\partial \xi_j} d\xi - \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial \xi_i} \nu_j(\xi) f(x) d\xi.$$

So at least formally

$$\int_{R^n} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} f(\xi) d\xi = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) d\xi - \tag{2.12}$$

$$-f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi)$$

where all the terms on the right hand side are well defined. Of course, equation (2.12) is utter non-sense since we aren't really sure how to define the left hand side. We therefore use the right hand side, which is defined, in the expression in the statement of step 3. However, we need to be very careful when we prove step 3 to make sure that all our integrals are well defined.

As we already remarked we will use the same method as in step 2 and consider approximate functions u_ϵ and prove that the approximate functions converge uniformly in C^2 to u.

Let us start with the real proof. Following step 2 we define the function

$$w_{ij}(x) = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) d\xi - \tag{2.13}$$

$$-f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi).$$

Since the integrand in the first integrand satisfy the estimate

$$\left| \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) \right| \leq \frac{C}{|x-\xi|^{n-\alpha}} \tag{2.14}$$

it follows that the first integrand is absolutely integrable on $B_R(x)$ for every $R > 0$ and the first integrand is therefore well defined. The second integral in (2.13) is also well defined since we integrate a continuous function over a compact set. It follows that w_{ij} is well defined.

We also define

$$v_\epsilon(x) = \int_{B_R(x)} \eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi. \tag{2.15}$$

Clearly $v_\epsilon(x) \to \frac{\partial u(x)}{\partial x_i}$ uniformly since

$$\left| v_\epsilon(x) - \frac{\partial u(x)}{\partial x_i} \right| = \left| \int_{B_R} \left(\eta_\epsilon(|x-\xi|) - 1\right) \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \right| \leq$$

$$\leq \sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)| \int_{B_{2\epsilon}(x)} \left| \frac{\partial N(x-\xi)}{\partial x_i} \right| d\xi \leq C\epsilon \sup_{\xi \in \mathbb{R}^n} |f(\xi)|,$$

where we used that $\eta_\epsilon(|x-\xi|) = 1$ for $|x-\xi| \geq 2\epsilon$.

Since the integrand in (2.15) is C^1 in x we may differentiate differentiate under the integral, Theorem 2.3 in the appendix, and deduce that

$$\frac{\partial v_\epsilon(x)}{\partial x_j} = \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) f(\xi) d\xi$$

which is continuous, since the integrand is and the set of integration is compact. As in step 2 we want to show that $\frac{\partial v_\epsilon(x)}{\partial x_j}$ converges uniformly as $\epsilon \to 0$.

To prove that we write $\frac{\partial v_\epsilon(x)}{\partial x_j}$ in a form similar to the form of w_{ij}:

$$\frac{\partial v_\epsilon(x)}{\partial x_j} = \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) f(\xi) d\xi -$$

$$-f(x) \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) d\xi +$$

$$+f(x) \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) d\xi =$$

$$= \left\{ \begin{array}{l} \text{integration} \\ \text{by parts} \end{array} \right\} =$$

$$= \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) (f(\xi) - f(x))\, d\xi -$$

$$-f(x) \int_{\partial B_R(x)} \eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi) =$$

$$= \int_{B_R(x)} \frac{\partial^2 N(x-\xi)\eta_\epsilon(|x-\xi|)}{\partial x_i \partial x_j} (f(\xi) - f(x))\, d\xi -$$

$$-f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi),$$

where we used that $\eta_\epsilon(|x-\xi|) = 1$ on $\partial B_R(x)$ if $\epsilon < R$ (which we may assume) in the last equality. Notice that now we have no problem to integrate by parts since we have "cut out" the singularity by multiplying by η_ϵ.

To prove that $\frac{\partial v_\epsilon}{\partial x_j}$ converges uniformly to w_{ij} we calculate

$$\left| \frac{\partial v_\epsilon(x)}{\partial x_j} - w_{ij}(x) \right| =$$

$$= \Bigg| \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_j \partial x_i} (f(\xi) - f(x)) (\eta_\epsilon(|x-\xi| - 1))\, d\xi +$$

$$+ \int_{B_R(x)} \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_j} \frac{\partial N(x-\xi)}{\partial x_i} (f(\xi) - f(x))\, d\xi \Bigg| \leq$$

$$\leq \int_{B_{2\epsilon}(x)} \left| \frac{\partial^2 N(x-\xi)}{\partial x_j \partial x_i} (f(\xi) - f(x)) \right| d\xi +$$

$$+ \frac{C}{\epsilon} \int_{B_{2\epsilon}(x) \setminus B_\epsilon(x)} \left| \frac{\partial N(x-\xi)}{\partial x_i} (f(\xi) - f(x)) \right| d\xi \leq$$

$$\leq C \int_{B_{2\epsilon}(x)} \frac{1}{|x-\xi|^{n-\alpha}} d\xi + \frac{C}{\epsilon} \int_{B_{2\epsilon}(x) \setminus B_\epsilon(x)} \frac{1}{|x-\xi|^{n-1-\alpha}} d\xi \leq$$

$$\leq C\epsilon^\alpha.$$

Thus $\frac{\partial v_\epsilon}{\partial x_j} \to w_{ij}$ uniformly. Since $v_\epsilon \to \frac{\partial u}{\partial x_i}$ uniformly we may use Theorem 2.2 conclude that

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} (f(\xi) - f(x))\, d\xi - f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi)$$

which finishes the proof of step 3.

Step 4. *The function $u(x)$ satisfies $\Delta u(x) = f(x)$.*

Proof of step 4: By step 3 we know that $u(x) \in C^2_{\text{loc}}(\mathbb{R}^n)$ and that

$$\frac{\partial^2 u(x)}{\partial x_i^2} = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i^2} \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_i(\xi) dA(\xi).$$

This implies in particular that

$$\Delta u(x) = \int_{B_R(x)} \sum_{i=1}^n \frac{\partial^2 N(x-\xi)}{\partial x_i^2} \left(f(\xi) - f(x)\right) d\xi -$$

$$-f(x) \int_{\partial B_R(x)} \sum_{i=1}^n \frac{\partial N(x-\xi)}{\partial x_i} \nu_i(\xi) dA(\xi) =$$

$$= \int_{B_R(x)} \Delta N(x-\xi) \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_R(x)} \nabla N(x-\xi) \cdot \nu(\xi) dA(\xi). \tag{2.16}$$

But $\Delta N(x-\xi) = 0$ at almost every point which implies that the first integral in (2.16) is zero. To calculate the second integral in (2.16) we notice that

$$\nu(\xi) = \frac{x-\xi}{|x-\xi|}$$

and

$$\nabla N(x-\xi) = \frac{1}{\omega_n} \frac{x-\xi}{|x-\xi|^n} = \frac{1}{\omega_n} \frac{\nu(\xi)}{R^{n-1}},$$

on $\partial B_R(x)$. Thus

$$\Delta u(x) = f(x) \int_{\partial B_R(x)} \nabla N(x-\xi) \cdot \nu(\xi) dA(\xi) =$$

$$= \frac{f(x)}{R^{n-1}\omega_n} \int_{\partial B_R(x)} |\nu(\xi)|^2 dA(\xi) = f(x)$$

since $|\nu(\xi)|^2 = 1$ and $\int_{\partial B_R(x)} dA(\xi) = \omega_n R^n$ by the definition of ω_n. This finishes the proof. $\square$

Remark: *The proof is rather long and difficult to overview. But the bulk of the proof consists in using the cut off function η_ϵ to make sure that the integrals involved are well defined. The real important step in the proof is in step 3 where we use the Hölder continuity of $f(x)$ to assure that the second derivatives of $u(x)$ are well defined. It is in the very last equation of the proof where we see why we choose the rather strange constant $\frac{1}{(n-2)\omega_n}$ in the definition of $N(x)$.*

2.2 Appendix: Some Integral Formulas and Facts from Analysis.

In this appendix we repeat some results form analysis.

Theorem 2.2. *Let $D \subset \mathbb{R}^n$ be a domain and assume that u_ϵ is a family of continuous functions on D such that*

1. *$u_\epsilon \to u$ locally uniformly as $\epsilon \to 0$,*
2. *$\frac{\partial u_\epsilon}{\partial x_i}$ is locally continuous on D.*
3. *$\frac{\partial u_\epsilon}{\partial x_i} \to w_i$ locally uniformly as $\epsilon \to 0$.*

Then $\frac{\partial u}{\partial x_i}$ exists, is locally continuous and $\frac{\partial u}{\partial x_i} = w_i$.

Proof:

Step 1: *The function w_i is locally continuous.*

Proof of step 1: We will argue by contradiction and assume that w_i has a discontinuity point $x^0 \in D$. That means that there exists two sequences $x^j \to x^0$ and $y^j \to x^0$ such that

$$\lim_{j\to\infty} |w_i(x^j) - w_i(y^j)| = \delta > 0. \tag{2.17}$$

Since D is open there exists some $r > 0$ such that $\overline{B_r(x^0)} \subset D$. Also, since $\frac{\partial u_\epsilon}{\partial x_i} \to w_i$ locally uniformly there exists an $\epsilon_0 > 0$ such that

$$\left| \frac{\partial u_\epsilon(x)}{\partial x_i} - w_i(x) \right| < \frac{\delta}{4} \tag{2.18}$$

for all $x \in B_r(x^0)$ and $\epsilon < 2\epsilon_0$.

Using that $\frac{\partial u_{\epsilon_0}}{\partial x_i}$ is locally continuous on D and that $\overline{B_r(x^0)}$ is compact we may conclude that there exists a $\mu > 0$ (for simplicity of notation we may assume that $\mu < r$) such that

$$\left| \frac{\partial u_{\epsilon_0}(x)}{\partial x_i} - \frac{\partial u_{\epsilon_0}(y)}{\partial x_i} \right| < \frac{\delta}{4} \tag{2.19}$$

for all $x, y \in B_\mu(x^0)$.

In particular we may conclude that for any j large enough such that $x^j, y^j \in B_\mu(x^0)$

$$\left|w_i(x^j) - w_i(x^j)\right| =$$

$$= \left| w_i(x^j) - \frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} - \left(w_i(y^j) - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right) + \left(\frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right) \right| \leq$$

$$\leq \left| w_i(x^j) - \frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} \right| + \left| w_i(y^j) - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right| + \left| \frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right| < \frac{3\delta}{4},$$

where we have used (2.18) and (2.19). This clearly contradicts (2.17) which finishes the proof of step 1.

Step 2: *Assume that $B_r(x) \subset D$. Then*

$$u(x + e_i h) = u(x) + \int_0^h w_i(x + s e_i) ds$$

for any $|h| < r$*. Here* e_i *is the* i *:th unit vector* $e_i = (0, 0, ..., 1, ..., 0)$ *where the* 1 *is in the* i *:th coordinate place.*

Proof of step 2: Since $u_\epsilon \to u$ locally uniformly it follows that

$$u(x + he_i) - u(x) = \lim_{\epsilon \to 0} (u_\epsilon(x + he_i) - u_\epsilon(x)) =$$

$$= \left\{ \begin{array}{l} \text{fundamental} \\ \text{Theorem} \\ \text{of calculus} \end{array} \right\} = \lim_{\epsilon \to 0} \int_0^h \frac{\partial u_\epsilon(x + se_i)}{\partial x_i} ds = \int_0^h w_i(x + se_i) ds,$$

where the last step follows form the uniform convergence $\frac{\partial u_\epsilon}{\partial x_i} \to w_i$. Step 2 follows.

Step 3: *The end of the proof.*

Form the fundamental Theorem of calculus and step 2 it follows that

$$\frac{\partial u(x)}{\partial x_i} = w_i(x)$$

which is continuous by step 1. $\square$

Theorem 2.3. *Let* D_0 *and* D_1 *be domains and assume that* $f(x, \xi)$ *is locally continuous on* $D_0 \times D_1 = \{(x, \xi);\ x \in D_0 \text{ and } \xi \in D_1\}$*. Assume furthermore that* $\frac{\partial f(x,\xi)}{\partial x_i}$ *is locally continuous on* $D_0 \times D_1$*.*

Then for any compact set $K \subset D_1$

$$f_i(x, \xi) \equiv \frac{\partial}{\partial x_i} \int_K f(x, \xi) d\xi = \int_K \frac{\partial f(x, \xi)}{\partial x_i} d\xi \tag{2.20}$$

and f_i *is locally continuous on* D_0*.*

Proof: Since D_0 is open and $x \in D_0$ there exists a ball $\overline{B_\mu(x)} \subset D_0$. Notice that $\overline{B_\mu(x)} \times K \subset D_0 \times D_1$ is compact since $\overline{B_\mu(x)}$ and K are[5].

Since $\overline{B_\mu(x)} \times K \subset D_0 \times D_1$ is compact it follows that $\frac{f(x,\xi)}{\partial x_i}$ is uniformly continuous on $\overline{B_\mu(x)} \times K \subset D_0 \times D_1$.[6]

By definition

$$\frac{\partial}{\partial x_i} \int_K f(x, \xi) d\xi = \lim_{h \to 0} \int_K \frac{f(x + he_i, \xi) - f(x, \xi)}{h} d\xi. \tag{2.21}$$

Next, using the mean value property for the derivative we see that there exists a $\gamma(x, \xi)$ such that $\gamma(x, \xi) \in [0, h]$ and

$$f(x + he_i, \xi) - f(x, \xi) = \frac{\partial f(x + \gamma(x, \xi) e_i)}{\partial x_i} h.$$

[5]Remember that the product of two compact sets are compact.

[6]Remember that continuous functions are uniformly continuous of compact sets.

We may conclude that

$$\left|\int_K \frac{f(x+he_i,\xi)-f(x,\xi)}{h}d\xi - \int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi\right| =$$
$$= \left|\int_K \frac{\partial f(x+\gamma(x,\xi)e_i,\xi)}{\partial x_i}d\xi - \int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi\right| \leq$$
$$\leq \int_K \left|\frac{\partial f(x+\gamma(x,\xi)e_i,\xi)}{\partial x_i}d\xi - \frac{\partial f(x,\xi)}{\partial x_i}\right| d\xi.$$

But since $\frac{\partial f(x,\xi)}{\partial x_i}$ is uniformly continuous there exists an $h_\epsilon > 0$ for each $\epsilon > 0$ such that

$$\left|\frac{\partial f(x+se_i,\xi)}{\partial x_i}d\xi - \frac{\partial f(x,\xi)}{\partial x_i}\right| < \epsilon$$

for each $|s| < h_\epsilon$. Since $|\gamma(x,\xi)| < h$ it follows, for $|h| < h_0$, that

$$\int_K \left|\frac{\partial f(x+\gamma(x,\xi)e_i,\xi)}{\partial x_i}d\xi - \frac{\partial f(x,\xi)}{\partial x_i}\right| d\xi < |K|\epsilon,$$

where $|K|$ denotes the volume of the set K.[7]

In particular we may conclude that for each $\epsilon > 0$

$$\lim_{h\to 0}\left|\int_K \frac{f(x+he_i,\xi)-f(x,\xi)}{h}d\xi - \int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi\right| < |K|\epsilon.$$

It follows that (2.20).

To see that $f_i(x,t)$ is locally continuous in x we again notice that $\frac{\partial f(x,\xi)}{\partial x_i}$ is uniformly continuous on $\overline{B_\mu(x)} \times K$ which implies that for every $\epsilon > 0$ there exists a $h_\epsilon > 0$ such that

$$\left|\frac{\partial f(x,\xi)}{\partial x_i} - \frac{\partial f(x,\xi)}{\partial x_i}\right| =$$
$$= \left|\int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi - \int_K \frac{\partial f(y,\xi)}{\partial x_i}d\xi\right| \leq$$
$$\int_K \epsilon d\xi \leq |K|\epsilon$$

for every $y \in B_\mu(x)$ such that $|x-y| < h_\epsilon$. Continuity follows. □

Let us remind ourselves of the following results from calculus.

Theorem 2.4. [The Divergence Theorem.] *Let Ω be a C^1 domain (that is the boundary $\partial\Omega$ is locally the graph of a C^1 function) in $\mathbb{R}^n$ and $v = (v^1, v^2, ..., v^n) \in C^1(\Omega;\mathbb{R}^n)$. Then*

$$\int_\Omega div(v)dx = \int_{\partial\Omega} v(x)\cdot\nu(x)dA(x)$$

where $div(v) = \sum_{j-1}^n \frac{\partial v^j}{\partial x_j}$ is the divergence of v and $\nu(x)$ is the outward pointing unit normal of $\partial\Omega$ the point x.

[7] Rather the measure of K (in case you have studied measure theory)

We will not prove this theorem.

Corollary 2.1. [INTEGRATION BY PARTS.] *Let Ω be a C^1 domain (that is the boundary $\partial\Omega$ is locally the graph of a C^1 function) in $\mathbb{R}^n$ and $v, w \in C^1(\Omega)$. Then*

$$\int_\Omega \frac{\partial v(x)}{\partial x_i} dx = -\int_\Omega v(x) \frac{\partial w(x)}{\partial x_i} dx + \int_{\partial\Omega} w(x)v(x)\nu_i(x)dA(x)$$

where $\nu_i(x)$ is the i :th component of the outward pointing unit normal of $\partial\Omega$ the point x.

Proof: If we apply the divergence theorem to the vector function $v(x)w(x)e_i$ we see that

$$\int_\Omega \mathrm{div}\,(v(x)w(x)e_i)\, dx = \int_{\partial\Omega} v(x)w(x)e_i \cdot \nu(x)dA(x).$$

The left hand side in the last expression is

$$\int_\Omega \frac{\partial v(x)}{\partial x_i} dx + \int_\Omega v(x) \frac{\partial w(x)}{\partial x_i} dx.$$

Putting these two expression together gives the Corollary. □

Theorem 2.5. [GREEN'S FORMULAS.] *Let Ω be a C^1 domain and $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ then*

1.
$$\int_\Omega v(x)\Delta u(x)dx + \int_\Omega \nabla v(x) \cdot \nabla u(x)dx = \int_{\partial\Omega} v(x) \frac{\partial u(x)}{\partial \nu} dA(x)$$

where ν is the outward pointing unit normal of Ω and $\frac{\partial u(x)}{\partial \nu} = \nu \cdot \nabla u(x)$ and $dA(x)$ is an area element of $\partial\Omega$.

2.

$$\int_\Omega (v(x)\Delta u(x) - u(x)\Delta v(x))\, dx = \int_{\partial\Omega} \left(v(x)\frac{\partial u(x)}{\partial \nu} - u(x)\frac{\partial v(x)}{\partial \nu} \right) dA(x).$$

Proof: For the first Green identity we apply the divergence theorem to $v(x)\nabla u(x)$. The second identity follows from interchanging u and v in the first identity and subtract the result. □

2.3 Appendix: An Excursion into the subject of Regularization.

In this appendix we remind ourselves of a fact from regularization theory. The goal of this section is to show that we may approximate any continuous function uniformly by a function in C^∞. This is an important tool in analysis to approximate irregular functions by infinitely differentiable functions. We start by introducing the standard mollifier.

Definition 2.4. *Let*

$$\phi(x) = \begin{cases} c_0 e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where c_0 is chosen so that $\int_{\mathbb{R}^n} \phi(x)dx = 1$.

We will, for $\epsilon > 0$, call $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon)$ the standard mollifier.

Slightly abusing notation we will at times write $\phi_\epsilon(x) = \phi_\epsilon(|x|)$.

Before we state the main theorem for mollifiers we need to introduce some notation.

Definition 2.5. *We say that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$ (where $\mathbb{N}_0 = \{0, 1, 2, ...\}$) is a multiindex. We will say that the length of α is $|\alpha| = \sum_{j=1}^n \alpha_j$. And for $u(x) \in C^k(\Omega)$ and $|\alpha| = l \leq k$ we will write*

$$\frac{\partial^{|\alpha|}u(x)}{\partial x^\alpha} \equiv \frac{\partial^l u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

Note that a multiindex is just a shorthand way of writing derivatives.

The standard mollifier is important because of the following Lemma.

Lemma 2.2. *Let $\epsilon > 0$ and $\phi_\epsilon(x)$ be the standard mollifier then*

1. *spt$(\phi_\epsilon) = \overline{B_\epsilon(0)}$ where spt$(\phi_\epsilon) = \overline{\{x \in \mathbb{R}^n;\ \phi_\epsilon(x) \neq 0\}}$ is the support of ϕ_ϵ,*
2. *$\phi_\epsilon \in C^\infty(\mathbb{R}^n)$,*
3. *$\int_{\mathbb{R}^n} \phi_\epsilon(x)dx = 1$,*
4. *if $u \in C(\Omega)$ (or if u is locally integrable) and we define*
$$u_\epsilon(x) = \int_\Omega u(y)\phi_\epsilon(x-y)dy$$
for $x \in \Omega_\epsilon = \{x \in \Omega;\ dist(x, \partial\Omega) > \epsilon\}$ then $u_\epsilon \in C^\infty(\Omega_\epsilon)$
5. *and if $u \in C(\Omega)$, where Ω is open, then $\lim_{\epsilon\to 0} u_\epsilon(x) \to u(x)$ uniformly on compact sets of Ω.*

Proof: We will prove each part individually.

Part 1: To show that the support of ϕ_ϵ is $\overline{B_\epsilon(0)}$ we notice that for $|x| \geq \epsilon$ we have

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon) = 0$$

since $|x/\epsilon| \geq 1$ and $\phi(x) = 0$ for $|x| \geq 1$. For $|x| < \epsilon$ we have $\phi_\epsilon(x) = \frac{c_0}{\epsilon^n} e^{-\frac{\epsilon^2}{\epsilon^2 - |x|^2}} > 0$. That is $\phi_\epsilon > 0$ in $B_\epsilon(0)$ and $\phi_\epsilon = 0$ in $\mathbb{R}^n \setminus B_\epsilon(0)$. By definition the support of ϕ_ϵ is the closure of the set where $\phi_\epsilon \neq 0$.

Part 2: In order to see that $\phi_\epsilon \in C^\infty(\mathbb{R}^n)$ we notice that it is enough to show that $\phi(x) \in C^\infty(\mathbb{R}^n)$. In particular we have

$$\frac{\partial^{|\alpha|}\phi_\epsilon(x)}{\partial x^\alpha} = \frac{1}{\epsilon^{n+|\alpha|}}\frac{\partial^{|\alpha|}\phi(x/\epsilon)}{\partial x^\alpha},$$

so if $\phi \in C^\infty$ then $\phi_\epsilon \in C^\infty$.

We will show, by induction, that, for any multiindex α,

$$\frac{\partial^{|\alpha|}\phi(x)}{\partial x^\alpha} = \frac{p_\alpha(x)}{q_\alpha(x)}\phi(x), \tag{2.22}$$

where $p_\alpha(x)$ and $q_\alpha(x)$ are polynomials, $q_\alpha(x) > 0$ in $B_1(0)$. When $|\alpha| = 0$ the representation is obviously true with $p_0(x) = q_0(x) = 1$. If we assume that (2.22) is true for all multiindexes α of length $k-1$ then for any multiindex β of length k we have some $j \in \{0, 1, 2, ..., n\}$ and multiindex α of length $k-1$ such that in $B_1(0)$

$$\frac{\partial^{|\beta|}\phi(x)}{\partial x^\beta} = \frac{\partial}{\partial x_j}\frac{\partial^{|\alpha|}\phi(x)}{\partial x^\alpha} = \frac{\partial}{\partial x_j}\left(\frac{p_\alpha(x)}{q_\alpha(x)}\phi(x)\right) =$$

$$= \frac{q_\alpha(x)\frac{\partial p_\alpha(x)}{\partial x_j} - p_\alpha(x)\frac{\partial q(x)}{\partial x_j}}{q_\alpha^2(x)}\phi(x) + \frac{p_\alpha(x)}{q_\alpha(x)}\frac{\partial\phi(x)}{\partial x_j} =$$

$$= \left(\frac{q_\alpha(x)\frac{\partial p_\alpha(x)}{\partial x_j} - p_\alpha(x)\frac{\partial q(x)}{\partial x_j}}{q_\alpha^2(x)} + \frac{x_j}{(1-|x|^2)^2}\right)\phi(x),$$

we may define the quantity in the brackets on the right hand side as $\frac{p_\beta(x)}{q_\beta(x)}$. Since p_α, q_α and $(1-|x|^2)^2$ are all polynomials it follows that p_β and q_β are polynomials. Moreover we see, by a simple induction, that q_β is a power of $1-|x|^2$ so $q_\beta(x) > 0$ in $B_1(0)$.

For $x \in \mathbb{R}^n \setminus \overline{B_1(0)}$ it follows that

$$\frac{\partial^{|\alpha|}\phi(x)}{\partial x^\alpha} = 0$$

since $\phi(x) = 0$ for $x \in \mathbb{R}^n \setminus \overline{B_1(0)}$.

Finally we notice that since $p_\alpha(x)$ is a polynomial we have for every multiindex α a constant C_α such that $\sup_{B_1(0)}|p_\alpha(x)| \le C_\alpha$. And for each multiindex α there is some k such that $q_\alpha(x) = (1-|x|^2)^k$. We may therefore estimate

$$\lim_{|x|\to 1}\left|\frac{p_\alpha(x)}{q_\alpha(x)}\phi(x)\right| \le \lim_{t\to 1}\left|c_n\frac{C_\alpha}{(1-t^2)^k}e^{-\frac{1}{1-t^2}}\right| = 0$$

since $e^{-\frac{1}{1-t^2}} \to 0$ with exponential speed as $t \to 1$ whereas $\frac{C_\alpha}{(1-t^2)^k} \to \infty$ with polynomial speed.

We have therefore shown that $\phi(x)$ is continuously differentiable for any α.

Part 3: This follows by a change of variables

$$\int_{\mathbb{R}^n} \phi_\epsilon(x)dx = \int_{\mathbb{R}^n} \frac{1}{\epsilon^n}\phi\left(\frac{x}{\epsilon}\right)dx = \left\{ \begin{array}{l} \text{set } x = \epsilon y \\ \text{then } dx = \epsilon^n dy \end{array} \right\} = \int_{\mathbb{R}^n} \phi(y)dy = 1.$$

Part 4: Follows from Theorem 2.3.

Part 5: Let $K \subset \Omega$ be compact. Since $u \in C(\Omega)$ it follows that for any $x \in \Omega$ and $\delta > 0$ there exists a $\frac{1}{2}\inf(1, \text{dist}(K, \partial\Omega)) > \kappa_\delta > 0$ such that

$$|u(x) - u(y)| < \delta$$

for all $x \in K$ and y such that $|x - y| < \kappa_\delta$. In particular if $\epsilon < \kappa_\delta$ then

$$\left|\int_\Omega \phi_\epsilon(x-y)u(y)dy - u(x)\right| = \left|\int_{B_\epsilon(x)} \phi_\epsilon(x-y)u(y)dy - u(x)\right| \leq$$

$$\left|\int_{B_\epsilon(x)} \phi_\epsilon(x-y)\big(u(y) - u(x)\big)dy\right| \leq \int_{B_\epsilon(x)} \phi_\epsilon(x-y)|u(y)-u(x)|dy <$$

$$< \delta \int_{B_\epsilon(x)} \phi_\epsilon(x-y)dy = \delta$$

where we used that $\phi_\epsilon(x-y) = 0$ in $\mathbb{R}^n \setminus B_\epsilon(x)$ in the first inequality, that $\int_{B_\epsilon(x)} \phi(x-y)dy = 1$ in the second and that $|u(y) - u(x)| < \delta$ in the forth and and that $\int_{B_\epsilon(x)} \phi(x-y)dy = 1$ again in the last equality. □

2.4 Exercises Chapter 3.

Exercise 1.

1. Show that all affine functions $u(x) = a + \mathbf{b} \cdot x$ are harmonic.

2. Let A be an $n\times n-$matrix and show that if $u(x) = \langle x, A\rangle \cdot x = \sum_{i,j=1}^n a_{ij}x_ix_j$ is harmonic if and only if $\text{trace}(A) = 0$.

3. Find all harmonic third order polynomials in $\mathbb{R}^2$.

4. Let $u(z)$ be an analytic function in a domain $D \subset \mathbb{C}$. Define the function $v(x,y) = \mathcal{RE}(u(x+iy))$ (the real part of the complex valued u) and prove that $\Delta v(x,y) = 0$ in the set $\{(x,y);\ x + iy \in D\}$. In particular there are polynomial harmonic functions of any order in $\mathbb{R}^2$.

   Hint: *The Cauchy-Riemann equations.*

Exercise 2:

A: Let $K \subset \mathbb{R}^n$ be a compact set and $x \in K^\circ$ (the interior of K). Prove the following

1. $\int_K \frac{1}{|x-y|^q} dy$ converges if and only if $q < n$.

2. If $|f(y)| \leq \frac{C}{|x-y|^q}$ and $q < n$ then $\int_K f(y)dy$ is well defined.

B: Let $f(x)$ be a continuous function defined on $\mathbb{R}^n$. Prove that if there exists a constant C such that $|f(x)| \leq C|x|^{-p}$ and $p > n$ then the integral $\int_{\mathbb{R}^n} f(x)dx$ is well defined.

C: Assume that $f(x)$ is a function defined on $\mathbb{R}^n$ and that $f(x)$ is continuous on $\mathbb{R}^n \setminus \{x^0\}$. Assume furthermore that there exists constants C_p, C_q, $0 \leq q < n$ and $p > n$ such that $|f(x)| \leq C_p|x|^{-p}$ for all $x \notin B_1(x^0)$ and $|f(x)| \leq C_q|x - x^0|^{-q}$ for all $x \in B_1(x^0)$. Prove that $\int_{\mathbb{R}^n} f(x)dx$ is well defined.

D: Prove that (2.7), (2.14) and (2.4) are well defined under the weaker assumption that $f(x)$ is continuous on $\mathbb{R}^n$ and satisfies $|f(x)| \leq C|x|^{-2+\epsilon}$ for some $\epsilon > 0$.

Exercise 3: In the informal discussion leading up to Theorem 2.1 we indicated that we needed to assume that $f \in C^\alpha_{\mathrm{loc}}$ in order to make sense of the second derivatives of $u(x)$ defined as in (2.4). It is always good in mathematics to make sure that our assumptions are necessary. In this exercise we will prove that the expression in Step 3 in the proof of Theorem 2.1 is not well defined under the assumption that $f(x)$ is continuous with compact support. We will also slightly weaken the assumption that $f \in C^\alpha$ in Theorem 2.1.

A: Define $f(x)$ in $B_{1/2}(0)$ according to

$$f(x) = \frac{x_1 x_2}{|x|^2 |\ln(|x|)|}.$$

Show that $f(x)$ is continuous.

B: Prove that we may extend $f(x)$ to a continuous function on $\mathbb{R}^2$ with support in $B_1(0)$.

HINT: *Can we find a function $g(x) \in C^\infty_c(B_1(0))$ such that $g(x) = 1$ in $B_{1/2}$? Then $f(x)g(x)$ would be a good candidate for a solution.*

C: Show that the right hand side in the expression

$$\frac{\partial^2 u(x)}{\partial x_1 \partial x_2} = \int_{B_2(x)} \frac{\partial^2 N(x-\xi)}{\partial x_1 \partial x_2} \left(f(\xi) - f(x)\right) d\xi -$$

$$-f(x) \int_{\partial B_2(x)} \frac{\partial N(x-\xi)}{\partial x_1} \nu_2(\xi) dA(\xi)$$

from step 3 in the proof of Theorem 2.1 is not well defined at $x = 0$ with the $f(x)$ defined in step **B**. Conclude that it is not enough to assume that $f(x)$ is continuous with compact support in order for our current proof of Theorem 2.1 to work.

D: In the theory of PDE one often uses Dini continuity. We say that a function $f(x)$ is Dini continuous if there exists a continuous function $\sigma \geq 0$ defined on $[0,1)$ such that $\sigma(0) = 0$ and

$$\int_0^1 \frac{\sigma(t)}{t} dt < \infty$$

such that

$$|f(x) - f(y)| \leq \sigma(|x-y|) \quad \text{ for all } x, y \text{ s.t. } |x-y| < 1.$$

Prove that Theorem 2.1 holds under the weaker assumption that $f(x)$ is Dini continuous with compact support in $\mathbb{R}^n$.

Remark on Exercise 3: *Notice that we have only proved that the expression in Step 3 in not well defined for this particular f. One might ask if there is another way to define a solution so that $\Delta u = f$. We will se later in the course that that is not the case. With our particular function f the only possible solutions to the Laplace equation are not C^2. But before we reach the point where we can understand how to define solutions that are not C^2 we need to develop more understanding of the Laplace equation.*

Exercise 4: Very often in PDE books one proves the weaker statement that if $f \in C_c^2(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi$$

is a $C^2_{\text{loc}}(\mathbb{R}^n)$ function and $\Delta u(x) = f(x)$. It might be a good exercise to prove this statement directly using the following steps.

A: Show that

$$u(x) = \int_{\mathbb{R}^n} N(\xi) f(x-\xi) d\xi.$$

B: Prove that

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} N(\xi) \frac{\partial^2 f(x-\xi)}{\partial x_i \partial x_j} d\xi$$

and that $u \in C^2_{\text{loc}}$.

C: Notice that

$$\Delta u(x) = \int_{\mathbb{R}^n} N(\xi) \Delta_x f(x-\xi) d\xi = \int_{\mathbb{R}^n} N(\xi) \Delta_\xi f(x-\xi) d\xi =$$

$$= \int_{B_\delta(x)} N(\xi) \Delta_\xi f(x-\xi) d\xi + \int_{B_R(x) \setminus B_\delta(x)} N(\xi) \Delta_\xi f(x-\xi) d\xi, \qquad (2.23)$$

if R is chosen large enough. Then show that for any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that the first integral in (2.23) has absolute value less than ϵ and the second integral differs from $f(x)$ by at most ϵ. Conclude the theorem.

HINT: *Use Green's second formula when you estimate the second integral.*

Chapter 3

Green's Functions.

In this section we will begin to understand how to solve the Dirichlet problem in a domain Ω. The Dirichlet problem consists of finding a $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solving

$$\begin{aligned} \Delta u(x) = f(x) \quad & \text{in } \Omega \\ u(x) = g(x) \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

where $f \in C^\alpha(\Omega)$ and $g \in C(\partial\Omega)$ are given functions.

In this chapter we will investigate what the theory from the previous chapter would imply for solutions to (3.1). This will lead to the concept of a Green's function which is similar to the fundamental solution - but for a given domain. However, we can not, in general, calculate the Green's function. But for certain simple domains, with much symmetry, it is possible to explicitly calculate the Green's function. We will calculate the Green's function for the upper half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_n > 0\}$ and for a ball $B_r(0)$.

3.1 An informal motivation for the concept of Green's functions.

To motivate the introduction of Green's functions we have to look at the theory we have developed so far - that is all we have. In particular we have shown that we may define a solution to $\Delta u(x) = f(x)$ for any $f \in C_c^\alpha(\mathbb{R}^n)$ by

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi. \tag{3.2}$$

Using that $\Delta u(x) = f(x)$ we arrive at

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) \Delta_\xi u(\xi) d\xi.$$

Let us try to see if the same argument applies to solutions to (3.1). To that end we assume that $u \in C^{2,\alpha}(\Omega)$ and define

$$\tilde{u}(x) = \int_{\Omega} N(x-\xi)\Delta u(x)dx.$$

If $u(x)$ where defined according to (3.2) for some $f \in C_c^{\alpha}(\mathbb{R}^n)$ and $\Omega = \mathbb{R}^n$ then $\tilde{u}(x) = u(x)$. But we don't expect, in general, that $\tilde{u}(x) = u(x)$ for any $u \in C^{2,\alpha}(\Omega)$ for an arbitrary Ω. The point is that, using the explicit expression for $\tilde{u}(x)$, we can calculate the difference $\tilde{u}(x)$ and see what mathematics gives us back and hopefully it will give us some information about u.

Therefore we estimate

$$\int_{\Omega} N(x-\xi)\Delta u(x)dx = \int_{B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi + \int_{\Omega\setminus B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi = I_1 + I_2. \tag{3.3}$$

$$+\int_{\Omega\setminus B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi = I_1 + I_2.$$

We expect I_1 to be small, as a matter of fact:

$$|I_1| \leq \sup_{B_\epsilon(x)} |\Delta u| \int_{B_\epsilon(x)} \frac{C}{|x-\xi|^{n-2}} d\xi \leq C \sup_{B_\epsilon(x)} |\Delta u| \epsilon^2,$$

where we assumed, for definiteness, that $n \geq 3$. To calculate I_2 we use the second Green formula and conclude

$$I_2 = \int_{\Omega\setminus B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi =$$

$$= \int_{\Omega\setminus B_\epsilon(x)} u(\xi)\Delta_\xi N(x-\xi)d\xi + \int_{\partial\Omega} \left(N(x-\xi)\frac{\partial u(\xi)}{\partial \nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial \nu(\xi)} \right) dA_{\partial\Omega}(\xi) +$$

$$+ \int_{\partial B_\epsilon(x)} \left(N(x-\xi)\frac{\partial u(\xi)}{\partial \nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial \nu(\xi)} \right) dA_{\partial B_\epsilon(x)}(\xi) = I_3 + I_4 + I_5.$$

Notice that $I_3 = 0$ since $\Delta_\xi N(x-\xi) = 0$ in $\Omega \setminus B_\epsilon(x)$. Next we look at I_5 and estimate

$$I_5 = \int_{\partial B_\epsilon(x)} -\frac{1}{(n-2)\omega_n}\frac{1}{\epsilon^{n-2}}\frac{\partial u(\xi)}{\partial \nu} d\xi + \int_{\partial B_\epsilon(x)} -\frac{1}{\omega_n}\frac{1}{\epsilon^{n-1}} u(\xi) d\xi.$$

That is

$$|I_5 - u(x)| \leq \left| \int_{\partial B_\epsilon(x)} -\frac{1}{(n-2)\omega_n}\frac{1}{\epsilon^{n-2}}\frac{\partial u(\xi)}{\partial \nu} d\xi \right| + \tag{3.4}$$

$$+ \left| \int_{\partial B_\epsilon(x)} \frac{1}{\omega_n}\frac{1}{\epsilon^{n-1}} \left(u(\xi) - u(x)\right) d\xi \right| \leq \frac{\sup_\Omega |\nabla u|}{n-2}\epsilon + \sup_{\partial B_\epsilon(x)} |u(\xi) - u(x)|,$$

where the first term goes to zero as $\epsilon \to 0$ since $|\nabla u|$ is bounded (we assume that $u \in C^2(\Omega)$) and the second term goes to zero as $\epsilon \to 0$ since u is continuous. We can thus conclude that $I_5 \to u(x)$ as $\epsilon \to 0$.

To summarize, we have proven that

$$\int_\Omega N(x-\xi)\Delta u(\xi)d\xi = I_1 + I_2 = 0 + I_3 + I_4 + I_5 =$$

$$= 0 + 0 + \int_{\partial\Omega}\left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}\right)dA_{\partial\Omega}(\xi) + u(x),$$

as $\epsilon \to 0$.

Rearranging terms we arrive at

$$u(x) = \int_\Omega N(x-\xi)\Delta u(\xi)d\xi - \int_{\partial\Omega}\left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}\right)dA_{\partial\Omega}(\xi). \tag{3.5}$$

So if $u(x)$ was a solution to (3.1) then

$$u(x) = \int_\Omega N(x-\xi)f(\xi)d\xi - \int_{\partial\Omega}\left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - g(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}\right)dA_{\partial\Omega}(\xi), \tag{3.6}$$

this is a rather good expression but it has one serious flaw. We do not know what value $\frac{\partial u}{\partial\nu}$ has on $\partial\Omega$. If we new that we could calculate $u(x)$ by just using $f(x)$, $g(x)$ and $\frac{\partial u}{\partial\nu}$. But if $N(x-\xi)$ happened to be equal to zero on $\partial\Omega$ then the troublesome term

$$\int_{\partial\Omega} N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)}dA_{\partial\Omega}(\xi)$$

in equation (3.5) would be equal to zero and (3.6) would become

$$u(x) = \int_\Omega N(x-\xi)f(\xi)d\xi + \int_{\partial\Omega} g(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}dA_{\partial\Omega}(\xi) \tag{3.7}$$

and we would have a representation formula for $u(x)$ in terms of the given data $f(x)$ and $g(x)$. This motivates us to define a function $G(x,\xi)$ that has similar properties as $N(x-\xi)$ but so that $G(x,\xi) = 0$ for $\xi \in \partial\Omega$.

Definition 3.1. *Let Ω be a domain with C^1 boundary and assume that for every $x \in \Omega$ we have a solution $\phi^x(\xi) \in C^2(\Omega)\cap C(\overline{\Omega})$ to*

$$\begin{aligned} &\Delta\phi^x(\xi) = 0 && \text{in } \Omega \\ &\phi^x(\xi) = N(x-\xi) && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

Then we say that

$$G(x,\xi) = N(x-\xi) - \phi^x(\xi)$$

is the Green's function in Ω.

Remark: Notice that until we can prove that $\phi^x(\xi)$ is a unique solution to (3.8) we have on right to say that $G(x,\xi)$ is *the* Green's function since there might be many Green's functions satisfying the definition. Later on we will prove that ϕ^x is indeed the unique solution and that we are therefore justified in calling G *the* Green's function.

Since the Green's function $G(x,\xi)$ has the same type of singularity as $N(x-\xi)$ at $x=\xi$ so there is some hope that the above calculations should work in the same way for $G(x,\xi)$ as they did for $N(x-\xi)$. Moreover, $G(x,\xi)=0$ for $\xi\in\partial\Omega$ which makes it reasonable to hope that the representation formula (3.7) would work for $G(x,\xi)$ in place of $N(x-\xi)$. However, we need to prove this.

3.2 The Green's function.

The main reason to introduce Green's functions is the following Theorem.

Theorem 3.1. *Assume that $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ and that $u(x)$ solves (3.1), where Ω is a bounded domain with C^1 boundary. Assume furthermore that $G(x,\xi)$ is the Green's function for Ω. Then*

$$u(x)=\int_\Omega G(x,\xi)f(\xi)d\xi+\int_{\partial\Omega}\left(g(\xi)\frac{\partial G(x,\xi)}{\partial\nu}\right)dA_{\partial\Omega}(\xi). \tag{3.9}$$

Proof: The proof is very similar to the calculations we did in the previous section. We will use use Green's second identity on $G(x,y)\equiv N(x-y)-\phi^x(y)$ and $u(y)$.

$$\int_\Omega G(x,\xi)\Delta u(\xi)d\xi=$$

$$=\int_{B_\epsilon(x)}G(x,\xi)\Delta u(\xi)d\xi+\int_{\Omega\setminus B_\epsilon(x)}\left(-u(y)\underbrace{\Delta_y G(x,y)}_{=0}+G(x,y)\Delta_y u(y)\right)dy=$$

$$=\left\{\begin{array}{l}\text{Green's Second}\\ \text{formula on the}\\ \text{second integal}\end{array}\right\}$$

$$=\int_{B_\epsilon(x)}G(x,\xi)\Delta u(\xi)d\xi+\int_{\partial\Omega}\left(u(\xi)-\frac{\partial G(x,\xi)}{\partial\nu}+\underbrace{G(x,\xi)}_{=0}\frac{\partial u(\xi)}{\partial\nu}\right)dA_{\partial\Omega}(\xi)+ \tag{3.10}$$

$$+\int_{\partial B_\epsilon(x)}\left(-u(\xi)\left(\frac{\partial N(x-\xi)}{\partial\nu}+\frac{\partial\phi^x(\xi)}{\partial\nu}\right)-\left(N(x-\xi)-\phi^x(\xi)\right)\frac{\partial u(\xi)}{\partial\nu}\right)dA_{\partial B_\epsilon(x)}(\xi)=$$

$$=\int_{B_\epsilon(x)}G(x,\xi)\Delta u(\xi)d\xi-\int_{\partial\Omega}u(\xi)\frac{\partial G(x,\xi)}{\partial\nu}dA_{\partial\Omega}(\xi)+$$

$$+\int_{\partial B_\epsilon(x)}\left(u(\xi)\frac{\partial\phi^x(\xi)}{\partial\nu}-\phi^x(\xi)\frac{\partial u(\xi)}{\partial\nu}\right)dA_{\partial B_\epsilon(x)}(\xi)+I_5,$$

where $I_5\to u(x)$ as $\epsilon\to 0$ is the same as the expression in the previous section, see formula (3.4).

It is easy to estimate the remaining terms, in particular

$$\left|\int_{B_\epsilon(x)} G(x,\xi)\Delta u(\xi)d\xi\right| \le \sup_\Omega |\Delta u(x)| \left(\int_{B_\epsilon(x)} |N(x-\xi)|\, d\xi + \int_{B_\epsilon(x)} |\phi(\xi)|\, d\xi\right) \le$$

$$\le C \sup_\Omega |\Delta u(x)| \left(\epsilon^2 + \sup_\Omega |\phi| \epsilon^n\right) \to 0$$

as $\epsilon \to 0$. And similarly

$$\left|\int_{\partial B_\epsilon(x)} \left(u(\xi)\frac{\partial \phi^x(\xi)}{\partial \nu} - \phi^x(\xi)\frac{\partial u(\xi)}{\partial \nu}\right) dA_{\partial B_\epsilon(x)}(\xi)\right| \le$$

$$\le C\left(\sup_\Omega |u| \sup_\Omega |\nabla \phi^x| + \sup_\Omega |\nabla u| \sup_\Omega |\phi^x|\right)\epsilon^{n-1} \to 0.$$

Using these estimates together with (3.10) we may conclude, after sending $\epsilon \to 0$, that

$$u(x) = \int_\Omega G(x,\xi) f(\xi) d\xi + \int_{\partial\Omega} \left(g(\xi)\frac{\partial G(x,\xi)}{\partial \nu}\right) dA_{\partial\Omega}(\xi),$$

this finishes the proof of the Theorem. □

Remarks: 1. Notice that (3.9) makes perfectly good sense even if we do not know that we have a solution. A good guess (which is indeed true) is that if we define u according to (3.9) then u solves the Dirichlet problem (3.1). To actually prove this will require some extra work.

2) In some sense we hide the difficulties in this Theorem. In particular we assume that we can solve (3.8) in order to define the Green's function. But to solve the Dirichlet problem is exactly what we are aiming to do. So we assume that we have a solution to one Dirichlet problem, namely (3.8), in order to find a representation for the solution to another Dirichlet problem (3.1).

The Theorem is however useful since the Dirichlet problem (3.8) has $f = 0$ and very special boundary data. So Theorem 3.1 states that if we can calculate a solution to the Dirichlet problem with simple boundary data (3.8) in Ω then we can find a representation for the solution to the Dirichlet problem in Ω with any boundary data $g \in C(\partial\Omega)$.

Our next goal will be to actually solve the Dirichlet problem (3.8) in some simple domains Ω.

3.3 The Dirichlet Problem in $\mathbb{R}^n_+$.

As pointed out in the last section, if we know that we have a solution u to the Dirichlet problem in Ω, it is enough to solve the Dirichlet problem

$$\begin{array}{ll} \Delta \phi^x(\xi) = 0 & \text{in } \Omega \\ \phi^x(\xi) = -N(\xi - x) & \text{on } \partial\Omega. \end{array} \tag{3.11}$$

for every x in order to find a representation formula for u.

If Ω is very complicated it will be very hard to find a solution to (3.11). But if Ω has some simple symmetries it is indeed possible to explicitly write down the solutions to (3.11). In this section we will consider $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_n > 0\}$.

We need to find a $\phi^x(\xi)$ solving

$$\begin{array}{ll} \Delta\phi^x(\xi) = 0 & \text{in } \mathbb{R}^n_+ \\ \phi^x(\xi) = N(\xi - x) & \text{on } \partial\mathbb{R}^n_+ = \{\xi \in \mathbb{R}^n;\ \xi_n = 0\}. \end{array}$$

Notice that

$$\begin{array}{ll} \Delta N(\xi - x) = 0 & \text{in } \mathbb{R}^n_- = \{\xi \in \mathbb{R}^n;\ \xi_n < 0\} \\ N(\xi - x) = N(\xi - x) & \text{on } \partial\mathbb{R}^n_- = \partial\mathbb{R}^n_+ = \{\xi \in \mathbb{R}^n;\ \xi_n = 0\}. \end{array}$$

So $N(y\xi - x)$ is a solution, but in the wrong half space! This is however very easy to fix by a simple reflection. We define

$$\phi^x(\xi) = N(\xi - \tilde{x}) \tag{3.12}$$

where $\tilde{x} = (x_1, x_2, ..., x_{n-1}, -x_n)$. Then we have, for $\xi \in \partial\mathbb{R}^n_+$ that is $\xi_n = 0$,

$$\phi^x(\xi) = -\frac{1}{(n-2)}\frac{1}{|\xi - x|^{n-2}} = -\frac{1}{(n-2)}\frac{1}{\left(|\xi' - x'|^2 + |x_n|^2\right)^{\frac{n-2}{2}}} = N(\xi - x),$$

where we have used the notation $x' = (x_1, x_2, ..., x_{n-1}, 0)$ and $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$. We also assumed that $n \geq 3$ for simplicity. The calculations for $n = 2$ is very similar.

We have thus proved the following Lemma.

Lemma 3.1. *The Green's function in* $\mathbb{R}^n_+$ *is*

$$G(x, \xi) = N(\xi - x) - N(\xi - \tilde{x})$$

where $\tilde{x} = (x_1, x_2, ..., x_{n-1}, -x_n)$.

3.3.1 The Poisson Kernel in $\mathbb{R}^n_+$.

We know that if we have a solution to the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

then we can represent the solution by the formula

$$u(x) = \int_\Omega G(x, \xi)\Delta u(\xi) d\xi + \int_{\partial\Omega} \left(u(\xi)\frac{\partial G(x, \xi)}{\partial\nu} \right) dA_{\partial\Omega}(\xi),$$

if Ω is C^1 and bounded and the Green's function $G(x, \xi)$ exists and is $C^1(\overline{\Omega}) \cap C^2(\Omega)$.

If $f = 0$ this reduces to

$$u(x) = \int_{\partial\Omega} \left(u(\xi) \frac{\partial G(x,\xi)}{\partial \nu} \right) dA_{\partial\Omega}(\xi).$$

This representation formula indicates that $\frac{\partial G(x,\xi)}{\partial \nu}$ is of special importance. We make the following definition.

Definition 3.2. *Let Ω be a domain and $G(x,\xi)$ be the corresponding Green's function. Call assume that the normal derivative of $G(x,\xi)$ exists on $\partial\Omega$ and callit*

$$K(x,\xi) = \frac{\partial G(x,\xi)}{\partial \nu}$$

the Poisson Kernel for Ω and the representation formula

$$u(x) = \int_{\partial\Omega} \left(u(\xi) \frac{\partial G(x,\xi)}{\partial \nu} \right) dA_{\partial\Omega}(\xi)$$

we call the Poisson formula.

Since we know the Green's function in $\mathbb{R}^n_+$ we are able to calculate the Poisson kernel for $\mathbb{R}^n_+$.

Lemma 3.2. *The Poisson kernel for $\mathbb{R}^n_+$ is*

$$K(x,\xi) = \frac{x_n}{\omega_n} \frac{1}{|x-\xi|^n}.$$

Proof: The Poisson kernel is by definition

$$\frac{\partial G(x,\xi)}{\partial \nu}.$$

The normal of $\mathbb{R}^n_+$ is $-e_n$ so the Poisson kernel is

$$K(x,\xi) = -\frac{\partial G(x,\xi)}{\partial \xi_n}$$

and

$$G(x,\xi) = N(\xi - x) - N(\xi - \tilde{x}).$$

The Lemma follows by a simple calculation. □

Lemma 3.3. *For every $x_n > 0$ we have*

$$\int_{\mathbb{R}^{n-1}} K(x,\xi')d\xi' = 1,$$

where $y' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$ and $d\xi' = d\xi_1 d\xi_2 ... d\xi_{n-1}$.

Proof: In order to evaluate the integral of the Poisson kernel we will resort to a trick. I am not particularly fond of tricks in mathematics but in this case it will save us some calculation (which I do not like any more than tricks).

First we notice that by translating $\xi' \to z + x'$ we get

$$\int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x|^n} d\xi' = \int_{\mathbb{R}^n} \frac{x_n}{|z' - x_n e_n|^n} dz' = \int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x_n e_n|^n} d\xi'$$

so we might assume that $x' = (0, 0, ..., 0)$ without changing the value of the integral.

Secondly, we notice that if we change variables $\xi' \to sz'$ for any $s > 0$ then we get

$$\int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x|^n} dy' = \int_{\mathbb{R}^{n-1}} \frac{s^{n-1} x_n}{|sz' - x|^n} dz' \int_{\mathbb{R}^n} \frac{\frac{x_n}{s}}{\left|z' - \frac{x}{s}\right|^n} dz'$$

which implies that the value of the integral of the Poisson kernel is independent of $x_n > 0$. So there is a constant c_n such that

$$c_n = \int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x|^n} d\xi',$$

where c_n is independent of x as long as $x_n > 0$.

The difficult part is to evaluate

$$\int_{\mathbb{R}^{n-1}} \frac{x_n}{|\xi' - x|^n} d\xi' = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|\xi'|^2 + |x_n|^2\right)^{n/2}} d\xi' = c_n. \tag{3.13}$$

In order to evaluate (3.13) we notice that

$$\int_0^\infty \frac{1}{1 + x_n^2} \left(\int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|\xi'|^2 + x_n^2\right)^{n/2}} d\xi' \right) dx_n \tag{3.14}$$

$$= c_n \int_0^\infty \frac{1}{1 + x_n^2} dx_n = c_n \left(\arctan(\infty) - \arctan(0)\right) = \frac{c_n \pi}{2}.$$

We may also evaluate (3.14)

$$\int_0^\infty \frac{1}{1 + x_n^2} \left(\int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|\xi'|^2 + x_n^2\right)^{n/2}} d\xi' \right) dx_n =$$

$$= \int_0^\infty \int_{\mathbb{R}_+^{n-1}} \frac{1}{1 + x_n^2} \frac{x_n}{\left(|\xi'|^2 + x_n^2\right)^{n/2}} d\xi' dx_n,$$

changing to polar coordinates $x_n = r\cos(\psi) r$, $r^2 = |\xi'|^2 + x_n^2$ we may continue the equality,

$$= \int_0^\infty \int_{\partial B_1^+(0)} \frac{r\cos(\psi)}{1 + r^2\cos^2(\psi)} \frac{1}{r^n} r^{n-1} dA_{\partial B_1^+(0)} dr =$$

$$= \int_{\partial B_1^+(0)} \int_0^\infty \frac{\cos(\psi)}{1+r^2\cos^2(\psi)} dA_{\partial B_r(0)} dr = \int_{\partial B_1^+(0)} \frac{\pi}{2} dA_{\partial B_1^+} = \frac{\pi \Omega_n}{2}, \quad (3.15)$$

where we again used that $\int \frac{a}{1+a^2x^2} = \arctan(ax)$ and that $r\cos(\psi) \to \infty$ as $r \to \infty$.

Comparing (3.15) and (3.14) we see that $c_n = \omega_n$ this implies that

$$\int_{\mathbb{R}^{n-1}} K(x,\xi')dy' = \frac{c_n}{\omega_n} = 1.$$

□

The next Theorem establishes that we may indeed use the Poisson kernel to calculate a solution to the Dirichlet problem in $\mathbb{R}^n_+$.

Theorem 3.2. *Let $g \in C_c(\partial\mathbb{R}^n_+)$ and define*

$$u(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi')}{|x-\xi'|^n} d\xi' \quad (3.16)$$

where $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$ and $d\xi' = d\xi_1 d\xi_2 ... d\xi_{n-1}$. Then

$$\begin{array}{ll} \Delta u(x) = 0 & \textit{in } \mathbb{R}^n_+ \\ \lim_{x_n \to 0^+} u(x', x_n) = g(x') & \textit{uniformly on compact sets } x' \in K \subset\subset \mathbb{R}^{n-1}. \end{array} \quad (3.17)$$

Remark: There is a slight abuse of notation in this Theorem. We use the notation $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$ as a vector in $\mathbb{R}^n$ with zero as its last component. But we also use $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}) \in \mathbb{R}^{n-1}$ without the zero in the n :th component when we write $g(\xi')$. It should be clear from context which convention we are using.

Proof: We will do the proof into two steps.

Step 1: *The function u defined in (3.16) is well defined and is harmonic in $\mathbb{R}^n_+$.*

That the function is well defined and that we may differentiate under the integral sign is clear since the integrand has compact support in ξ' and is C^∞ in x for each $x \in \mathbb{R}^n_+$. To show that $\Delta u(x) = 0$ follows from a simple calculation.

Step 2: *Showing that $\lim_{x_n \to 0^+} u(x', x_n) = g(x')$ uniformly on compact sets.*

To show that $\lim_{x_n \to 0^+} u(x', x_n) = g(x')$ uniformly on compact sets we notice that since $g \in C(\mathbb{R}^{n-1})$ it follows that g is uniformly continuous on compact sets $K \subset\subset \mathbb{R}^{n-1}$. Fix a compact set $K \subset\subset \mathbb{R}^{n-1}$. For technical reasons that will become clear later we will define

$$K^1 = \cup_{x \in K} \overline{B_1(x)},$$

that is K^1 is the closed set containing all points that are at a distance at most one from K. Since K is compact it is closed and bounded which implies that K^1

is closed and bounded and thus compact. Therefore g is uniformly continuous on K^1.

In particular for every $\epsilon > 0$ we have a $\delta_{\epsilon/2} > 0$, which we may assume to satisfy $\delta_{\epsilon/2} < 1$, such that

$$|g(x') - g(\xi')| < \frac{\epsilon}{2} \tag{3.18}$$

for every $x' \in K$ such that $|x' - \xi'| < \delta_{\epsilon/2}$. Here we use that g is uniformly continuous on K^1, notice that if $x' \in K$ and $|x' - \xi'| < \delta_{\epsilon/2} < 1$ then $x', \xi' \in K^1$ and we may use the same $\delta_{\epsilon/2}$ for all $x' \in K$.

Using that $\int_{\mathbb{R}^{n-1}} K(x,\xi')dy = 1$ (Lemma 3.3) we see that for any $x' \in K$

$$\begin{aligned}|u(x', x_n) - g(x')| &= \left|\int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n}\frac{g(\xi') - g(x')}{|x - \xi'|^n} dy'\right| \leq \\ \leq \left|\int_{\mathbb{R}^{n-1}\setminus B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n}\frac{g(\xi') - g(x')}{|x - \xi'|^n} dy'\right| &+ \left|\int_{B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n}\frac{g(\xi') - g(x')}{|x - \xi'|^n} d\xi'\right| = \\ &= I_{\epsilon/2} + J_{\epsilon/2}\end{aligned} \tag{3.19}$$

It is easy to see that

$$J_{\epsilon/2} \leq \int_{B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n}\frac{|g(\xi') - g(x')|}{|x - \xi'|^n} d\xi' < \frac{\epsilon}{2}\int_{B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n}\frac{1}{|x - \xi'|^n} d\xi' < \frac{\epsilon}{2} \tag{3.20}$$

since $\int_{\mathbb{R}^{n-1}} K(x,\xi')dy = 1$, $|g(\xi') - g(x')| < \epsilon$ for all $\xi' \in B_{\delta_\epsilon}(x')$ and $K(x,\xi) > 0$. Noticew that the estimate (3.20) is independent of $x_n > 0$.

Also, if we chose R so large that $g(\xi') = 0$ outside of $B_R(x')$ for all $x' \in K$,

$$\begin{aligned}\left|I_{\epsilon/2}\right| &= \left|\int_{\mathbb{R}^{n-1}\setminus B_{\delta_{\epsilon/2}}} \frac{x_n}{\omega_n}\frac{g(\xi') - g(x')}{|x - \xi'|^n} d\xi'\right| \leq \\ \leq \sup_{\xi' \in \mathbb{R}^{n-1},\, x' \in K} |g(\xi') - g(x')| &\left|\int_{B_R(0)\setminus B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n}\frac{1}{|x - \xi'|^n} d\xi'\right| \leq \\ &\leq \frac{2x_n \sup_{y\in\mathbb{R}^{n-1}}|g(y)|}{\omega_n}\left|\int_{B_R(0)\setminus B_{\delta_{\epsilon/2}}(x')} \frac{1}{\delta_{\epsilon/2}^n} d\xi'\right| \leq \\ &\leq \left(\frac{2\sup_{y\in\mathbb{R}^{n-1}}|g(y)|R^n}{n\delta_{\epsilon/2}^n}\right) x_n.\end{aligned} \tag{3.21}$$

From (3.21) it follows that $\left|I_{\epsilon/2}\right| < \frac{\epsilon}{2}$ if $x_n < \tilde{\delta}_\epsilon$ where

$$\tilde{\delta}_\epsilon = \frac{n\delta_{\epsilon/2}^n}{4\sup_{y\in\mathbb{R}^{n-1}}|g(y)|R^n}$$

only depend on g and the dimension.[1]

Putting (3.19), (3.20) and (3.21) together we have shown that for each compact set K and each $\epsilon > 0$ there is a $\tilde{\delta}_\epsilon$ such that for each $x' \in K$

$$|u(x', x_n) - g(x')| < \epsilon \quad \text{for all } x_n < \tilde{\delta}_\epsilon.$$

It follows that

$$\lim_{x_n \to 0^+} u(x', x_n) = g(x')$$

uniformly on compact sets. □

Corollary 3.1. *Theorem 3.2 is still true under the assumption that $g(\xi')$ is continuous and bounded.*

Sketch of the Proof: The proof of the corollary is almost the same as the proof of the Theorem. We only need to make sure that the integrals are convergent. We will show that the integral in (3.16) is convergent and leave the rest of the details to the reader.

Notice that $u(x)$ is still well defined if $g(\xi)$ is bounded and integrable. In particular, under those assumptions there exists a constant C, depending only on x_n, $\sup_{\mathbb{R}^{n-1}} |g(\xi')|$ and the dimension such that

$$\left| \frac{x_n}{\omega_n} \frac{g(\xi')}{|x - \xi'|^n} \right| \le C \text{ in } B_1(x'),$$

and

$$\left| \frac{x_n}{\omega_n} \frac{g(\xi')}{|x - \xi'|^n} \right| \le \frac{C}{|x' - \xi'|^n} \text{ in } \mathbb{R}^{n-1} \setminus B_1(x'),$$

it follows that the integral (3.16) is convergent under the assumptions in the Corollary, see exercise 2C in the previous set of lecture notes. □

Notice that we can now solve the Dirichlet problem in $\mathbb{R}^+_+$. In particular if $f \in C^\alpha_c(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^{n-1})$ then

$$u^1(x) = \int_{\mathbb{R}^n} N(x - \xi) f(\xi) d\xi$$

solves

$$\Delta u^1(x) = f(x) \text{ in } \mathbb{R}^n$$

and

$$u^2(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi') - u^1(\xi')}{|x - \xi'|^n} dy'$$

solves

$$\begin{array}{ll} \Delta u^2(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u^2(x', 0) = g(x') - u^1(x', 0) & \text{on } \partial\mathbb{R}^{n-1}, \end{array}$$

[1] In particular, $\tilde{\delta}_\epsilon$ only depend on g through $\sup|g|$, the support of g and the continuity properties of g, that is on $\delta_{\epsilon/2}$.

where the second identity is interpreted in the sense of limits as in Theorem 3.2. In particular $u(x) = u^1(x) + u^2(x)$ will solve

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \mathbb{R}^n_+ \\ u(x', 0) = g(x') & \text{on } \partial\mathbb{R}^{n-1}, \end{array}$$

that is we now know how to solve the Dirichlet problem in $\mathbb{R}^n_+$.

In Theorem 3.1 we made an assumption that Ω was bounded. Obviously $\mathbb{R}^n_+$ is not a bounded set so we can not apply Theorem 3.1 to $\mathbb{R}^n_+$. We can however, Theorem 3.2, construct a solution in $\mathbb{R}^n_+$.

The difference between Theorem 3.1 and Theorem 3.2 is that in Theorem 3.1 we assume that we have a solution and we find a representation formula for that solution. In Theorem 3.2 we do not assume that we have a solution - we prove that we have a solution.

However to state that the solution we construct in Theorem 3.2 is the same as the any given solution we would have to know that the solutions are unique. In bounded domains Ω it is indeed the case that solutions that are $C^2(\Omega)\cap C(\overline{\Omega})$ are unique, a fact that we will prove later. In unbounded domains, in particular in $\mathbb{R}^n_+$, the solutions are not uniquely determined by the boundary data. A simple example is that $u(x) = ax_n$ is a solution to

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u(x', 0) = 0 & \text{for every } x' \in \mathbb{R}^{n-1}. \end{array} \tag{3.22}$$

for any $a \in \mathbb{R}$. Clearly the Dirichlet problem (3.22) does not admit a unique solution.

Before we end our discussion about the Dirichlet problem in $\mathbb{R}^n_+$ we should mention something about the conclusion in Theorem 3.2 that $\lim_{x_n\to 0^+} u(x', x_n) = g(x')$ uniformly on compact sets. We start by an example.

Example: *There are infinitely many solutions to the following Dirichlet problem in $\mathbb{R}^n_+$:*

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^{n-1} \\ \lim_{x_n\to 0^+} u(x', x_n) = 0 & \text{for all } x' \in \mathbb{R}^{n-1} \lim_{x\to\infty, x_n>0} u(x) = 0. \end{array} \tag{3.23}$$

Only one of these solutions, the trivial solution $u(x) = 0$, is bounded.

To see that there are infinitively many solutions we just notice that for each $a \in \mathbb{R}$ and $i = 1, 2, ..., n-1$ the function

$$u_a(x) = a\frac{x_i x_n}{|x|^{n+2}}$$

solves (3.23). In particular, u_a is just a constant multiple of the derivative of the Poisson kernel $\frac{\partial K(x,0)}{\partial x_i}$ which is harmonic in $\mathbb{R}^n_+$. That $\lim_{x\to\infty, x_n>0} u(x) = 0$ follows easily from $|u_a(x)| \le \frac{a}{|x|^n} \to 0$ as $|x| \to \infty$. The proof that $u(x', x_n) \to 0$ as $x_n \to 0^+$ splits up into two cases. If $|x'| = \delta \ne 0$ then $|u_a(x)| \le \frac{a x_n}{|\delta|^{n+1}} \to 0$ as $x_n \to 0^+$ and if $|x'| = 0$ then $x_i = 0$ and thus $u_a(x) = 0$.

Clearly, if $a \neq 0$ then the limit $\lim_{x_n \to 0^+} u_a(x', x_n) = 0$ is not uniform since u_a is not bounded close to $x = 0$.

We have not developed enough theory, yet, to show that $u(x) = 0$ is the only bounded solution. But we will in the next few weeks.

This example shows that the solution defined in Theorem 3.2 is a particularly good solution. And that we have to be very careful when we investigate uniqueness properties of the solutions. In general, it is not enough that the boundary values are obtained in a limit sense for the solution to be unique. That is we need the solution to the Dirichlet problem in Ω is continuous up to the boundary for the solution to be unique.

3.4 The Green's function in B_r.

In this section we will repeat the analysis in the previous section for the domain $\Omega = B_r(0)$. We will leave some calculations for the reader.

For every $x \in B_r(0)$ we need to find a solution to

$$\begin{array}{ll} \Delta \phi^x(\xi) = 0 & \text{in } B_r(0) \\ \phi^x(\xi) = N(\xi - x) & \text{on } \partial B_r(0). \end{array}$$

As before we want to use the particular symmetry of the domain to explicitly calculate ϕ^x. To do that we need the following definition and Lemma.

Definition 3.3. *For any $x \in \mathbb{R}^n$ we denote*

$$x^* = \frac{r^2 x}{|x|^2} \qquad \text{if } |x| \neq 0.$$

We say that x^* is the reflection of x in $\partial B_r(0)$. And if u a function defined in Ω then we say that

$$u^*(x) = \frac{r^{n-2}}{|x|^{n-2}} u(x^*) \text{ for } x^* \in \Omega$$

is the Kelvin transform of u.

Lemma 3.4. *The * operator maps the ball $B_r(0)$ onto $\mathbb{R}^n \setminus \overline{B_r(0)}$:*

$$\{x^*;\ x \in B_r(0) \setminus \{0\}\} = \mathbb{R}^n \setminus \overline{B_r(0)}.$$

Furthermore $(x^*)^* = x$ for all $x \neq 0$.

Proof: Clearly if $|x| < r$ then

$$|x^*| = \left| \frac{r^2}{|x|} \right| > r.$$

Also

$$(x^*)^* = \left(\frac{r^2 x}{|x|^2}\right)^* = \frac{r^2 \frac{r^2 x}{|x|^2}}{r^4 \frac{|x|^2}{|x|^4}} = x.$$

$\square$

Lemma 3.5. *Assume that u^* is harmonic on Ω. Then u^* is harmonic on*

$$\Omega^* = \{x;\ x^* \in \Omega\}.$$

Proof: The proof is a straightforward, although rather tedious, calculation

$$\Delta u^*(x) = \Delta\left(\frac{r^{n-2}}{|x|^{n-2}} u\left(\frac{r^2 x}{|x|^2}\right)\right) = 0.$$

$\square$

Next we notice that for $x \in B_r(0)$ we have that $N(\xi - x)$ is harmonic in both x and ξ for $x \neq \xi$. We want to do the Kelvin transform of $N(y-x)$ with respect to $x \in B_r(0)$. That is

$$N^*(\xi - x) = \begin{cases} -\frac{r^{n-2}}{|x|^{n-2}} \frac{1}{(n-2)\omega_n} \frac{1}{|\xi - x^*|^{n-2}} = -\frac{r^{n-2}}{(n-2)\omega_n} \frac{1}{(|x||x^* - \xi|)^{n-2}} & \text{if } x \neq 0 \\ -\frac{1}{(n-2)\omega_n} & \text{if } x = 0, \end{cases}$$

when $n > 2$.

By Lemma 3.5 that $\Delta_\xi N^*(\xi - x) = 0$ whenever $\xi \neq x^*$. In particular, since $x \in B_r(0)$ so $x^* \notin B_r(0)$, we have that for every $\xi \in B_r(0)$ $x^* \neq xi$ and thus $\Delta_\xi N^*(\xi - x) = 0$.

We have that if $\xi \in \partial B_r(0)$ then

$$\begin{aligned} N^*(\xi - x) &= -\frac{1}{(n-2)\omega_n} \frac{r^{n-2}}{(|x|^2 |x^* - \xi|^2)^{(n-2)/2}} = \\ &= -\frac{1}{(n-2)\omega_n} \frac{r^{n-2}}{\left(\left|\frac{r^2 x}{|x|} - \xi|x|\right|^2\right)^{(n-2)/2}} = \\ &= -\frac{1}{(n-2)\omega_n} \frac{r^{n-2}}{(r^4 + 2r^2 x \cdot \xi + r^2|x|^2)^{(n-2)/2}}, \end{aligned} \tag{3.24}$$

where we used that $|\xi|^2 = r^2$ since $\xi \in \partial B_r(0)$. Again using that $|\xi|^2 = r^2$ we deduce that $r^4 + 2r^2 x \cdot \xi + r^2|x|^2 = r^2|\xi - x|^2$. So we may write (3.24) as

$$N^*(\xi - x) = -\frac{1}{(n-2)\omega_n} \frac{1}{|\xi - x|^{n-2}} = N(\xi - x)$$

for all $\xi \in \partial B_r(0)$. In particular $N^*(\xi - x)$ satisfies the criteria of being the corrector $\phi^x(\xi)$ in the definition of the Green's function. All these calculations also works for $n = 2$. We have thus proved the following Lemma

Lemma 3.6. *The Green's function for $B_r(0)$ is*

$$G(x,\xi) = N(\xi - x) - N^*(\xi - x),$$

where

$$N^*(\xi - x) = \begin{cases} -\frac{1}{(n-2)\omega_n}\frac{r^{n-2}}{(|x||x^*-\xi|)^{n-2}} & \text{if } x \neq 0 \\ -\frac{1}{(n-2)\omega_n} & \text{if } x = 0, \end{cases}$$

when $n > 2$ and

$$N^*(\xi - x) = \begin{cases} -\frac{1}{2\pi}\left(\ln(|\xi - x^*|) - \ln(r^2)\right) = & \text{if } x \neq 0 \\ -\frac{1}{2\pi} & \text{if } x = 0, \end{cases}$$

when $n = 2$.

When we know the Green's function we can calculate the Poisson kernel for the ball.

Lemma 3.7. *The Poisson kernel for the ball $B_r(0)$ is*

$$K(x,\xi) = \frac{r^2 - |x|^2}{\omega_n r}\frac{1}{|x-\xi|^n}.$$

Proof: The proof is a simple calculation. We know, for $n > 2$, that

$$G(x,\xi) = N(\xi - x) - N^*(\xi - x) =$$

$$-\frac{1}{(n-2)\omega_n}\frac{1}{|x-\xi|^{n-2}} + \frac{r^{n-2}}{(n-2)\omega_n}\frac{1}{(|x|\,|\tilde{x} - \xi|)^{n-2}}.$$

The outward normal of $B_r(0)$ is $\nu = \frac{\xi}{|\xi|} = \frac{\xi}{r}$ which implies that for $|\xi| = r$ we have

$$K(x,\xi) = \frac{\xi}{r}\cdot\nabla_\xi G(x,\xi) = \sum_{j=1}^n \frac{\xi_j}{r}\left(\frac{1}{\omega_n}\frac{x_j - \xi_j}{|x-\xi|^n} - \frac{r^n}{\omega_n}\frac{x_j - |x|^2\xi_j}{(|x||\tilde{x}-\xi)|)^n}\right) =$$

$$= \frac{\xi}{r}\cdot\nabla_\xi G(x,\xi) = \sum_{j=1}^n \frac{\xi_j}{r}\left(\frac{1}{\omega_n}\frac{x_j - \xi_j}{|x-\xi|^n} - \frac{1}{\omega_n}\frac{x_j - |x|^2\xi_j}{(|x-\xi)|)^n}\right) \tag{3.25}$$

where we used the same argument as in and the lines following (3.24). Simplifying (3.25) we get

$$K(x,\xi) = \frac{r^2 - |x|^2}{\omega_n r}\frac{1}{|x-\xi|^n}.$$

□

Continuing as we did with the Poisson's equation in $\mathbb{R}^n_+$ we need the following Lemma.

Lemma 3.8. *Let $K(x,\xi)$ be the Poisson kernel for a $B_r(0)$ then for each $x \in B_r(0)$*

$$\int_{\partial B_r(0)} K(x,\xi) dA_{\partial B_r(0)}(\xi) = 1.$$

Proof: We know, Theorem 3.1, that if $\Delta u(x) = 0$ in $B_r(0)$ and $u(x) = g(x)$ on $\partial B_r(0)$ then

$$u(x) = \int_{\partial\Omega} \left(g(\xi) \frac{\partial G(x,\xi)}{\partial \nu} \right) dA_{\partial\Omega}(\xi) = \tag{3.26}$$
$$= \int_{\partial\Omega} g(\xi) K(x,\xi) dA_{\partial\Omega}(\xi),$$

where we also used the definition of $K(x,\xi)$ in the last equality.

Clearly $u(x) = 1$ is a C^2 solution to the Dirichlet problem with $g(x) = 1$. Inserting this in (3.26) gives the lemma. □

We are now ready to state the main Theorem of this section. The proof is parallel to the proof of Theorem (3.2) and left to the reader (see the exercises).

Theorem 3.3. *Let $g \in C(\partial B_r(0))$ and define*

$$u(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{\omega_n r} \frac{1}{|x-y|^n} g(y) dA_{\partial B_r(0)}(y). \tag{3.27}$$

Then $u \in C^2(B_r 0)$ and

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } B_r(0) \\ \lim_{s\to 1^-} u(sx) = g(x) & \text{uniformly for every } x \in \partial B_r(0). \end{array} \tag{3.28}$$

Notice that the second line in (3.28) only says that u satisfies the boundary conditions in some sense.

3.5 Exercises:

Exercise 1: Let $K((\xi,x)$ be the Poisson kernel for the half space $\mathbb{R}^n_+$. Prove that $\Delta_x K(\xi,x) = 0$ for all $\xi \in \partial\mathbb{R}^n_+$. Conclude that an integral of the kind $\int_{\mathbb{R}^{n-1}} K(x,\xi')g(\xi')d\xi'$ is nothing more that a summation of harmonic functions $K(x,\cdot)$.

Exercise 2: Verify that $v(x) = x_n$ is a solution to

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u(x) = 0 & \text{on } \partial\mathbb{R}^n_+. \end{array}$$

Define $u(x)$ as in Theorem 3.2 and verify that $u(x)+av(x)$ is a solution to (3.17) for any $a \in \mathbb{R}$.

Draw the conclusion that the solution to (3.17) are not unique.

Exercise 3: We say that the functions $K_\epsilon(x,\xi)$ defined for every $\epsilon > 0$ and $x,\xi \in \mathbb{R}^n$ is a family of "Good Kernels" if

1. For every $\epsilon > 0$ and every $x \in \mathbb{R}^n$ the function $K_\epsilon(x, \xi)$ is integrable in ξ and
$$\int_{\mathbb{R}^n} K_\epsilon(x, \xi) d\xi = 1.$$
2. For every $\epsilon > 0$ and $x \in \mathbb{R}^n$ there is a constant C that is independent of ϵ such that
$$\int_{\mathbb{R}^n} |K_\epsilon(x, \xi)| \, d\xi \leq C.$$
3. For every, $\delta > 0$, and every $x \in \mathbb{R}^n$
$$\int_{\mathbb{R}^n \setminus B_\delta(x)} |F(x, \xi)| d\xi \to 0 \text{ as } \epsilon \to 0.$$

A: Prove the Poisson kernel $K(x, \xi')$ for $\mathbb{R}^n_+$ is a family of "Good Kernels" on $\mathbb{R}^{n-1}$ if we interpret $K(x, \xi') = \tilde{K}_{x_n}(x', \xi')$ with $x_n > 0$ playing the role of ϵ.

B: Prove that if $F_\epsilon(x, \xi)$ is a family of "Good Kernels", g is continuous and bounded on $\mathbb{R}^n$ and
$$u_\epsilon(x) = \int_{\mathbb{R}^n} K_\epsilon(x, \xi) g(\xi) d\xi.$$
Then $\lim_{\epsilon \to 0^+} u(x) = g(x)$.

Hint: *Look at step 2 in the proof of Theorem 3.2. As a matter of fact, my main reason for putting this exercise here is to force you to think through that proof.*

C: Can you formulate what it would mean for $K_\epsilon(x, \xi)$ to be a family of "Good Kernels" on the unit sphere $\partial B_1(0)$? Use this to prove Theorem 3.3.

Exercise 4: Assume that $u(x) \in C^2_{\text{loc}}(D)$ and that $\Delta u(x) = 0$ in D. Assume furthermore that $B_r(x^0) \subset D$ is any ball. Prove that
$$u(x^0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(x) dA_{\partial B_r(x^0)}(x).$$
This is known as *"The mean value property for harmonic functions"* since it states that if $u(x)$ is harmonic in a domain then $u(x^0)$ is equal to the mean value of u on the boundary of any ball with center at x^0.

Hint: *Can you use Theorem 3.3?*

Exercise 5: Show that if $u, v \in C^2(B_r(0)) \cap C^1(\overline{B_1(0)})$ both solve the Dirichlet problem
$$\begin{array}{ll} \Delta w(x) = 0 & \text{in } B_1(0) \\ w(x) = g(x) & \text{on } \partial B_1(0) \end{array}$$
then $u(x) = v(x)$ for all $x \in B_1(0)$. This shows that $C^2(B_r(0)) \cap C^1(\overline{B_1(0)})$ solutions to the Dirichlet problem in $B_1(0)$ are unique.

Hint: *Representation formulas are great! They tell us exactly what the solutions to the problem are.*

Exercise 6: Derive a representation formula for the solutions to the following Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } B_1^+(0) = \{x \in B_1(0);\ x_n > 0\} \\ u(x) = f(x) & \text{for } x \in (\partial B_1(0))^+ = \{x \in \partial B_1(0);\ x_n > 0\} \\ u(x) = g(x) & \text{for } x \in B_1(0) \cap \{x;\ x_n = 0\} \end{array}$$

Where $f(x)$ and $g(x)$ are given functions.

HINT: *First prove that if we define the function $\hat{f}$ on $\partial B_1(0)$ according to*

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } x \in (\partial B_1(0))^+ \\ -f(x', -x_n) & \text{for } x \in (\partial B_1(0))^- . \end{cases}$$

Then the solution, $v(x)$, to the Dirichlet problem in $B_1(0)$ that satisfies $u(x) = \hat{f}(x)$ on $\partial B_1(0)$ also satisfies $v(x', 0) = 0$. Use this together with Theorem 3.2 to get your representation formula - it doesn't have to be pretty.

Chapter 4

An interlude on the Path we will take - why go abstract?

So far we have been able to prove that for any $f \in C_c^\alpha(\mathbb{R}^n)$ we can solve $\Delta u(x) = f(x)$ in $\mathbb{R}^n$. Also, by using very similar ideas, we where able to solve the simple Dirichlet problems

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } D \\ u(x) = g(x) & \text{on } \partial D \end{array} \tag{4.1}$$

for the simple domains $D = \mathbb{R}^n_+$ and $D = B_r(0)$. With a little bit of work we could also, Exercise 6 from last installment of the lecture notes, solve the Dirichlet problem for the simple domain $D = B_1^+(0)$.

However, in many applications we would like to solve a PDE on a very complicated domain. For instance, if we want to solve a problem involving turbulence we might want to solve a PDE describing the motion of air in a domain D that consists of $\mathbb{R}^3$ minus the shape of an airplane.

The method of solving a PDE by means of a Green's function involves finding the functions $\phi^x(y)$, that is solving the Dirichlet problem with boundary data $N(x-y)$, which we could only do for very simple domains. Even for fairly simple domains such as the one consisting of three overlapping circles in figure 4 we do not know how to calculate $\phi^x(y)$ - and thus not how to calculate the Green's function. We need to move into the abstract theory and give up any hope of finding explicit representation formulas.

Since the only way we know (at least from this course) to solve PDE in a domain is by means of a Green's function that is the only thing we can use in solving the Dirichlet problem for a more complicated domain, such as the domain consisting of three circles.

So let us try to hammer out an approach on how to solve (4.1) for $D =$ "*the union of three circles* $\mathbf{A}_1$, $\mathbf{A}_2$ *and* $\mathbf{A}_3$" and $f(x) = 0$. We may use the Green's

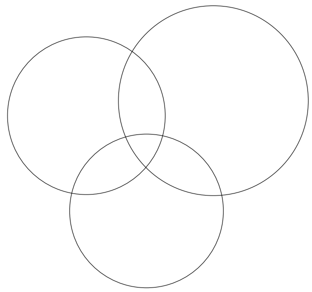

Figure 4.1: Domain consisting of Three Circles.

function to find a solution, lets call it $u^1(x)$, to the Dirichlet problem in $\mathbf{A}_1$ with boundary data $g(x)$ on the part of $\partial\mathbf{A}_1$ where $g(x)$ is defined and 0 on the other part of $\partial\mathbf{A}_1$. If we let v^1 be the function u^1 extended by 0 to the rest of D we have created a function v^1 that is harmonic in $D \setminus (\partial\mathbf{A} \cap D)$.

We may continue and use the Green's function to find a harmonic function, lets call it $u^2(x)$, in $\mathbf{A}_2$ with boundary data $g(x)$ on the part of $\partial\mathbf{A}_2$ where $g(x)$ is defined and $u^1(x)$ on the other part of $\partial\mathbf{A}_2 \cap D$ and boundary data equal to zero on the rest of $\partial\mathbf{A}_1 \cap D$. We may define

$$v^2(x) = \begin{cases} v^1(x) & \text{in } D \setminus \mathbf{A}_2 \\ u^2(x) & \text{in } \mathbf{A}_2. \end{cases}$$

Inductively we may create sequences $u^k(x)$ and $v^k(x)$ such that for any $l = 0, 1, 2, 3, ...$ and $j \in \{1, 2, 3\}$ the function u^{3l+j} solves the Dirichlet problem in $\mathbf{A}_j$:[1]

$$\begin{array}{ll} \Delta u^{3l+j}(x) = 0 & \text{in } \mathbf{A}_j \\ u^{3l+j}(x) = g(x) & \text{on } \partial\mathbf{A}_j \setminus \partial D \\ u^{3l+j}(x) = v^{3l+j-1}(x) & \text{on } \partial\mathbf{A}_j \cup D \end{array} \tag{4.2}$$

and

$$v^{3l+j} = \begin{cases} v^{3l+j-1}(x) & \text{in } D \setminus \mathbf{A}_j \\ u^{3l+j}(x) & \text{in } \mathbf{A}_j. \end{cases}$$

Notice that $v^{3l+j}(x)$ is then harmonic in $\mathbf{A}_j$ and that $v^{3l+j} = g(x)$ on ∂D for every $l \geq 1$ and $j = 1, 2, 3$. So if $\lim_{k\to\infty} v^k(x)$ converges to some function $u(x)$ then $u(x) = \lim_{l\to\infty} v^{3l+j}(x)$ in $\mathbf{A}_j$ for $j = 1, 2, 3$. That is $u(x)$ would be the limit of a sequence of harmonic functions in $\mathbf{A}_j$ for $j = 1, 2, 3$.

This leads to two questions:

1. Can we show that $\lim_{k\to\infty} v^k(x)$ exists?

2. Is harmonicity preserved under limits? That is, if a sequence of harmonic functions $v^{3l+j}(x) \to u(x)$ as $l \to \infty$ will it follow that $u(x)$ is harmonic?

[1]Since $\mathbf{A}_j$ is a ball we have no difficulties to solve this Dirichlet problem by means of a Green's function.

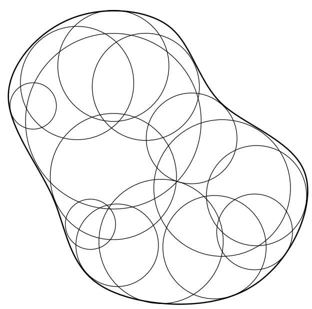

Figure 4.2: An open Domain covered by Balls.

If the answer to both these questions are affirmative then we know how to construct a solution to the Dirichlet problem, even though we don't have an explicit solution formula.

Before we try to make a brief outline of the theory that lies ahead. We will indicate that the three balls domain described above isn't as special as it looks. We could have used the same approach for a domain consisting of four, five on N balls. And if we can solve the Dirichlet problem for a domain that is the union of a finite number of balls then we should be able to use some analysis to to solve the Dirichlet problem for any domain that is the union of an infinite number of balls. Observe that any open domain is the union of all the balls in its interior.

So let us briefly indicate how we could attack the Dirichlet problem for a general domain D using the strategy used for the domain consisting of three balls. The natural way to approach this problem would be to we start with a function $v^0(x)$ defined on that domain with boundary data $g(x)$. Then we define a new function

$$v^k(x) = \begin{cases} u^k(x) & \text{in } B_r(x^0) \subset D \\ v^{k-1}(x) \text{ in } D \setminus B_r(x^0) \end{cases} \tag{4.3}$$

for some ball $B_r(x^0) \subset D$ and u^k being a harmonic function, constructed by means of a Green's function, in $B_r(x^0)$ with boundary data $v^{k-1}(x)$. This way we can construct a sequence $v^k(x)$ that hopefully converge to a harmonic function.

The problem with this approach in a general domain is that the choice of the ball $B_r(x^0)$ was a quite arbitrary choice among infinitively many balls $B_r(x^0) \subset D$. With this arbitrariness we can not expect that $v^k(x)$ converges to a unique solution.[2] So we can not rely on an arbitrary choice of the ball $B_r(x^0)$.

Before we explain how to get rid of the problem with the arbitrary choice of the ball $B_r(x^0)$ in (4.3) let us say something brief about the convergence of

[2]If we choose the ball $B_{r/2^k}(x^0)$ in the construction of $v^k(x)$ then every function $v^k(x)$ would equal v^0 in $D \setminus B_r(0)$ so unless our starting function was harmonic in $D \setminus B_r(x^0)$ there is no chance that the limit would be harmonic in D.

$v^k(x)$. There are many ways to prove convergence of sequences of functions, but one of the simplest ways to assure convergence is to have a bounded and monotone sequence. So if, for every $k = 1, 2, 3, ...$, $v^{k-1}(x)$ had the property that $v^k(x) \geq v^{k-1}(x)$ then the convergence of the sequence $v^k(x)$ would be easy.

If we could identify some functions $\mathcal{S}$ that has the property that if $v^{k-1}(x) \in \mathcal{S}$ then $v^k(x)$ defined as in (4.3) would satisfy $v^k(x) \geq v^{k-1}(x)$ and $v^k(x) \in \mathcal{S}$ for any ball $B_r(x^0) \subset D$ then it would follow that $v^0(x) \leq v^1(x) \leq ... \leq v^k(x) \leq ....$ So if $v^k(x)$ would be bounded then it would be pointwise convergent to some function $u(x)$.

But if every function in $\mathcal{S}$ is bounded then we could define

$$u(x) = \sup_{v \in \mathcal{S}} v(x) = \sup_{v \in \mathcal{S}} \tilde{v}(x) \tag{4.4}$$

where

$$\tilde{v}(x) = \begin{cases} w(x) & \text{in } B_r(x^0) \subset D \\ v(x) \text{ in } D \setminus B_r(x^0) \end{cases} \tag{4.5}$$

where w is harmonic in $B_r(x^0)$ and equal to v on $\partial B_r(x^0)$. That (4.4) holds, for any ball $B_r(x^0) \subset D$, would follow from $v \leq \tilde{v} \in \mathcal{S}$ if $v \in \mathcal{S}$.

Notice that by considering the supremum over $\mathcal{S}$ we no longer make any choice of $B_r(x^0)$. The supremum assures that we take all balls $B_r(x^0) \subset D$ into consideration simultaneously.

So the strategy to show existence of solutions in a general domain would involve:

1. To identify a class $\mathcal{S}$ such that if $v \in \mathcal{S}$ then $v(x) \leq \tilde{v}(x)$ and $\tilde{v} \in \mathcal{S}$ where $\tilde{v}(x)$ is defined by (4.5). The class $\mathcal{S}$ will be all the sub-harmonic functions.

2. Since we will be taking a supremum over $\mathcal{S}$ we will have to understand the limit properties[3] of harmonic functions. In particular, we have to prove that if $\tilde{v}^k$ is harmonic in $B_r(x^0)$ and $\tilde{v}^k \to u$ in $B_r(x^0)$ will u be harmonic?

Step 1 of the strategy: We would like to define $\mathcal{S}$ so that $v \in \mathcal{S}$ implies $v \leq \tilde{v}$ for any ball $B_r(x^0)$. It is easy to find such a condition on $\mathcal{S}$. In particular, if $v(x) > \tilde{v}(x)$ for some point in $x \in B_r(x^0)$ then, since $v = \tilde{v}$ on $\partial B_r(x^0)$, the function $v(x) - \tilde{v}(x)$ has a strictly positive maximum at some point $\hat{x} \in B_r(x^0)$. At $\hat{x}$ we have, by first year calculus, that $\frac{\partial^2 v(\hat{x}) - \tilde{v}(\hat{x})}{\partial x_i^2} \leq 0$. Summing from $i = 1, ..., n$ we deduce that $0 \geq \Delta(v(\hat{x}) - \tilde{v}(\hat{x})) = \Delta v(\hat{x})$. This implies that if $\Delta v(x) \geq 0$ then there are no $x \in B_r(x^0)$ such that $v(x) > \tilde{v}(x)$. So we are tempted to define the class $\mathcal{S}$ to be the class of all functions $v(x)$ such that $\Delta v(x) \geq 0$.

But the other condition we impose on $\mathcal{S}$ is that $\tilde{v} \in \mathcal{S}$. For us to state that $\Delta \tilde{v} \geq 0$ we need to know that $\tilde{v} \in C^2(D)$. But even if $v(x) \in C^2(D)$ it will

[3] In taking the supremum we may consider a sequence $v^k \in \mathcal{S}$ such that $v^k(x) \to \sup_{v \in \mathcal{S}} v(x)$. So taking the supremum and a limit are more or less equivalent.

not follow that $\tilde{v}$ is C^2, or even differentiable, on $\partial B_r(x^0)$.[4] We will therefore have to find another way to define the class $\mathcal{S}$ without using derivatives. As a matter of fact we will find a way to define harmonic functions without referring to derivatives.

Step 2 of the strategy: Secondly we need to understand the convergence properties of harmonic functions. To that end we can not use monotonicity but we have to rely on compactness. We want to show that if $\tilde{v}^k(x)$ is a sequence of harmonic, and thus C^2, functions in $B_r(x^0)$ that converges to $u(x)$ then $u(x)$ is harmonic. It is enough to show that the second derivatives of $\tilde{v}^k$ converges.

In general, by the Arzela-Ascoli Theorem, it is enough for a bounded sequence of continuous functions to be equicontinuous in order for a subsequence to converge to a continuous function. Therefore we need to show that the second derivatives of $\tilde{v}^k$ are equicontinuous. This leads us to one of the more complicated aspects of the theory of partial differential equations: the regularity theory. Regularity theory involves proving that the solutions to partial differential equations are regular, that is have a certain number of derivatives defined - preferably also being able to say that the derivatives are bounded in terms of the given data.[5]

In this case we will prove that the third derivatives of $\tilde{v}^k$ are bounded uniformly which implies that the second derivatives are equicontinuous and thus convergent.

Once we have understood sub-harmonic functions and the convergence properties of harmonic functions we will be able to prove existence of solutions for general domains using the strategy outlined above - a method called **Perron's method**. That proof will be quite long and complicated.

When we consider the $\sup_{v\in\mathcal{S}} v(x)$ we do not address the issue of the boundary values. So we have to prove that our solution satisfy the boundary values[6] in a separate Theorem.

When we solve the Dirichlet problem in a general domain D we can not hope to find an explicit solution. Imagine how complicated such an explicit solution would have to be, it would have to be a function from the set of domains D, functions f and g and points $x \in D$ to the value $u(x)$ where $\Delta u(x) = f(x)$ in D and $u(x) = g(x)$ on ∂D. Just to find a reasonable way to define the space of all domains D would be rather complicated. We have to move into an abstract theory because the Dirichlet problem is very complicated and we have very few tools.

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[4] Take for instance $v(x) = |x|^2 - 1$ and $D = B_1(0)$, then $\Delta v(x) = 2n \geq 0$ but if we use $x^0 = 0$ and $r = \frac{1}{2}$ in the definition of $\tilde{v}(x)$ we will get

$$\tilde{v}(x) = \begin{cases} -\frac{3}{4} & \text{in } B_{1/2}(0) \\ |x|^2 - 1 & \text{in } B_1(0) \setminus B_{1/2}(0) \end{cases}$$

which is clearly not differentiable on $\partial B_{1/2}(0)$. However, as a distribution $\Delta \tilde{v}$ is well defined and $\Delta \tilde{v}(x) \geq 0$. But we will not discuss the theory of distributions in this course.

[5] By data I mean the domain D, the right hand side f and the boundary data g.

[6] Or does it? Under what assumptions on the domain?

Chapter 5

The Mean value Property.

If $u \in C^2(B_r(0)) \cap C(\overline{B_r(0)})$ is given by the Poisson integral then

$$u(0) = \int_{\partial B_r(0)} K(x,y)u(y)dA_{\partial B_r(0)}(y) = \int_{\partial B_r(0)} \frac{r^2}{\omega_n r}\frac{1}{|y|^n}u(y)dA_{\partial B_r(0)}(y) = \tag{5.1}$$

$$= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(0)} u(y)dA_{\partial B_r(0)}(y),$$

where we have used that $|y| = r$ on $\partial B_r(0)$. In particular, it follows that $u(0)$ equals the mean value of $u(y)$ on the boundary of $\partial B_r(0)$. This is a very powerful property and it is true for all harmonic functions.

Theorem 5.1. [The Mean Value Theorem.] *Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in the domain Ω and that $B_r(x^0) \subset \Omega$. Then*

1.

$$u(x^0) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x^0)} u(y)dA_{\partial B_r(x^0)}(y)$$

2. and

$$u(x^0) = \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u(y)dy.$$

Remark: *The calculation in (5.1) constitutes a proof of the first statement. We will however provide a different proof that directly uses that $\Delta u = 0$. First of all this proof is classical and should be included in the course. Secondly, we will have reason to investigate the mean value property for solutions to $\Delta u(x) \geq 0$ and for those solutions the proof given here will be easier to utilize.*

Proof: By translation invariance of the Laplace equation we may assume that $x^0 = 0$. That is the function $\tilde{u}(x) = u(x + x^0)$ is harmonic if u is. It is therefore enough to prove the theorem for $\tilde{u}(x)$ with $x^0 = 0$. By this, there is not loss of generality to assume that $x^0 = 0$ from the start.

Assuming that $x^0 = 0$ and making a change of variables in the mean value formula $rz = y$ we see that, defining the function $\Psi(r)$,

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} u(y) dA_{\partial B_r(x)}(y) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(rz) dA_{\partial B_1(0)}(z).$$

Taking the derivative with respect to r we see that

$$\Psi'(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} z \cdot \nabla u(rz) dA_{\partial B_1(0)}(z) = \tag{5.2}$$

$$= \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{\partial u(rz)}{\partial \nu} dA_{\partial B_1(0)}(z) = \frac{1}{\omega_n} \int_{B_1(0)} \Delta u(rz) dz = 0$$

since u is harmonic. We also used the divergence theorem in the second to last equality. In particular $\Psi(r)$ =constant= $\lim_{r\to 0} \Psi(r)$. Since $u \in C(\Omega)$ we have

$$\Psi(r) = \lim_{r \to 0} \Psi(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} \lim_{r\to 0} u(rz) dA_{\partial B_1(0)}(z) =$$

$$= \frac{1}{\omega_n} \int_{\partial B_1(0)} u(0) dA_{\partial B_1(0)}(z) = u(0).$$

This proves the first version of the mean value Theorem.

To prove the second part of the mean value Theorem we use polar coordinates.

$$\frac{n}{\omega_n r^n} \int_{B_r(x^0)} u(y) dy = \frac{1}{n\omega_n r^n} \int_0^r \left(\int_{\partial B_s(0)} u(y) dA_{\partial B_s(0)}(y) \right) ds.$$

Using the mean value Theorem on spheres we see that the integral in the brackets can be evaluated

$$\int_{\partial B_s(0)} u(y) dA_{\partial B_s(0)}(y) = \omega_n s^{n-1} u(0).$$

This implies that

$$\frac{n}{\omega_n r^n} \int_{B_r(x^0)} u(y) dy = \frac{n}{\omega_n r^n} \int_0^r \omega_n s^{n-1} u(0) ds = u(0).$$

This concludes the proof. □

As a matter of fact the mean value property characterises harmonic functions as the following Corollary shows.

Corollary 5.1. *Assume that $u \in C^2(\Omega)$ and that u satisfies the mean value property in Ω. That is, for every ball $\overline{B_r(x^0)} \subset \Omega$ the following equality holds*

$$u(x^0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y). \tag{5.3}$$

Then u is harmonic in Ω.

Proof: We will argue by contradiction and assume that there exist an $x^0 \in \Omega$ such that $\Delta u(x^0) \neq 0$ and derive a contradiction. For definiteness we assume that $\Delta u(x^0) = \delta > 0$.

Since $u \in C^2(\Omega)$ there exist an $r_\delta < \mathrm{dist}(x^0, \partial\Omega)$ such that

$$\left|\Delta u(x^0) - \Delta u(y)\right| < \frac{\delta}{2}$$

for all y such that $|x^0 - y| < r_\delta$. In particular $\Delta u(y) > \delta/2$ in $B_{r_\delta}(x^0)$.

Define $\Psi(r)$ according to

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y).$$

That is, using the mean value property (5.3), $\Psi(r) = u(x^0)$. It follows, for $r < r_\delta$ and using the calculation in the proof of the mean value property,

$$0 = \Psi'(r) = \frac{1}{\omega_n} \int_{B_1(x^9)} \Delta u(rz) dz > \frac{1}{\omega_n} \int_{B_1(x^0)} r^2 \frac{\delta}{2} dz > 0.$$

This is a contradiction. It follows that $\Delta u(x) = 0$ in Ω. □

Remark: *Something important, but subtle, happens in this section. We show that there is a property that is equivalent to $\Delta u(x) = 0$ for C^2 functions - the mean value property. But the mean value property is in itself independent of the function being C^2. So could we define any function, regardless of whether it is C^2 or not, to be harmonic if it satisfies the mean value property? Indeed we can, it even turns out as we will see later that the mean value property for a function u implies that $u \in C^2$. It is also through the mean value property that we will be able to define something like $\Delta v(x) \geq 0$ without assuming that $v \in C^2$ which will be a crucial step in defining the class $\mathcal{S}$ of sub-harmonic functions.*

Chapter 6

The maximum Principle.

From the mean value Theorem it follows that if $u(x)$ is harmonic in a domain Ω and if $u(x)$ equals its supremum at a point $x^0 \in \Omega$ then u must equal its supremum in every ball contained in Ω with center at x^0. It is a direct consequence that only constant harmonic functions achieve their maximum in their domain of harmonicity (if the domain is bounded and connected). The next Theorem proves this.

Theorem 6.1. [The Strong Maximum Principle.] *Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in the bounded domain Ω. Then*

$$\sup_{x \in \Omega} u(x) = \sup_{x \in \partial\Omega} u(x).$$

Furthermore if Ω is also connected and there exist a point $x^0 \in \Omega$ such that $u(x^0) = \sup_{x \in \overline{\Omega}} u(x)$ then $u(x)$ is a constant.

Proof: Lets denote $M = \sup_{x \in \overline{\Omega}} u(x)$. Since $u \in C(\overline{\Omega})$ it follows that the set

$$\Omega_M = \{x \in \Omega;\ u(x) = M\}$$

is a relatively closed set in Ω. Now assume that there is a point $x^0 \in \Omega$ such that $u(x^0) = M$ then for any r such that $0 < r < \mathrm{dist}(x^0, \partial\Omega)$ we have by the mean value property

$$M = u(x^0) = \frac{n}{\omega_n r^n} \int_{B_r(x^0)} u(y)dy \leq \frac{n}{\omega_n r^n} \int_{B_r(x^0)} M dy = M, \tag{6.1}$$

where the inequality is an equality (which it obviously is) if and only if $u(y) = M$ for all $y \in B_r(x^0)$. It follows that for any $x \in \Omega_M$ there is a ball $B_r(x) \subset \Omega_M$, that is Ω_M is an open set in Ω. Since Ω_M is both open and relatively closed in Ω it follows that Ω_M is either empty or a component of Ω.

If Ω_M is the empty set it follows that the supremum of u is attained on the boundary of Ω. If Ω_M is a component of Ω it still follows that $u(x) = M$ on the boundary of that component of Ω.

Finally, if Ω is connected and there exist an $x^0 \in \Omega$ such that $u(x^0) = M$ then it follows that $\emptyset \neq \Omega_M$ and therefore $\Omega_M = \Omega$, that is $u(x) = M$ in Ω. $\square$

Remark: If $u(x)$ is harmonic so is $-u(x)$. It is therefore an immediate consequence of this theorem that if Ω is bounded and u is harmonic in Ω then

$$\inf_{x\in\Omega} u(x) = \inf_{x\in\partial\Omega} u(x).$$

If Ω is also connected and if u attains its infimum at a point $x^0 \in \Omega$ then u is constant.

The maximum principle has many consequences, one of the most important consequences is that it implies that solutions to the Dirichlet problem are unique.

Theorem 6.2. *Let Ω be a bounded domain and suppose that $u^1, u^2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be two solutions to*

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

Then $u^1 = u^2$ in Ω.

Proof: Define $v = u^1 - u^2$ then

$$\begin{array}{ll} \Delta v(x) = 0 & \text{in } \Omega \\ v(x) = 0 & \text{on } \partial\Omega. \end{array}$$

So by the maximum principle it follows that $\sup_{x\in\Omega} v(x) \leq \sup_{x\in\partial\Omega} v(x) = 0$. Applying the maximum principle on $-v(x)$ we see that

$$-\inf_{x\in\Omega} v(x) = \sup_{x\in\Omega}(-v(x)) \leq \sup_{x\in\partial\Omega}(-v(x)) = 0.$$

It follows that $0 \leq v(x) \leq 0$, that is $v(x) = 0$ or $u^1(x) = u^2(x)$ in Ω. $\square$

Chapter 7

Sub-harmonic functions.

If we assume that $\Delta u(x) \geq 0$ in Ω and define

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} u(y) dA_{\partial B_r(0)}(y)$$

for all r such that $\overline{B_r(0)} \subset \Omega$. Then we see, following the calculation in (5.2), that

$$\Psi'(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} z \cdot \nabla u(rz) dA_{\partial B_1(0)}(z) =$$

$$= \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{\partial u(rz)}{\partial \nu} dA_{\partial B_1(0)}(z) = \frac{1}{\omega_n} \int_{B_1(0)} \Delta u(rz) dz \geq 0.$$

In particular $\Psi(r)$ is a non-decreasing function and since u is continuous we have

$$u(0) = \lim_{r \to 0^+} \Psi(r) \leq \Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} u(y) dA_{\partial B_r(0)}(y). \tag{7.1}$$

We will say that u satisfies the sub-mean value property if it satisfy (7.1).

Definition 7.1. *We say that $u \in C(\Omega)$ is sub-harmonic if it satisfies the sub-mean value property:*

$$u(x^0) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y)$$

for all $x^0 \in \Omega$ and $r \geq 0$ such that $\overline{B_r(x^0)} \subset \Omega$.

We say that $u \in C(\Omega)$ is super-harmonic if $-u(x)$ is sub-harmonic. Equivalently, $u \in C(\Omega)$ is super-harmonic if it satisfies the super-meanvalue property:

$$u(x^0) \geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y)$$

for all $x^0 \in \Omega$ and $r \geq 0$ such that $\overline{B_r(x^0)} \subset \Omega$.

Since u being sub-harmonic implies that $-u$ is super-harmonic it follows that every theorem for subharmonic functions have a corresponding theorem for super-harmonic functions.

Many of the theorems in the previous two sections have versions for sub and super-harmonic functions with very similar proofs. In particular we have the following, corresponding to Corollary 5.1.

Lemma 7.1. *Assume that $u \in C^2(\Omega)$ and that u is sub-harmonic in Ω. Then $\Delta u(x) \geq 0$ in Ω.*

Conversely if $u \in C^2(\Omega)$ and $\Delta u(x) \geq 0$ in Ω then $u(x)$ is subharmonic in Ω.

Proof: The proof is very similar to the proof of Corollary 5.1.

We will argue by contradiction and assume that there exist an $x^0 \in \Omega$ such that $\Delta u(x^0) < 0$ and derive a contradiction. For definiteness we assume that $\Delta u(x^0) = -\delta < 0$.

Since $u \in C^2(\Omega)$ there exist an $r_\delta < \text{dist}(x^0, \partial\Omega)$ such that

$$\left|\Delta u(x^0) - \Delta u(y)\right| < \frac{\delta}{2}$$

for all y such that $|x^0 - y| < r_\delta$. In particular $\Delta u(y) < -\delta/2$ in $B_{r_\delta}(x^0)$.

Define $\Psi(r)$ according to

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y).$$

It follows, for $r < r_\delta$, that

$$\Psi'(r) = \frac{1}{\omega_n} \int_{B_1(0)} \Delta u(rz + x^0) dz < -\frac{1}{\omega_n} \int_{B_1(0)} r^2 \frac{\delta}{2} dz < 0. \tag{7.2}$$

Since $u \in C(\Omega)$ it also follows that $\lim_{r \to 0^+} \Psi(r) = u(x^0)$. Using (7.2) we see that $\psi(r) < u(x^0)$ for $r \in (0, r_\delta)$. This contradicts the sub-mean value property.

The second part follows by the calculation in the beginning of this section.

$\square$

Remark: Here we use a wonderful technique of mathematics. In principle we could define $u(x)$ to be sub-harmonic if $u \in C^2(\Omega)$ and $\Delta u(x) \geq 0$. Instead we use the sub-mean value property and are able to define sub-harmonicity for functions that are only in $C(\Omega)$ which is a much more flexible class of functions. In particular, which we will show and use later, if u and v are subharmonic so is $\max(u(x), v(x))$ (this would not be true if we demanded that subharmonic functions had to be in C^2).

The Lemma shows that we are not giving up anything in our definition based on the sub-mean value property. If a sub-harmonic function happens $u(x)$ to be in $C^2(\Omega)$ then it satisfies the equation $\Delta u(x) \geq 0$.

Since our proof of the maximum principle was based on the mean value property it is not surprising that the same result holds for sub-harmonic functions

Theorem 7.1. The Strong Maximum Principle for Sub-Harmonic Functions. *Suppose that $u \in C(\overline{\Omega})$ is sub-harmonic in the bounded domain Ω. Then*

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x).$$

Furthermore if Ω is also connected and there exist a point $x^0 \in \Omega$ such that $u(x^0) = \sup_{x\in\overline{\Omega}} u(x)$ then $u(x)$ is a constant.

Proof: The proof is exactly the same as for the strong maximum principle. The only difference is that the second equality in (6.1) should be an inequality and every time we referred to the mean value property we now have to refer to the sub-mean value property. □

Next we state a theorem that will be very important in our proof of existence of solutions for the Dirichlet problem in a general domain. We will state it for super-harmonic functions, but a similar statement is also true for sub-harmonic functions.

Theorem 7.2. *Let $u, v \in C(\Omega)$ be super-harmonic functions. Define*

$$w(x) = min(u(x), v(x)).$$

Then $w(x)$ is super-harmonic.

Proof: It is clear that $w(x)$ is continuous so we only need to show that w satisfies the super-mean value property. Notice that by definition $w(x) \leq u(x)$ and $w(x) \leq v(x)$, with one of the inequalities being an equality. We will fix an arbitrary point $x^0 \in \Omega$ and for definiteness assume that $w(x^0) = u(x^0)$. Then since $u(x)$ is super-harmonic we have, for any ball $\overline{B_r(x^0)} \subset \Omega$, that

$$w(x^0) = u(x^0) \geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y) \geq$$

$$\geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} w(y) dA_{\partial B_r(x^0)}(y),$$

where we used that $w(x) \leq u(x)$ for all x. But this shows that w satisfies the super-mean value property. □

7.1 Sub and Super-Solutions.

It is possible to extend the concept of sub and super-harmonic functions to general solutions to the Dirichlet problem.

Definition 7.2. *We say that $w(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a sub-solution to*

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array} \tag{7.3}$$

for $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ if

$$\begin{array}{ll} \Delta u(x) \geq f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

Similarly we say that $w(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a super-solution to (7.3) if

$$\begin{array}{ll} \Delta u(x) \leq f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

Remark: Notice that if $u \in C^2(\Omega)$ is sub-harmonic then by Lemma 7.1 u is a sub-solution to $\Delta u(x) = 0$.

When we defined sub-harmonicity, we only needed to assume that $u \in C(\Omega)$ (see Definition 7.1) whereas we demand general sub-solutions to be C^2. It is noteworthy that there are other definitions of sub-solutions that require less stringent assumptions[1] - and most of the Theorems we show for sub-solutions would still be true. For simplicity we will assume that sub and super-solutions are $C^2(\Omega)$ for now.

The following Theorem will be important in our proof of existence of solutions to the Dirichlet problem.

Theorem 7.3. [THE COMPARISON PRINCIPLE.] *Let Ω be a bounded domain and suppose that $u^1(x) \in C^2(\Omega) \cap C(\Omega)$ be a sub-solution and $u^2(x) \in C^2(\Omega) \cap C(\Omega)$ be a super-solution to*

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array} \tag{7.4}$$

Then $u^1(x) \leq u^2(x)$ in Ω.

Proof: Notice that $w(x) = u^1(x) - u^2(x)$ solves

$$\begin{array}{ll} \Delta w(x) \geq 0 & \text{in } \Omega \\ w(x) = 0 & \text{on } \partial\Omega. \end{array}$$

That is $w(x)$ is sub-harmonic. By the maximum principle for sub-harmonic functions it follows that $w(x) \leq 0$ which implies that $u^1(x) \leq u^2(x)$. □

Notice that if $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to (7.4) then u is both a sub and a super-solution so this Theorem directly implies Theorem 6.2.

[1] See for instance Definition 24.1.

Chapter 8

Interior Regularity of Harmonic Functions.

A major part of the study of partial differential equations (PDEs), a part that can be a little difficult to grasp, is the regularity theory. Regularity theory is the branch of PDE studies that investigates how regular a solution is, basically how many derivatives the solution has and if one can bound those derivatives.

We have already seen that the mean value property is equivalent to harmonicity for C^2 functions. But we only need the function to be continuous in order to define the mean value property. So if we would define a function to be harmonic if it is continuous and satisfies the mean value property could we still make sense of the equation $\Delta u(x) = 0$?

There are many different definitions of a function being a solution to a PDE; classical solutions (solutions that are continuously differentiable), weak solutions (defined by means of integration by parts), variational solutions (functions that minimise a certain energy), viscosity solutions (solutions defined by the comparison principle) etc. The only solutions that a priori have enough derivatives to satisfy the equation in the classical sense are classical solutions. These are the solutions that we have been working with so far, we assume that a solution to $\Delta u(x) = 0$ are in C^2 which makes it unproblematic to interpret whether a given function is a solution or not.

There are several reasons that regularity theory is so important for the study of partial differential equations. One reason is that it is often easier to prove the existence of a, say, weak solution than it would be to show the existence of a classical solution. But it is of obvious interest to know if the weak solution, once we have it, is in fact a classical solution. Other reasons for doing regularity theory is that one can use regularity theory to show properties of solutions, something that we will exemplify by by the Liouiville Theorem below. Regularity is also strongly related to existence theory, often it is only possible to show that a solution exists by approximating the PDE and by a limit procedure for which we need compactness. As a final motivation we should mention that only in very special cases are we able to write down a solution to a PDE. Instead we rely on numerical analysis to calculate approximate solutions with computers. In order to verify that we actually get a good approximation, and to say how good our approximation is, we need to know something about the regularity of the solutions.

In this section we will start to do some easy regularity theory. Our first theorem states that if $u(x)$ satisfies the mean value property in Ω then $u \in C^\infty(\Omega)$.

Theorem 8.1. *Let Ω be a domain. Suppose that $u \in C(\Omega)$ and satisfies the mean value property in Ω. Then $u \in C^\infty(\Omega)$ and $\Delta u(x) = 0$ in Ω.*

Proof: It is enough to show that $u \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$ where

$$\Omega_\epsilon = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \epsilon\}.$$

Fix an $\epsilon > 0$ and define u_ϵ by means of the standard mollifier

$$u_\epsilon(x) = \int_\Omega u(y)\phi_\epsilon(x-y)dy,$$

where $\phi_\epsilon(x)$ is a standard mollifier.It follows that $u_\epsilon \in C^\infty(\Omega_\epsilon)$.[1]

We will show that $u_\epsilon(x) = u(x)$ in Ω_ϵ. This is established in the following calculation

$$u_\epsilon(x) = \int_{B_\epsilon(x)} \phi_\epsilon(x-y)u(y)dy = \left\{ \begin{array}{l} \text{change to} \\ \text{polar coordinates} \end{array} \right\} =$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \phi\left(\frac{s}{\epsilon}\right)\left(\int_{\partial B_s(x)} u(y)dA_{\partial B_s(x)}\right)ds =$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \phi\left(\frac{r}{\epsilon}\right)\omega_n s^{n-1}u(x)ds,$$

where we have used the mean value property in the last equality. Noticing that $\int_{\partial B_s(0)} dA_{\partial B_s(0)}(y) = \omega_n s^{n-1}$ we may continue the calculation

$$\frac{1}{\epsilon^n}\int_0^\epsilon \phi\left(\frac{r}{\epsilon}\right)\omega_n s^{n-1}u(x)ds =$$

$$= u(x)\int_0^r \int_{\partial B_s(0)} \frac{1}{\epsilon^n}\phi\left(\frac{r}{\epsilon}\right)dA_{\partial B_s(0)}(y)ds = u(x)\int_{B_\epsilon(0)} \phi_\epsilon(y)dy = u(x),$$

where we used that $\int \phi_\epsilon(y)dy = 1$ by the definition of ϕ in the final step. In particular it follows that $u(x) = u_\epsilon(x) \in C^\infty(\Omega_\epsilon)$ for every $x \in \Omega_\epsilon$. Since Ω is open it follows that $u \in C^\infty\left(\cup_{\epsilon>0}\Omega_\epsilon\right) = C^\infty(\Omega)$. $\square$

[1] See the appendix in the first part of these lecture notes for the definition of standard mollifiers and the C^∞-proof.

Naively, one might think that the above result, that if u is harmonic then $u \in C^\infty$, is the best possible result in regularity theory, which is after all about showing that solutions to partial differential equations have derivatives.

There are two reasons why this result is not the best possible. The first reason is that one can show (but we will not) that harmonic functions are in fact analytic (can be expressed in a Taylor series). That is if $\Delta u(x) = 0$ in the domain Ω and $x^0 \in \Omega$ then there is a ball $B_r(x^0) \subset \Omega$ such that $u(x)$ equals it Taylor expansion

$$u(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha (x - x^0)^\alpha \text{ in } B_r(x^0),$$

where we have used the multiindex notation again; $\alpha = (\alpha_1, ..., \alpha_n)$ in a multiindex and $(x - x^0)^\alpha = (x_1 - x_1^0)^{\alpha_1}(x_2 - x_2^0)^{\alpha_2}...(x_n - x_n^0)^{\alpha_n}$. That analyticity is stronger than C^∞ is easy to see since the standard mollifier $\phi(x) \in C^\infty(\mathbb{R}^n)$ but the Taylor expansion at any point on $\partial B_1(0)$ must be identically zero since all derivatives vanish on $\partial B_1(0)$. Thus we can not express ϕ by means of a Taylor series.

The other reason why the above C^∞ result is not the best possible (in every respect) is more subtle. We already know that *any* continuous function f may be approximated as closely as we want by a C^∞ function, namely f_ϵ (see Lemma 2 in the Lectures from week 5). This means that a function being in C^∞ does not mean very much, in particular convergence and compactness properties of C^∞ functions are not good.

We need *estimates* in order to deduce desirable compactness properties of solutions. By estimates we mean some inequality where we control higher derivatives by means of lower derivatives. A typical, and important, estimate is presented in the following theorem where we show that derivatives on any order of a harmonic function can be controlled by the integral of the function (that is higher derivatives are controlled by the zeroth order derivatives). Before we state the theorem we need a definition.

Definition 8.1. *If u is a function whose absolute value is integrable in Ω we write*

$$\|u\|_{L^1(\Omega)} = \int_\Omega |u(x)| dx.$$

More generally, if $|u|^p$ is integrable in Ω we write

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)| dx \right)^{1/p}.$$

Remark: We can consider the space of all integrable functions v such that $\|v\|_{L^p(\Omega)} < \infty$, call this space $L^p(\Omega)$. If $1 \le p < \infty$ then $\| \cdot \|_{L^p(\Omega)}$ is a norm on $L^p(\Omega)$. The most important result in integration theory is that $L^p(\Omega)$ is a complete space with the norm $\| \cdot \|_{L^p(\Omega)}$ if we interpret the integral in the Lebesgue sense. These considerations are not important for us in this course.

Theorem 8.2. *Suppose that $u \in C^2(\Omega)$ is harmonic in Ω. Then for each ball $B_r(x^0) \subset \Omega$ and each multiindex α of length $|\alpha| = k \geq 1$ we have the following estimate*

$$\left| \frac{\partial^{|\alpha|} u(x^0)}{\partial x^\alpha} \right| \leq \frac{n(2^{n+1}nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_r(x^0))}.$$

Proof: Since u is harmonic in Ω we know that $u \in C^\infty(\Omega)$. Writing $\frac{\partial u}{\partial x_i} = u_i$ we see by changing the order of differentiation that

$$\Delta u_i(x) = \frac{\partial}{\partial x_i}\big(\Delta u(x)\big) = \frac{\partial}{\partial x_i}\big(0\big) = 0.$$

So u_i is harmonic and satisfies therefore the mean value property. In particular for $B_r(x^0) \subset \Omega$ we may apply the mean value property to the ball $B_{r/2}(x^0)$:

$$u_i(x^0) = \frac{n2^n}{\omega_n r^n} \int_{B_{r/2}(x^0)} u_i(y)dy.$$

Taking the absolute values and integrating by parts we get

$$|u_i(x^0)| = \left| \frac{n2^n}{\omega_n r^n} \int_{B_{r/2}(x^0)} \frac{\partial u(y)}{\partial x_i} dy \right| =$$

$$= \left| \frac{n2^n}{\omega_n r^n} \int_{\partial B_{r/2}(x^0)} u(y)\nu_i dA_{\partial B_{r/2(x^0)}}(y) \right| \leq \frac{2n}{r} \sup_{\partial B_{r/2}(x^0)} \big(|u|\big) \tag{8.1}$$

where we used the notation $\nu_i = \nu \cdot e_i$ where ν is the unit normal of $\partial B_{r/2}(x^0)$ and that

$$\left| \int_{\partial B_{r/2}(x^0)} u(y) dA_{\partial B_{r/2}(x^0)}(y) \right| \leq \sup_{\partial B_{r/2}(x^0)} |u| \int_{\partial B_{r/2}(x^0)} dA_{\partial B_{r/2}(x^0)}(y) =$$

$$\frac{1}{\omega(r/2)^{n-1}} \sup_{\partial B_{r/2}(x^0)} |u|$$

in the last inequality. To estimate $\sup_{y \in \partial B_{r/2}(x^0)} \big(|u(y)|\big)$ we use the mean value formula again. Since $B_r(x^0) \subset \Omega$ we have that $B_{r/2}(y) \subset \Omega$ for each $y \in \partial B_{r/2}(x^0)$. We can therefore apply the mean value formula to the ball $B_{r/2}(y) \subset \Omega$:

$$|u(y)| \leq \frac{n2^n}{\omega_n r^n} \left| \int_{B_{r/2}(y)} u(z)dz \right| \leq \frac{n2^n}{\omega_n r^n} \int_{B_{r/2}(y)} |u(z)|dz \leq \tag{8.2}$$

$$\leq \frac{n2^n}{\omega_n r^n} \int_{B_r(x^0)} |u(z)|dz = \frac{n2^n}{\omega_n r^n} \|u\|_{L^1(B_r(x^0))},$$

where we used that $\int_{B_{r/2}(y)} |u(z)|dz \leq \int_{B_r(x^0)} |u(z)|dz$ since $B_{r/2}(y) \subset B_r(x^0)$ and that the integrand is non negative.

Taking the supremum over $\partial B_{r/2}(x^0)$ on both sides in (8.2) and inserting this in (8.1) we get

$$|u_i(x^0)| \leq \frac{n^2 2^n}{\omega_n r^{n+1}} \|u\|_{L^1(B_r(x^0))}$$

which proves the theorem for $|\alpha| = 1$.

In order to prove the Theorem for general α we will use induction on the length of $|\alpha|$. We will assume that we have proved the theorem for all multiindexes α of length $k-1$. Now fix a multiindex β of length k and assume that $\frac{\partial^{|\beta|}}{\partial x^\beta} = \frac{\partial}{\partial x_i}\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ where α is a multiindex of length $k-1$. Writing $u_\gamma(x) = \frac{\partial^{|\gamma|}u(x)}{\partial x^\gamma}$ for any multiindex γ we have for any $B_r(x^0) \subset \Omega$ that

$$|u_\beta(x^0)| = \left| \frac{nk^n}{\omega_n r^n} \int_{B_{r/k}(x^0)} \frac{\partial u_\alpha(y)}{\partial x_i} dy \right| =$$

$$= \left| \frac{nk^n}{\omega_n r^n} \int_{\partial B_{r/k}(x^0)} u_\alpha(y) \nu_i dA_{\partial B_{r/k(x^0)}}(y) \right| \leq \frac{kn}{r} \sup_{\partial B_{r/k}(x^0)} \big(|u_\alpha|\big). \tag{8.3}$$

Using the induction hypothesis we see that, for $y \in \partial B_{r/k}(x^0)$

$$|u_\alpha(y)| \leq \frac{n\big(2^{n+1}n(k-1)\big)^{k-1} k^{n+k-1}}{\omega_n\big((k-1)r\big)^{n+k-1}} \left| \int_{B_{(k-1)r/k}(y)} u(z) dz \right| \leq$$

$$\leq \frac{n\big(2^{n+1}n\big)^{k-1} k^{n+k-1}}{\omega_n (k-1)^n r^{n+k-1}} \int_{B_{(k-1)r/k}(y)} |u(z)| dz \leq \tag{8.4}$$

$$\leq \frac{n\big(2^{n+1}n\big)^{k-1} k^{n+k-1}}{\omega_n (k-1)^n r^{n+k-1}} \|u\|_{L^1(B_r(x^0))},$$

Putting (8.3) and (8.4) together we see that

$$|u_\beta(x^0)| \leq \frac{n^2\big(2^{n+1}n\big)^{k-1} k^{n+k}}{\omega_n (k-1)^n r^{n+k}} \|u\|_{L^1(B_r(x^0))} =$$

$$= \left(\frac{k^n}{2^{n+1}(k-1)^n} \right) \left(\frac{n(2^{n+1}nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_r(x^0))} \right),$$

noticing that the first bracket to the right in the last equation is less than one gives the desired estimate. □

As a direct consequence of Theorem 8.2 we state the following theorem.

Theorem 8.3. [The Liouiville Theorem] *Suppose that $u \in C_0(\mathbb{R}^n)$ is harmonic. Then if $|u(x)| \leq C_0$ for every $x \in \mathbb{R}^n$ and for some constant C_0 (C_0 is independent of x) then $u(x)$ is constant in $\mathbb{R}^n$.*

Proof: We use Theorem 8.2 for $k = 1$ and deduce that for any $j \in \{1, 2, ..., n\}$

$$\left|\frac{\partial u(x^0)}{\partial x_j}\right| \leq \frac{n^2 2^n}{\omega_n r^{n+1}} \|u\|_{L^1(B_r(x^0))} = \frac{n^2 2^n}{\omega_n r^{n+1}} \int_{B_r(x^0)} |u(y)| dy \leq \frac{n 2^n}{r} C_0. \quad (8.5)$$

If we let $r \to \infty$ in (8.5) we can deduce that

$$\left|\frac{\partial u(x^0)}{\partial x_j}\right| = 0$$

for every $x^0 \in \mathbb{R}^n$ and j. It follows that u is constant. $\square$

Corollary 8.1. *Suppose that $u \in C_0(\mathbb{R}^n)$ is harmonic. Then if $|u(x)| \leq C_0(1+|x|^{k+\alpha})$ for every $x \in \mathbb{R}^n$ and for some constant C_0, $k \in \mathbb{N}$ and $0 \leq \alpha < 1$ then $u(x)$ is a polynomial of degree at most k in $\mathbb{R}^n$.*

Proof: The argument is similar to the argument in Theorem 8.3. From Theorem 8.2 we deduce that

$$\frac{\partial^{|\beta|} u(x)}{\partial x^\beta} \leq \frac{C}{r^{1-\alpha}}$$

for any multiindex β of length $k+1$. In particular, sending $r \to \infty$ we see that the $(k+1)$:st derivatives of $u(x)$ are zero. That is the k :th derivatives of u are constant. It follows that u is a polynomial of degree at most k. $\square$

8.1 The Harnack Inequality.

In this section we will state a very important important theorem known as the Harnack inequality. At this point I am not sure if we are going to further explore its consequences in this course. We will certainly not talk more about it in the first part of the course.

Theorem 8.4. (The Harnack Inequality.) *Let Ω be a domain. Then for every connected compact set $K \subset \Omega$ there exist a constant C_K such that*

$$\sup_{x \in K} u(x) \leq C_K \inf_{x \in K} u(x)$$

for all non-negative harmonic functions u in Ω.

Proof: From the mean value property (used both in the first and in the last equality) and standard estimates we may conclude that

$$u(x) = \frac{n}{\omega_n (2r)^n} \int_{B_{2r}(x)} u(z) dz \geq \frac{n}{\omega_n (2r^n} \int_{B_r(y)} u(z) dz = \frac{1}{2^n} u(y) \qquad (8.6)$$

for any $y \in B_r(x)$. Notice that we use that $u \geq 0$ in the inequality of (8.6).

We have thus shown that

$$u(x) \geq \frac{1}{2^n} u(y) \tag{8.7}$$

for any $y \in B_r(x)$.

Next we let $r_0 = \frac{1}{4}\mathrm{dist}(K, \partial\Omega)$ and notice that for any $z \in K$ there is a path of balls (that will be chosen below), for $j = 1, 2, ..., j_0$, $B_{r_0}(y^j)$ such that $B_{r_0}(x) \cup B_{r_0}(y^j) \neq \emptyset$ and $B_{r_0}(y^j) \cup B_{r_0}(y^{j+1}) \neq \emptyset$ and $z \in B_{r_0}(y^{j_0})$. Since K is compact we see that j_0 is finite. In particular, the set $\cup_{z\in K} B_{r_0}(z)$ is an open cover of K so there is a finite sub-cover $K \subset \cup_{k=1}^N B_{r_0}(z^k)$. It follows that we may choose $y^j = z^{k_j}$ for some k_j and conclude that $j_0 \leq N$.

We may pick a sequence $\tilde{x}^0 = x$, $\tilde{x}^j \in B_{r_0}(y^j) \cap B_{r_0}(y^{j+1})$ and $\tilde{x}^{j_0+1} = z$ and apply (8.7) with $\tilde{x}^j$ in place of x and $\tilde{x}^{j+1}$ in place of y and $r = 2r_0$. Since $\tilde{x}^j, \tilde{x}^{j+1} \in B_{r_0}(y^{j+1}) \subset B_{2r_0}(\tilde{x}^j)$ we it is justified to apply (8.7).

In particular we have shown that

$$u(x) = u(\tilde{x}^0) \geq 2^{-n} u(\tilde{x}^1) \geq 2^{-n}\left(2^{-n} u(\tilde{x}^2)\right) \geq ... \geq$$

$$\geq 2^{-(j_0+1)n} u(\tilde{x}^{j_0+1}) = 2^{-(j_0+1)n} u(z).$$

But this holds for arbitrary $x, z \in K$. In particular we can choose x such that $u(x) = \inf_{y\in K}\left(u(y)\right)$ and z such that $u(z) = \sup_{y\in K}\left(u(y)\right)$. The theorem follows. $\square$

Remark: Notice that we may view the Harnack inequality as a quantitative version of the strong maximum principle. In particular if $u \geq 0$ is a harmonic function in the bounded connected domain Ω. Then by the strong maximum principle we know that $-u$ (which is also harmonic) satisfies either $-u(x) < 0$ in Ω or there exist a point $x^0 \in \Omega$ such that $-u(x^0) = 0$ in which case $-u(x) = 0$ in Ω. However the strong maximum principle says nothing if $-u(x^0) < 0$ but $|u(x^0)|$ is small.

But if we assume that $u(x^0) = \epsilon$ for some $x^0 \in \Omega$ then the Harnack inequality states that $0 \leq u(x) \leq C_K\epsilon$ for all $x \in K$, where K is some compact connected set containing x^0. If $\epsilon = 0$ then it follows that $u = 0$ on every compact set in Ω, that is $u = 0$ in Ω so we recover the strong maximum principle.

But the estimate $0 \leq u(x) \leq C_K\epsilon$ is stronger than the strong maximum principle in that it provides information even if $\epsilon > 0$.

Chapter 9

Exercises.

Exercise 1. The following Theorem is known as the weak maximum principle

Theorem: *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ where Ω is a bounded domain. Furthermore assume that $\Delta u(x) \geq 0$ in Ω. Then*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x).$$

Prove this Theorem using the following steps:

Step 1: Assume that $x \in \Omega$ and that x is a local maximum of $u(x)$ show that $\Delta u(x) \leq 0$.

(Hint: What do we know about the second derivatives at a local maximum?)

Step 2: Prove the Theorem under the assumption that $\Delta u(x) > 0$.

(Hint: If the Theorem is false can you find a contradiction to step 1?)

Step 3: Define $u_\epsilon(x) = u(x) - \epsilon|x|^2$ and show that the Theorem is true for u_ϵ. Pass to the limit $\epsilon \to 0$ and conclude that the Theorem is true for u.

Exercise 2. Assume that $u \in C(\mathbb{R}^n)$ and that for every $\phi \in C_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u(x)\phi(x) = 0.$$

Show that $u(x) = 0$.

(Hint: Assume that $u(x^0) > 0$ and let $\phi(x) = \max(\delta - |x - x^0|, 0)$ chose δ small enough and derive a contradiction.)

Exercise 3 a) Let $u \in C^2(\Omega)$ solve $\Delta u = f(x)$ in Ω, where $\Omega \subset \mathbb{R}^n$ is some domain in $\mathbb{R}^n$ and $f \in C(\Omega)$. Show that

$$\int_\Omega \big(\nabla u(x) \cdot \nabla\phi(x) + \phi(x)f(x)\big)dx = 0 \tag{9.1}$$

for every $\phi \in C_0^1(\Omega) \equiv \{\phi \in C^1(\Omega);\ \phi = 0 \text{ on } \partial\Omega\}$.

(Hint: Use Green's formula.)

b) Let $u \in C^2(\mathbb{R}^n)$ and assume that (9.1) holds for every $\phi \in C^1(\Omega)$. Prove that $\Delta u = f$.

(Hint: Look at Exercise 2.)

c) Note that the equation (9.1) makes perfectly good sense even if $f \notin C(\Omega)$ and in particular (9.1) makes sense even if $u \in C^1(\Omega)$ but $u \notin C^2(\Omega)$. We will say that u is a *weak solution* of $\Delta u = f$ if $u \in C^1(\Omega)$ and if (9.1) holds for every $\phi \in C_0^1(\Omega)$

Try to find a weak solution in $\mathbb{R}^3$ to

$$\Delta u = \begin{cases} 1 & \text{when } |x| \le 1 \\ 0 & \text{when } |x| > 1. \end{cases}$$

(Hint: Look for a radial $u(x)$, that is $u(x) = u(|x|) = u(r)$.)

d) Let u be your weak solution form c) and define

$$v(x) = \begin{cases} u(x) & \text{when } |x| \le 1 \\ u(x) + \frac{1}{|x|} - 1 & \text{when } |x| > 1. \end{cases}$$

Then v is continuous and $\Delta v = 1$ when $|x| < 1$ and $\Delta v = 0$ when $|x| > 1$. However, $v \notin C^1(\mathbb{R}^3)$ prove that v does not satisfy (9.1) and v is therefore not a weak solution.

Remark: Notice that what we do in this exercise is very similar to what we did when we defined sub-harmonic functions. Both solutions and sub-harmonic functions can be defined by using C^2. But we may relax the C^2 assumption when we define sub-harmonic functions by using the mean value formula. In the same way we can relax the notion of solution to weak solution where a weak solution is defined in a bigger function space (C^1 instead of C^2). This allows us to talk about solutions with discontinuous right hand sides such as the solution in part c).

Exercise 4. Let $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)})$ and

$$\Delta u = f(x) \quad \text{in } B_1(0) \subset \mathbb{R}^n \text{ and}$$

$$u(x) = g(x) \quad \text{on } \partial B_1(0).$$

Where $f, g \in C(\mathbb{R}^n)$ are some given functions. Show that

$$\sup_{B_1} u \le \sup_{\partial B_1(0)} g + \frac{1}{2n} \sup_{B_1} f^-(x),$$

where $f^-(x) = \max(0, -f(x))$.

(HINT: What equation will $v = u + \alpha|x|^2 - \alpha$ solve when α is a constant, when is v sub-harmonic?)

Exercise 5. Hopf's Boundary Lemma. Let $u \in C^2(\overline{B_1^+})$, where $B_1^+ = \{x \in B_1;\ x_n > 0\}$, and

$$\begin{array}{ll} \Delta u = 0 & \text{in } B_1^+ \\ u = g \in C^2 & \text{on } \partial B_1 \cap \{x_n > 0\} \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{array}$$

Assume furthermore that $0 \leq g$ and that g is not identically zero.

Then the Hopf boundary lemma states that

$$\frac{\partial u(0)}{\partial x_n} > 0,$$

the important point is that the inequality is strict. The aim of this exercise is to prove this.

a.) Let u be as above. Show that the maximum principle implies that

$$\frac{\partial u(0)}{\partial x_n} \geq 0$$

b.) Define $v(x)$ to be

$$v(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, ..., x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Show that $v \in C^2(B_1)$ and that $\Delta v = 0$ in B_1.

c.) Use the mean value formula to express $\frac{\partial v(0)}{\partial x_n}$. Use this expression to show that $\frac{\partial v(0)}{\partial x_n} > 0$.

(Hint: Let $e_n = (0, 0, 0..., 0, 1)$ as usual, then $\int_{B_1} div(e_n v(x))dx = \int_{B_1} \frac{\partial v}{\partial x_n} dx$, also if ν is the normal of $\partial B_1(0)$ then $\nu \cdot e_n > 0$ at points on ∂B_1 where $x_n > 0$...)

d.) Use b.) and c.) to prove Hopf's lemma.

Exercise 6. Suppose that $u \in C^2(\overline{\mathbb{R}^n_+})$ and that $\Delta u(x) = 0$ in $\mathbb{R}^n_+$ and $u(x_1, x_2, ..., x_{n-1}, 0) = 0$. Furthermore assume that $\lim_{|x|\to\infty} \left(|x|^{-1}|u(x)|\right) = 0$ uniformly.

a) Define

$$v(x) = \begin{cases} u(x) & \text{for } x_n \geq 0 \\ -u(x_1, x_2, ..., x_{n-1}, -x_n) & \text{for } x_n < 0. \end{cases}$$

Show that $\Delta v = 0$ in $\mathbb{R}^n$.

b) Use the estimates on the first derivatives of v to prove that $\nabla v(x) = 0$. Conclude that $u(x) = 0$.

Exercise 7. Let Ω be an unbounded domain and assume that

$$\begin{aligned} \Delta u^i(x) &= f(x) && \text{in } \Omega \\ u^i(x) &= g(x) && \text{on } \partial\Omega, \end{aligned}$$

for $i = 1, 2$.

a) Show that if $\lim_{\Omega \ni x \to \infty} |u^1(x) - u^2(x)| = 0$ uniformly then $u^1 = u^2$.

b) Assume that $\Omega = \mathbb{R}^n_+$ and show that if $\lim_{|x| \to \infty} \left(|x|^{-1} |u^1(x) - u^2(x)| \right) = 0$ then $u^1 = u^2$.

(HINT: Look at Exercise 6.)

c) Assume that $\Omega \subset \mathbb{R}^2$ and that, in polar coordinates, $\Omega = \{(r, \phi);\ \phi \in (0, \phi_0)\}$ for some $\phi_0 \in (0, 2\pi)$. Show that if $\lim_{|x| \to \infty} \left(|x|^{-\pi/\phi_0} |u^1(x) - u^2(x)| \right) = 0$ then $u^1 = u^2$.

(HINT: Let $w(x) = u^1(x) - u^2(x) - \epsilon r^{\pi/\phi_0} \sin(\pi\phi/\phi_0)$. Is $w(x)$ harmonic? Does $w(x)$ have a sign on $\partial\left(\Omega \cap B_r(0)\right)$ if R is large enough?)

Exercise 8. Use the Harnack inequality to show that if $\{u^j\}_{j=1}^\infty$ is an increasing sequence of harmonic functions in the connected domain Ω then if $u^j(x^0)$ converges for some $x^0 \in \Omega$ then there exist a harmonic function u^0 such that $u^j \to u^0$ uniformly on compact sets $K \subset\subset \Omega$.

(HINT: What can you say about $u^{j+k} - u^j$ for $k \geq 1$?)

Exercise 9. Let $\phi_\epsilon(x)$ be the standard mollifier. Use the estimate

$$\sup_{\mathbb{R}^n} \left| \frac{\partial^{|\alpha|} \phi(x)}{\partial x^\alpha} \right| \leq C_\alpha$$

for any multiindex α together with

$$\frac{\partial^{|\alpha|} \phi_\epsilon(x)}{\partial x^\alpha} = \frac{1}{\epsilon^{n+|\alpha|}} \frac{\partial^{|\alpha|} \phi(x/\epsilon)}{\partial x^\alpha}$$

to directly show that if u is harmonic in Ω then for any $x^0 \in \Omega$

$$\left| \frac{\partial^{|\alpha|} u(x^0)}{\partial x^\alpha} \right| \leq C_0 C_\alpha \frac{1}{\mathrm{dist}(x^0, \partial\Omega)^{n+|\alpha|}} \|u\|_{L^1(B_{\mathrm{dist}(x^0,\partial\Omega)}(x^0))},$$

for some constant C_0.

Exercise 10. Suppose that u is harmonic in Ω. Prove that u^2 is sub-harmonic in Ω.

(HINT: Is $u^2 \in C^2(\Omega)$?)

Exercise 11. Show that the following definition is equivalent to our definition of sub-harmonicity:

We say that $u \in C(\Omega)$ is sub-harmonic if for any $D \subset \Omega$ we have $u \leq h$ for all h that are harmonic in D and $h \geq u$ on ∂D.

Chapter 10

Compactness Properties of Harmonic Functions.

One of the stated reasons for the importance to develop a regularity theory for harmonic functions is that estimates implies compactness for harmonic functions. With the Arzela-Ascoli Theorem at hand (see the appendix) we can to prove the following version of Weierstrass theorem.

Theorem 10.1. *Let $\{u^j\}_{j=1}^\infty$ be a uniformly bounded sequence of harmonic functions in the domain Ω. That is, $u^j \in C^2(\Omega)$, $\Delta u^j(x) = 0$ in Ω and there exist a constant C_0 (independent of j) such that $\sup_{x\in\Omega} |u^j(x)| \le C_0$.*

Then there exists a sub-sequence $\{u^{j_k}\}_{k=1}^\infty$ of $\{u^j\}_{j=1}^\infty$ that is uniformly convergent on compact sets in Ω and the limit $u^0(x) = \lim_{k\to\infty} u^{j_k}$ is harmonic in Ω.

Proof: We want to show that the sequence $\{u^j\}_{j=1}^\infty$ is equicontinuous in Ω. Then the Arzela-Ascoli Theorem assures that there is a sub-sequence converging uniformly on compact sets of Ω.

To show that the sequence is equicontinuous we notice that for every point $x \in \Omega_{2r} = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > 2r\}$ we have $B_{2r}(x) \subset \Omega$. In particular for $y \in B_r(x)$ we have the estimate

$$\left|\nabla u^j(y)\right| \le \sqrt{n}\frac{n^2 2^{n+1}}{\omega_n r^{n+1}} \|u^j\|_{L^1(B_r(y))}.$$

Using that $|u^j| \le C_0$ we see that

$$\|u^j\|_{L^1(B_r(y))} = \int_{B_r(y)} |u(z)| dz \le \frac{\omega_n r^n}{n} C_0.$$

So for any $x \in \Omega_{2r}$ we have

$$\left|\nabla u^j(y)\right| \le \frac{n^{3/2} 2^{n+1}}{r} C_0, \tag{10.1}$$

for every $y \in B_r(x)$.

To show that $\{u^j\}_{j=1}^\infty$ is equicontinuos at x we need, for every $\epsilon > 0$, to find a $\delta_\epsilon > 0$ such that

$$|u^j(x) - u^j(y)| < \epsilon \qquad \text{for all } y \in B_{\delta_\epsilon}(x),$$

where δ_ϵ is independent of j. There is no loss of generality to assume that $\delta_\epsilon < r$.

By the mean value Theorem (from analysis, not the mean value Theorem for harmonic functions) we get for some $t \in (0,1)$

$$|u^j(x) - u^j(y)| = \left|(y-x)\cdot \nabla u^j\big(x + t(y-x)\big)\right| \le \frac{n^{3/2}2^{n+1}}{r} C_0 |x-y| \quad (10.2)$$

if $|x-y| < r$ where we also used the estimate (10.1). If we set

$$\delta_\epsilon = \inf\left(\frac{r}{C_0 n^{3/2} 2^{n+1}}\epsilon, r\right),$$

then (10.2) implies that

$$|u^j(x) - u^j(y)| < \epsilon \quad (10.3)$$

for $|x-y| < \delta_\epsilon$. Since (10.3) is independent of j it follows that $\{u^j\}_{j=1}^\infty$ is equicontinious in Ω.

By the Arzela-Ascoli Theorem it follows that we can find a sub-sequence $\{u^{j_k}\}_{k=1}^\infty$ of $\{u^j\}_{j=1}^\infty$ that converges uniformly on compact sets of Ω to some $u^0 \in C(\Omega)$.

We still need to show that u^0 is harmonic. We could do that by applying the Arzela-Ascoli Theorem to the second derivatives (using estimates on the third derivatives to show that the second derivatives of $\{u^j\}_{j=1}^\infty$ forms an equicontinuous sequence). But we will use another argument based on the mean value Theorem.

Let $x^0 \in \Omega$ and $\overline{B_r(x^0)} \subset \Omega$. Then since $\overline{B_r(x^0)}$ is a compact set we know that $u^{j_k} \to u^0$ uniformly $\overline{B_r(x^0)}$. In particular for every $\epsilon > 0$ there exists an N_ϵ such that $|u^0(x) - u^{j_k}(x)| < \epsilon$ for all $k > N_\epsilon$ and $x \in \overline{B_r(x^0)}$.

Using this and the mean value property for u^{j_k} we see that when $k > N_\epsilon$

$$\epsilon > \left|u^0(x^0) - u^{j_k}(x^0)\right| = \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^{j_k}(y)dy\right| =$$

$$= \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^0(y)dy - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} \big(u^{j_k}(y) - u^0(y)\big)\,dy\right| \ge$$

$$\ge \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^0(y)dy\right| - \left|\frac{n}{\omega_n r^n}\int_{B_r(x^0)} \big(u^{j_k}(y) - u^0(y)\big)\,dy\right| \ge$$

$$\ge \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^0(y)dy\right| - \epsilon$$

where we used that $|u^0(x) - u^{j_k}(x)| < \epsilon$ in $\overline{B_r(x^0)}$ in the last inequality.

In particular

$$\left| u^0(x^0) - \frac{n}{\omega_n r^n} \int_{B_r(x^0)} u^0(y)dy \right| < 2\epsilon,$$

for any $\epsilon > 0$. That is u^0 satisfies the mean value property and is therefore harmonic. □

10.1 Appendix: The Arzela-Ascoli Theorem.

One of the main reasons that we are interested in estimating the derivatives of a harmonic function is that it gives good compactness properties of solutions, that is we can show that bounded sequences of solutions converge in C^k. One of the main compactness theorems for functions is the Arzela-Ascoli Theorem which we will prove presently. We begin with a definition.

Definition 10.1. *Let $\mathcal{F}$ be a set of functions defined in Ω. We say that $\mathcal{F}$ is equicontinuous at $x \in \Omega$ if for every $\epsilon > 0$ there exist an $\delta_{x,\epsilon} > 0$ such that*

$$|f(x) - f(y)| \leq \epsilon$$

for all $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon}$ and all $f \in \mathcal{F}$.

We also say that $\mathcal{F}$ is equicontinuous in Ω if $\mathcal{F}$ is equicontinuous at every $x \in \Omega$.

Naturally, we may consider a sequence of functions $\{f_j\}_{j=1}^{\infty}$ defined on Ω as a set $\mathcal{F} = \{f_j;\ j \in \mathbb{N}\}$ and we may therefore say that a sequence $\{f_j\}_{j=1}^{\infty}$ is equicontinuous at x or in Ω.

Theorem 10.2. *Let $\{f_j\}_{j=1}^{\infty}$ be a uniformly bounded sequence of functions defined on Ω, that is $\sup_{x\in\Omega} |f_j(x)| \leq C$ for some C independent of j. Assume furthermore that $\{f_j\}_{j=1}^{\infty}$ is equicontinuous in Ω. Then there exist a sub-sequence $\{f_{j_k}\}_{k=1}^{\infty}$ such that $f_{j_k}(x)$ converges pointwise.*

If we define $f_0(x) = \lim_{k\to\infty} f_{j_k}(x)$ then $f_{j_k} \to f_0$ uniformly on compact subsets and $f_0 \in C(\Omega)$.

Proof: The proof is rather long so we will divide it into several steps.

Step 1: *There is a sub-sequence $\{f_{j_k}\}_{k=1}^{\infty}$ that converges pointwise on a countable dense set of Ω.*

Consider the intersection of Ω and the points with rational coordinates $\Omega_{\mathbb{Q}} \equiv \mathbb{Q}^n \cap \Omega$. Since $\mathbb{Q}^n$ is countable it follows that $\Omega_{\mathbb{Q}}$ is countable. Say $\Omega_{\mathbb{Q}} = \{y^j;\ j \in \mathbb{N},\ y^j \in \mathbb{Q}^n\}$.

We will inductively define the sub-sequence $\{f_{j_k}\}_{k=1}^{\infty}$ so that it converges pointwise on $\Omega_{\mathbb{Q}}$.

Consider the sequence $\{f_j(y^1)\}_{j=1}^{\infty}$. Since $|f_j| \leq C$ in Ω it follows that $|f_j(y^1)| \leq C$. In particular $\{f_j(y^1)\}_{j=1}^{\infty}$ is a bounded sequence of real numbers.

We may thus extract a convergent sub-sequence which we will denote $\{f_{1,j}\}_{j=1}^{\infty}$ where the sub-script 1 indicates that the sequence converges at y^1.

Next we make the induction assumption that we have extracted sub-sequences $\{f_{l,j}\}_{j=1}^{\infty}$ for each $l \in \{1, 2, 3, ..., m\}$, such that

1. $\{f_{l,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{l-1,j}\}_{j=1}^{\infty}$ for $l = 2, 3, 4, ..., m$
2. and $f_{m,j}(y^l)$ converges for $l = 1, 2, 3, ..., m$.

In order to complete the induction we need to show that we can find a sub-sequence $\{f_{m+1,j}\}_{j=1}^{\infty}$ of $\{f_{m,j}\}_{j=1}^{\infty}$ such that $\{f_{m+1,j}(y^{m+1})\}_{j=1}^{\infty}$ converges.

Arguing as before, we see that $\{f_{m,j}(y^{m+1})\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ and we may thus extract a sub-sequence, that we denote $\{f_{m+1,j}\}_{j=1}^{\infty}$, that converges.

By induction it follows that for each $m \in \mathbb{N}$ there exist a sequence $\{f_{m,j}\}_{j=1}^{\infty}$ such that $\{f_{m,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m-1,j}\}_{j=1}^{\infty}$ and $\{f_{m,j}(y^m)\}_{j=1}^{\infty}$ is convergent.

Notice that since $\{f_{m,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m-1,j}\}_{j=1}^{\infty}$ and $\{f_{m-1,j}(y^l)\}_{j=1}^{\infty}$ converges for $1 \le l \le m-1$ it follows that $\{f_{m,j}(y^l)\}_{j=1}^{\infty}$ converges to the same limit for $1 \le l \le m-1$. In particular, $\{f_{m,j}(y^l)\}_{j=1}^{\infty}$ converges for all $l \le m$.

Now we define the sequence $\{f_{j_k}\}_{k=1}^{\infty}$ by a diagonalisation procedure

$$f_{j_k} = f_{k,k}.$$

Noticing that $\{f_{j_k}\}_{k=m}^{\infty} = \{f_{k,k}\}_{k=m}^{\infty}$ is a sub-sequence of $\{f_{m,j}\}_{j=1}^{\infty}$. This follows from the fact that $f_{k,k}$ is an element of the sequence $\{f_{k,j}\}_{j=1}^{\infty}$. But $\{f_{k,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m,j}\}_{j=1}^{\infty}$ for $k \ge m$.

We may conclude that $\{f_{j_k}\}_{k=m}^{\infty}$ converges at y^l for all $l \le k$. But k is arbitrary so $f_{j_k}(y^l)$ converges for every $l \in \mathbb{N}$. This proves step 1.

Step 2: *The sequence $\{f_{j_k}\}_{k=1}^{\infty}$ converges pointwise in Ω.*

It is enough to show that $\{f_{j_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence for every $x \in \Omega$. To that end we fix an $\epsilon > 0$. We need to show that there exist an $N_\epsilon \in \mathbb{N}$ such that $|f_{j_k}(x) - f_{j_l}(x)| < \epsilon$ for all $k, l > N_\epsilon$.

Since $\{f_{j_k}\}_{k=1}^{\infty}$ is equicontinuous at $x \in \Omega$ there exist a $\delta_{x,\epsilon/3}$ such that

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \quad \text{for all } k \in \mathbb{N}, \tag{10.4}$$

and $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon/3}$.

Moreover since $\Omega_{\mathbb{Q}}$ is dense in Ω there exist an $y^x \in \Omega_{\mathbb{Q}}$ such that $|x - y^x| < \delta_{x,\epsilon/3}$. In step 1 we showed that $f_{j_k}(y)$ was convergent for all $y \in \Omega_{\mathbb{Q}}$ in particular it follows that $\{f_{j_k}(y^x)\}_{k=1}^{\infty}$ is a Cauchy sequence. That is, there exist an $N_{y^x,\epsilon/3} \in \mathbb{N}$ such that

$$|f_{j_k}(y^x) - f_{j_l}(y^x)| < \frac{\epsilon}{3} \quad \text{for all } k, l > N_{y^x,\epsilon/3}. \tag{10.5}$$

From (10.4) and (10.5) we can deduce that

$$|f_{j_k}(x)-f_{j_l}(x)| \le |f_{j_k}(x)-f_{j_k}(y^x)|+|f_{j_l}(x)-f_{j_l}(y^x)|+|f_{j_k}(y^x)-f_{j_l}(y^x)| < \epsilon,$$

for all $k,l > N_{y^x,\epsilon/3}$. It follows that $\{f_{j_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence and this finishes the proof of step 2.

Step 3: *Define* $f_0(x) = \lim_{k\to\infty} f_{j_k}(x)$, *then* $f_0 \in C(\Omega)$.

Since $f_{j_k}(x)$ is convergent for every $x \in \Omega$ by step 2 it follows that f_0 is well defined in Ω. To show continuity we need to show that for every $x \in \Omega$ and $\epsilon > 0$ there exist a $\delta_\epsilon > 0$ such that

$$|f_0(x) - f_0(y)| < \epsilon$$

for every $y \in \Omega$ such that $|x - y| < \delta_\epsilon$. By equicontinuity there exist a $\delta_{x,\epsilon/3}$ such that

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{10.6}$$

for every $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon/3}$ and all $j \in \mathbb{N}$.

Also by step 2 there exist an $N_{x,\epsilon/3}$ such that

$$|f_0(x) - f_{j_k}(x)| < \frac{\epsilon}{3} \tag{10.7}$$

for all $k \ge N_{x,\epsilon/3}$. And an $N_{y,\epsilon/3}$ such that

$$|f_0(y) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{10.8}$$

for all $k \ge N_{y,\epsilon/3}$.

From (10.6), (10.7) and (10.8) we can deduce that for $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon/3}$

$$|f_0(x) - f_0(y)| \le |f_0(x) - f_{j_k}(x)| + |f_0(y) - f_{j_k}(y)| + |f_{j_k}(x) - f_{j_k}(y)| < \epsilon$$

if $k > \max(N_{x,\epsilon/3}, N_{y,\epsilon/3})$.

This proves step 3.

Step 4: $\{f_{j_k}\}_{k=1}^{\infty}$ *converges uniformly on compact sets.*

We fix a compact set $K \subset \Omega$. We need to show that for every $\epsilon > 0$ there exist an N_ϵ such that when $k > N_\epsilon$ then $|f_0(x) - f_{j_k}(x)| < \epsilon$ for all $x \in K$.

Notice that by equicontinuity there exist a $\delta_{x,\epsilon/3}$ for each $x \in K$ such that for all $k \in \mathbb{N}$

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{10.9}$$

for all $y \in B_{\delta_{x,\epsilon/3}}(x) \cap \Omega$.

Notice that the balls $B_{\delta_{x,\epsilon/3}}(x)$ forms an open cover of K: $K \subset \cup_{x\in K} B_{\delta_{x,\epsilon/3}}(x)$. Since K is compact there exist a finite sub-cover $B_{\delta_{x^l,\epsilon/3}}(x^l)$, for $l = 1, 2, 3, ..., l_0$ for some $l_0 \in \mathbb{N}$. That is $K \subset \cup_{l=1}^{l_0} B_{\delta_{x^l,\epsilon/3}}(x^l)$.

Also, using that $\lim_{k\to\infty} f_{j_k}(x^l) = f_0(x^l)$, we see that there exist an $N_{x^l,\epsilon/3}$ such that
$$|f_{j_i}(x^l) - f_{j_k}(x^l)| < \frac{\epsilon}{3} \tag{10.10}$$
for all $i, k > N_{x^l,\epsilon/3}$. We choose $N_\epsilon = \max\left(N_{x^1,\epsilon/3}, N_{x^2,\epsilon/3}, ..., N_{x^{l_0},\epsilon/3}\right)$.

Since $K \subset \cup_{l=1}^{l_0} B_{\delta_{x^l,\epsilon/3}}(x^l)$ it follows that for every $x \in K$ that $x \in B_{\delta_{x^l,\epsilon/3}}(x^l)$ for some l. Using this and (10.9) and (10.10) we see that
$$\begin{aligned}|f_{j_i}(x) - f_{j_k}(x)| &\le |f_{j_i}(x) - f_{j_i}(x^l)| + |f_{j_k}(x) - f_{j_k}(x^l)| + |f_{j_i}(x^l) - f_{j_k}(x^l)| < \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned} \tag{10.11}$$
for all $k \ge N_\epsilon$. Taking the limit $i \to \infty$ in (10.11) we see that
$$|f_0(x) - f_{j_k}(x)| < \epsilon$$
for all $k > N_\epsilon$. This finishes the proof of the Theorem. $\square$

Chapter 11

Existence of Solutions.

11.1 The Perron Method.

We are now ready to prove the existence of solutions to the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array} \tag{11.1}$$

The idea of the proof is to consider the largest subharmonic function that is smaller than g on $\partial\Omega$. If a solution to (11.1) exists then, by the maximum principle, that solution has to be the largest subharmonic function that is less than or equal to g on $\partial\Omega$. This gives some hope that the largest sub-harmonic function should be the solution to (11.1).

Before we prove that the largest sub-harmonic function is harmonic we need to prove a Lemma that shows us that we can change a sub-harmonic function into a harmonic function in part of the domain without destroying the sub-harmonicity.

Lemma 11.1. *Suppose that $v \in C(\Omega)$ is sub-harmonic in Ω. Moreover, we assume that $\overline{B_{r_0}(x^0)} \subset \Omega$. If we define $\tilde{v}$ to by the harmonic replacement of v in $B_{r_0}(x^0)$:*

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \setminus B_{r_0}(x^0) \\ \int_{\partial B_{r_0}(x^0)} \frac{r^2 - |x-x^0|^2}{\omega_n r} \frac{1}{|x-x^0-y|^n} v(y) dA_{\partial B_{r_0}(x^0)}(y) & \text{for } x \in B_{r_0}(x^0). \end{cases}$$

Then $\tilde{v}$ is sub-harmonic in Ω.

Remark: We say that $\tilde{v}$ is defined by the harmonic replacement in $B_{r_0}(x^0)$. This language usage is natural since $\tilde{v}$ equals v outside of $B_{r_0}(x^0)$ and is defined by Poisson's formula in $B_{r_0}(x^0)$. We know that functions defined by Poisson's formula are harmonic so $\tilde{v}$ is defined by replacing the values of v by the harmonic function with boundary data v in $B_{r_0}(x^0)$.

At times the harmonic replacement is referred to as the harmonic lifting in $B_{r_0}(x^0)$. The reason for that terminology is that, by the maximum principle,

$\tilde{v} \geq v$ in $B_{r_0}(x^0)$. So $\tilde{v}$ is defined by increasing, or lifting, the values of v in $B_{r_0}(x^0)$.

Proof: Since $\tilde{v}$ is defined by Poisson's formula in $B_r(x^0)$ it follows that $\tilde{v}$ is harmonic in $B_{r_0}(x^0)$. Since $\tilde{v}$ is harmonic in $B_{r_0}(x^0)$ it follows that $\tilde{v}$ is sub-harmonic in $B_{r_0}(x^0)$.

Also $\tilde{v} = v$ in $\Omega \setminus B_{r_0}(x^0)$ so $\tilde{v}$ is sub-harmonic in $\Omega \setminus B_{r_0}(x^0)$.

This does not imply that $\tilde{v}$ is sub-harmonic in Ω. We need to show that $\tilde{v}$ satisfies the sub-meanvalue property for every ball $\overline{B_{r_0}(y)} \subset \Omega$.

To that end we fix an arbitrary ball $\overline{B_r(y)} \subset \Omega$. If $B_r(y) \subset \Omega \setminus B_{r_0}(x^0)$ then $\tilde{v}$ satisfies the sub-meanvalue property for the ball $B_r(y)$ since $\tilde{v} = v$ in $B_r(y)$. Similarly, if $B_r(y) \subset B_{r_0}(x^0)$ then $\tilde{v}$ satisfies the sub-meanvalue property (and even the mean value property) for the ball $B_r(y)$ since $\tilde{v}$ is harmonic in $B_r(y) \subset B_{r_0}(x^0)$.

We therefore only need to prove that $\tilde{v}$ satisfies the sub-meanvalue property for balls $\overline{B_r(y)} \subset \Omega$ such that $B_r(y) \cap B_{r_0}(x^0) \neq \emptyset$ and $B_r(y) \cap \big(\Omega \setminus B_{r_0}(x^0)\big) \neq \emptyset$. Fix such a ball $B_r(y)$. We continue the proof in several steps.

Step 1: *Let $\tilde{h}$ be the harmonic function in $B_r(y)$ with $\tilde{h}(x) = \tilde{v}(x)$ on $\partial B_r(y)$. We claim that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \setminus B_{r_0}(x^0)$.*

Notice that $v - \tilde{v}$ is sub-harmonic in $B_{r_0}(x^0)$ and that $v - \tilde{v} = 0$ on $\partial B_{r_0}(x^0)$. So by the maximum principle for sub-harmonic functions $v \leq \tilde{v}$ in $B_{r_0}(x^0)$.

Also, if we let h solve

$$\begin{aligned} \Delta h = 0 \quad & \text{in } B_r(y) \\ h = v \quad & \text{on } \partial B_r(y), \end{aligned}$$

then again, by the sub-harmonicity of v and the maximum principle $v \leq h$ in $B_r(y)$.

Since $v \leq \tilde{v}$ we have that $h \leq \tilde{h}$ on $\partial B_r(y)$ and since both h and $\tilde{h}$ are harmonic it follows that $\tilde{h} \geq h$ in $B_r(y)$. That is $v \leq h \leq \tilde{h}$ in $B_r(y)$.

Using that $\tilde{v} = v$ in $B_r(y) \setminus B_{r_0}(x^0)$ the claim in step 1 follows.

Step 2: *Let, as in step 1, $\tilde{h}$ be the harmonic function in $B_r(y)$ with $\tilde{h}(x) = \tilde{v}(x)$ on $\partial B_r(y)$. We claim that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \cap B_{r_0}(x^0)$.*

By step 1 we know that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \setminus B_{r_0}(x^0)$. Since $\tilde{v}$ and $\tilde{h}$ are continuous functions it follows that $\tilde{h} \geq \tilde{v}$ on $\big(\partial B_{r_0}(x^0)\big) \cap B_r(y)$. On $\big(\partial B_r(y)\big) \cap B_{r_0}(x^0)$ we have that $\tilde{h} = \tilde{v}$ by the definition of $\tilde{h}$.

In particular, $\Delta \tilde{v} = \Delta \tilde{h} = 0$ in $B_{r_0}(x^0) \cap B_r(y)$ and $\tilde{v} \leq \tilde{h}$ on $\partial\big(B_{r_0}(x^0) \cap B_r(y)\big)$. It follows that $w = \tilde{v} - \tilde{h}$ solves

$$\begin{aligned} \Delta w = 0 \quad & \text{in } B_{r_0}(x^0) \cap B_r(y) \\ w \leq 0 \quad & \text{on } \partial\big(B_{r_0}(x^0) \cap B_r(y)\big). \end{aligned}$$

By the maximum principle $w \leq 0$ in $B_{r_0}(x^0) \cap B_r(y)$, that is $\tilde{v} \leq \tilde{h}$ in $B_{r_0}(x^0) \cap B_r(y)$.

Step 3: *$\tilde{v}$ satisfies the sub-meanvalue property in Ω.*

Pick any ball $\overline{B_r(y)} \subset \Omega$. If $B_r(y) \cap B_{r_0}(x^0) = \emptyset$ we have already shown that $\tilde{v}$ satisfies the sub-meanvalue property in $B_r(y)$. So we may assume that $B_r(y) \cap B_{r_0}(x^0) \neq \emptyset$.

Define h as in step 1 and 2, then

$$\tilde{v}(y) \leq h(y) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} \tilde{h}(z) dA_{\partial B_r(y)}(z) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} \tilde{v}(z) dA_{\partial B_r(y)}(z),$$
$$= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} \tilde{v}(z) dA_{\partial B_r(y)}(z), \tag{11.2}$$

where we have used step 1 if $y \in B_r(y) \backslash B_{r_0}(x^0)$ and step 2 if $y \in B_r(y) \cap B_{r_0}(x^0)$ in the first inequality, the meanvalue property for the harmonic function h in and finally that $\tilde{h} = \tilde{v}$ on $\partial B_r(y)$.

Notice that (11.2) is nothing by the sub-meanvalue property for $\tilde{v}$.

We have thus shown that $\tilde{v}$ satisfies the sub-meanvalue property in Ω and is thus sub-harmonic. □

Definition 11.1. *Let $g \in C(\partial\Omega)$ where Ω is a bounded domain. We define $S_g(\Omega)$ to be the class of sub-harmonic functions $v \in C(\overline{\Omega})$ such that $v(x) \leq g(x)$ on $\partial\Omega$. That is*

$$S_g(\Omega) = \big\{ v \in C(\overline{\Omega});\ v \text{ is sub-harmonic in } \Omega \text{ and } v(x) \leq g(x) \text{ on } \partial\Omega \big\}.$$

Since $g \in C(\partial\Omega)$ it follows that the constant $\inf_{x\in\partial\Omega} g(x) \in S_g(\Omega)$. That is $S_g(\Omega) \neq \emptyset$.

The first part of the existence theorem is:

Theorem 11.1. [PERRON'S METHOD.] *Suppose that Ω is a bounded domain and $g(x) \in C(\partial\Omega)$. Define*

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

Then $u(x)$ is harmonic in Ω.

Proof: As we remarked before $S_g(\partial\Omega) \neq \emptyset$. Also by the maximum principle for sub-harmonic functions we have

$$\sup_{x\in\Omega} \big(v(x) - \sup_{y\in\partial\Omega} g(y) \big) \leq \sup_{x\in\partial\Omega} \big(v(x) - \sup_{y\in\partial\Omega} (g(y)) \big) = \sup_{x\in\partial\Omega} \big(v(x) \big) - \sup_{y\in\partial\Omega} g(y) \leq 0$$

for every $v \in S_g(\Omega)$ since if $v \in S_g(\Omega)$ then v is sub-harmonic and $v \leq g$ on $\partial\Omega$.

It follows that

$$\sup_{v\in S_g(\Omega)} v(x) \leq \sup_{y\in\partial\Omega} g(y).$$

Using that a non-empty set of real numbers that is bounded from above has a supremum (the completeness property of $\mathbb{R}$) we may conclude that $u(x) =$

$\sup_{v\in S_g(\Omega)} v(x)$ is well defined. Moreover, for every $x \in \Omega$ we can find a sequence $\{v^k\}_{k=1}^\infty$ in $S_g(\Omega)$ so $u(x) = \lim_{k\to\infty} v^k(x)$. We fix an arbitrary $x^0 \in \Omega$ and sequence $\{v^k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} v^k(x^0) = u(x^0)$. Since Ω is a domain, in particular Ω is open, there exist an $r > 0$ such that $B_r(x^0) \subset \Omega$.

We may assume that

$$v^k \geq \inf_{x\in\partial\Omega} g(x). \tag{11.3}$$

If (11.3) where not true then we could consider the sequence $\max\left(v^k(x), \inf_{x\in\partial\Omega} g(x)\right) \in S_g(\Omega)$ instead.

In order to proceed we define the harmonic replacement of v^k in $B_r(x^0)$ according to

$$\tilde{v}^k(x) = \begin{cases} v^k(x) & \text{if } x \notin B_r(x^0) \\ \int_{\partial B_r(x^0)} \frac{r^2-|x-x^0|^2}{\omega_n r} \frac{1}{|x-x^0-y|^n} v^k(y) dA_{\partial B_r(x^0)}(y) & \text{for } x \in B_r(x^0). \end{cases}$$

Notice that $\tilde{v}^k$ is defined by the Poisson integral in $B_r(x^0)$. It follows that

$$\begin{aligned} \Delta\tilde{v}^k(x) = 0 &\quad \text{in } B_r(x^0) \\ \tilde{v}^k(x) = v^k(x) &\quad \text{on } \partial B_r(x^0). \end{aligned}$$

By Lemma 11.1 it follows that $\tilde{v}^k$ is sub-harmonic in Ω and since $\tilde{v}^k = v^k \leq g$ on $\partial\Omega$ it follows that $\tilde{v}^k \in S_g(\Omega)$.

Moreover, since v^k is sub-harmonic and $\tilde{v}^k$ is harmonic in $B_r(x^0)$ and $\tilde{v}^k = v^k$ on $\partial B_r(x^0)$ we can conclude that $\tilde{v}^k \geq v^k$ in $B_r(x^0)$. Also $u(x^0) \geq \tilde{v}^k(x^0)$ since $\tilde{v}^k \in S_g(\Omega)$.

It follows that

$$u(x^0) = \lim_{k\to\infty} v^k(x^0) \leq \lim_{k\to\infty} \tilde{v}^k(x^0) \leq u(x^0),$$

so $\tilde{v}^k(x^0) \to u(x^0)$.

From the compactness Lemma 10.1 we know that there exist a sub-sequence $\{\tilde{v}^{k_j}\}_{j=1}^\infty$ of $\{\tilde{v}^k\}_{k=1}^\infty$ such that $\tilde{v}^{k_j} \to \tilde{v}^0$ uniformly on compact sets in $B_r(x^0)$ and that $\tilde{v}^0$ is harmonic in $B_r(x^0)$.We claim that $\tilde{v}^{k_j}(x) \to u(x)$ uniformly on compact sets in $B_r(x^0)$.

By the definition of u it follows that $u \geq \tilde{v}^0$ in $B_r(x^0)$.

Claim: *We claim that $u(x) = \tilde{v}^0(x)$ for all $x \in B_r(x^0)$. Since x^0 i arbitrary this implies that $\Delta u(x) = 0$ in any ball $B_s(y) \subset \Omega$ and finishes the proof of the theorem.*

We prove this claim by an argument of contradiction. Aiming for a contradiction we assume that there exist a $z \in B_r(x^0)$ such that $\tilde{v}^0(z) < u(z)$. Since $u(z) = \sup_{w\in S_g(\Omega)} w(z)$ there exist a $w \in S_g(\Omega)$ such that $\tilde{v}^0(z) < w(z)$.

We define $w^j = \sup(w, \tilde{v}^{k_j})$. Then w^j is sub-harmonic since w and $\tilde{v}^{k_j}$ are sub-harmonic.

We also define the harmonic lifting of w^j according to

$$\tilde{w}^j(x) = \begin{cases} w^j(x) & \text{if } x \notin B_r(x^0) \\ \int_{\partial B_r(x^0)} \frac{r^2-|x-x^0|^2}{\omega_n r} \frac{1}{|x-x^0-y|^n} w^j(y) dA_{\partial B_r(x^0)}(y) & \text{for } x \in B_r(x^0) \end{cases}$$

Arguing as before, using the maximum principle, we see that $\tilde{w}^j \geq w^j$. Notice that

$$\tilde{w}^j \geq w^j = \sup(w, \tilde{v}^{k_j}) \geq \tilde{v}^{k_j}. \tag{11.4}$$

Using Lemma 10.1 again we can extract a sub-sequence $\{\tilde{w}^{j_l}\}_{l=1}^{\infty}$ of $\{\tilde{w}^j\}_{j=1}^{\infty}$ such that $\{\tilde{w}^{j_l}\}_{l=1}^{\infty}$ converges uniformly on compact sets to some harmonic function $\tilde{w}^0$. Notice that

$$\tilde{w}^0(z) = \lim_{l\to\infty} \tilde{w}^{j_l}(z) \geq \lim_{l\to\infty} w^{j_l}(z) = \lim_{l\to\infty} \sup(w(z), \tilde{v}^{k_{j_l}}(z)) = w(z). \tag{11.5}$$

In particular this imples that

$$\tilde{w}^0(z) > \tilde{v}^0(z). \tag{11.6}$$

Also, since

$$u(x^0) \geq \tilde{w}^{j_l}(x^0) \geq \tilde{v}^{k_{j_l}}(x^0) \to u(x^0)$$

we get that $\tilde{w}^0(x^0) = u(x^0)$. Using (11.4), (11.5) and that $\tilde{w}^0(x^0) = \tilde{v}^0(x^0) = u(x^0)$ we get

$$\begin{aligned}
&\Delta(\tilde{v}^0(x) - \tilde{w}^0(x)) = 0 \qquad && \text{in } B_r(x^0)\\
&\tilde{v}^0(x) - \tilde{w}^0(x) \leq 0 \text{ on } \partial B_r(x^0) &&\\
&\tilde{v}^0(x^0) - \tilde{w}^0(x^0) = 0. &&
\end{aligned}$$

From the last two lines and the strong maximum principle we can conclude that $\tilde{v}^0(x) - \tilde{w}^0 = 0$ in $B_r(x^0)$. This contradicts (11.6).

We have thus finished our contradiction argument and proved that $\tilde{v}^{k_j} \to u^0$ uniformly on compact sets in $B_r(x^0)$. But $\lim_{j\to\infty} \tilde{v}^{k_j} = \tilde{v}^0$ where $\Delta\tilde{v}^0 = 0$ in $B_r(x^0)$. It follows that $\Delta u(x) = 0$ in $B_r(x^0)$. But $x^0 \in \Omega$ was arbitrary so we may conclude that $\Delta u(x) = 0$ in Ω. □

11.2 Attaining the Boundary Data.

In the previous section we showed that

$$u(x) = \sup_{v\in S_g(\Omega)} v(x)$$

is harmonic in Ω. This is not enough in order to show existence of solutions to the Dirichlet problem

$$\begin{aligned}
\Delta u(x) &= 0 \qquad && \text{in } \Omega\\
u(x) &= g(x) && \text{on } \partial\Omega.
\end{aligned}$$

Perron's method gives a harmonic function but it does not prove that the harmonic function actually attain the boundary values $u(x) = g(x)$. In order to solve the Dirichlet problem we need to show that the solution attained from the Perron process actually satisfy the boundary data.

We will show that, at least in some cases, by the method of using barriers.

Definition 11.2. *Let Ω be a domain and $\xi \in \partial\Omega$. We say that w is a barrier at ξ relative to Ω if*

1. *$w \in C(\overline{\Omega})$,*
2. *$w > 0$ in $\overline{\Omega} \setminus \{\xi\}$, $w(\xi) = 0$ and*
3. *w is super-harmonic in Ω.*

If Ω is a domain and there exist a barrier at ξ relatively to Ω then we say that ξ is a regular point of $\partial\Omega$.

Theorem 11.2. *Let Ω be a bounded domain and $g \in C(\partial\Omega)$. Furthermore let*

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

If ξ is a regular point of $\partial\Omega$ then

$$\lim_{x \to \xi} u(x) = g(\xi).$$

Proof: We need to find, for each $\epsilon > 0$, a $\delta_\epsilon > 0$ such that

$$\sup_{x \in B_{\delta_\epsilon}(\xi) \cap \Omega} |u(x) - g(\xi)| < \epsilon.$$

Since $g \in C(\partial\Omega)$ there exist an $\delta_{g,\epsilon/2}$ such that

$$\sup_{x \in \partial\Omega \cap B_{\delta_{g,\epsilon/2}}(\xi)} |g(x) - g(\xi)| < \frac{\epsilon}{2}.$$

Let w be a barrier at ξ and define

$$\kappa = \inf_{x \in \partial\Omega \setminus B_{\delta_{g,\epsilon/2}}(\xi)} w(x).$$

Using that $w > 0$ in $\overline{\Omega} \setminus \{\xi\}$, $\partial\Omega$ is compact and $w \in C(\overline{\Omega})$ we see that $\kappa > 0$.

If we define

$$k = \frac{1}{\kappa} \sup_{x \in \partial\Omega} |g(x) - g(\xi)|$$

then it follows that

$$-\frac{\epsilon}{2} - kw(x) \leq g(x) - g(\xi) \leq \frac{\epsilon}{2} + kw(x). \tag{11.7}$$

We know that $w \in C(\overline{\Omega})$ and $w(\xi) = 0$ so there is a $\delta_{w,\epsilon/(2k)}$ such that

$$\sup_{x \in B_{\delta_{w,\epsilon/(2k)}}(\xi)} |w(x)| < \frac{\epsilon}{2k}. \tag{11.8}$$

Since w is super-harmonic it follows from the comparison principle and (11.7) that

$$v(x) \leq g(\xi) + \frac{\epsilon}{2} + kw(x) \tag{11.9}$$

for every $v \in S_g(\Omega)$.

From (11.9) it follows that

$$u(x) \leq g(\xi) + \frac{\epsilon}{2} + kw(x). \tag{11.10}$$

Since w is super-harmonic it follows that $g(\xi) - \frac{\epsilon}{2} - kw(x)$ is sub-harmonic so by (11.7) it follows that $-\frac{\epsilon}{2} - kw(x) \in S_g(\Omega)$. In particular

$$u(x) \geq g(\xi) - \frac{\epsilon}{2} - kw(x). \tag{11.11}$$

From (11.10) and (11.11) we deduce that

$$|u(x) - g(\xi)| \leq \frac{\epsilon}{2} + kw(x). \tag{11.12}$$

Finally we see that if $\delta < \delta_{w,\epsilon/2k}$ and $x \in B_\delta(\xi) \cap \Omega$ then, from (11.12) we may estimate

$$|u(x) - g(\xi)| \leq |\frac{\epsilon}{2} + kw(x)| < \frac{\epsilon}{2} + \frac{k\epsilon}{2k} = \epsilon$$

where we used (11.8) in the strict inequality. This proves the Theorem. □

We are now in the position to create harmonic functions by the Perron method and we also have a criteria to assure that the function so created satisfies the boundary values.

The criteria that assures that the Perron solution assumes the boundary data at $\xi \in \partial\Omega$ is that there exists a barrier at ξ relative to Ω. Since the definition of a barrier is rather abstract we need to develop some adequate theory for the existence of barriers.

The simplest condition that assures the existence of a barrier is the exterior ball condition.

Definition 11.3. *Let Ω be a domain. We say that Ω satisfies the exterior ball condition at ξ if there exist a ball $B_s(y^\xi) \subset \Omega^c$ such that $\xi \in \overline{B_s(y^\xi)} \cap \overline{\Omega}$*

We say that the domain Ω satisfies the exterior ball condition if Ω satisfies the exterior ball condition at every $\xi \in \partial\Omega$.

We say that that the domain Ω satisfies the exterior ball condition uniformly if Ω satisfies the exterior ball condition at every $\xi \in \partial\Omega$ and the radius of the touching balls have radius $s > 0$ independent of ξ.

Lemma 11.2. *Let Ω be a bounded domain and assume that Ω satisfies the exterior ball condition at $\xi \in \partial\Omega$.*

Then ξ is a regular point of $\partial\Omega$. That is there exist a barrier at ξ relatively to Ω.

Proof: Let the touching ball at ξ be $B_s(y)$. We define the the following function that is zero on $\partial B_{s/2}((\xi+y)/2)$

$$w(x)=\left\{\begin{array}{ll} \ln\left|x-\frac{y+\xi}{2}\right|-\ln\left(\frac{s}{2}\right) & \text{if } n=2\\ \frac{2^{n-2}}{s^{n-2}}-\frac{1}{\left|x-\frac{y+\xi}{2}\right|^{n-2}} & \text{if } n\geq 3.\end{array}\right.$$

Since w is a multiple of the Newtonian potential plus a constant it is clear that $\Delta w(x)=0$ in $\mathbb{R}^n\setminus\{(y-\xi)/2\}$.

Notice that $B_{s/2}((\xi+y)/2)$ touches $\partial\Omega$ at only the point $\xi\in\partial\Omega$. The original ball $B_s(y)$ might touch at a larger set.

Moreover $w(x)=0$ on $\partial B_{s/2}((y+\xi)/2)$, $w>0$ in $\mathbb{R}^n\setminus\overline{B_{s/2}((y+\xi)/2)}$. Finally notice that $\xi\in\partial B_{s/2}((y^\xi-\xi)/2)$ so $w(\xi)=0$. It follows that w is a barrier. $\square$

A somewhat stronger sufficient (but not necessary) condition for a the existence of a barrier is the exterior cone condition.

Definition 11.4. *Let Ω be a domain. We say that Ω satisfies the exterior cone condition at ξ relative to $B_r(\xi)$ if $\Omega^c\cap B_r(\xi)$ contains a circular cone. That is if there exist a $\kappa>0$ and a unit vector η such that*

$$\{x\in B_r(\xi);\ \eta\cdot(x-\xi)>0 \text{ and } |x-\eta\cdot(x-\xi)|<\kappa|x-\xi|\}\subset\Omega^c.$$

We say that the domain Ω satisfies the exterior cone condition if Ω satisfies the exterior cone condition at every $\xi\in\partial\Omega$ with respect to some ball $B_{r_\xi}(\xi)$ and $r_\xi>0$ and some $\kappa_\xi>0$.

We say that that the domain Ω satisfies the exterior cone condition uniformly if Ω satisfies the exterior cone condition at every $\xi\in\partial\Omega$ with respect to some ball $B_r(\xi)$ and $r>0$ and some $\kappa>0$ where r and κ is independent of ξ.

Proposition 11.1. *Let Ω be a bounded domain and assume that Ω satisfies the exterior cone condition at $\xi\in\partial\Omega$.*

The ξ is a regular point of $\partial\Omega$. That is there exist a barrier at ξ relatively to Ω.

Proof (only in $\mathbb{R}^2$ and $\mathbb{R}^3$ (sort of)): There is no loss of generality to assume that $\xi=0$. If $\xi\neq 0$ we may simply translate the coordinate system by the translation $x\to x-\xi$ to attain this situation.

By assumption there exist a unit vector η and $r,\kappa>0$ such that

$$K_\kappa=\{x\in B_s(0);\ \eta\cdot x>0 \text{ and } |x-\eta\cdot x|<\kappa|x|\}\subset\Omega^c.$$

By rotation the coordinate system we may assume that $\eta=e_n$.

Proof in $\mathbb{R}^2$: If we change to polar coordinates $x_1=r\sin(\phi)$ and $x_2=r\cos(\phi)$ then the cone becomes

$$K_\kappa=\{(r,\phi);\ |\sin(\phi)|<\kappa,\ \sin(\phi)>0\}.$$

Recalling that Laplace's equation in polar coordinates is

$$\Delta w(r,\phi) = \frac{\partial^2 w(r,\phi)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\phi)}{\partial \phi^2} = 0$$

it is easy to verify that

$$w(r,\phi) = \sqrt{r}\sin\left(\frac{\phi}{2}\right)$$

is harmonic in $\mathbb{R}^2 \setminus K$ and $w(r,\phi) = \sqrt{r}\kappa > 0$ on ∂K. So $w(r,\phi)$ is a barrier at $\xi = 0$ relative to Ω.

Sketch of the proof in Proof in $\mathbb{R}^3$: Laplace equation in spherical coordinates, $(x_1, x_2, x_3) = r(\sin(\psi)\cos(\phi), \sin(\psi)\sin(\phi), \cos(\psi)$, is

$$\frac{\partial^2 w(r,\phi,\psi)}{\partial r^2} + \frac{2}{r}\frac{\partial w(r,\phi,\psi)}{\partial r} + \frac{1}{r^2\sin(\psi)}\frac{\partial}{\partial \psi}\left(\sin(\psi)\frac{\partial w(r,\phi,\psi)}{\partial \psi}\right) +$$

$$+\frac{1}{r^2\sin^2(\psi)}\frac{\partial^2 w(r,\phi,\psi)}{\partial \phi^2} = 0.$$

To simplify the expression somewhat we assume that we can find a solution $w(r,\phi,\psi) = r^\alpha \tilde{w}(\psi)$ that is independent of ϕ and homogeneous in r. Taking into consideration that we also want our barrier to be zero on $\partial K_{\kappa/2}$ we end up with the following ordinary differential equation

$$\begin{array}{ll} r^{\alpha-2}\left(\alpha(\alpha+1)\tilde{w}(r,\psi) + \frac{1}{\sin(\psi)}\frac{\partial}{\partial\psi}\left(\sin(\psi)\frac{\partial \tilde{w}(r,\psi)}{\partial\psi}\right)\right) = 0 & \text{in } K_{\kappa/2} \\ \tilde{w}(r,\psi) = 0 & \text{on } \partial K_{\kappa/2}. \end{array}$$

It turns out that this ordinary differential equation is solvable and that there exists a unique $\alpha_\kappa > 0$ such that the solution is positive in $\mathbb{R}^3 \setminus K_\kappa$. It follows that $w(r,\phi,\psi) = r^{\alpha_\kappa}\tilde{w}(r,\phi,\psi)$. □

Knowing that we have barriers in some cases it is natural to ask the question if we always have barriers. The answer to that is: No.

Example of a non-regular point: Consider the domain $\Omega = B_1(0) \setminus \{0\}$, for simplicity we assume that $n \geq 3$. We want to solve the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array}$$

where

$$\begin{array}{ll} g(x) = 0 & \text{on } \Omega \setminus \{0\} \\ g(0) = -1. & \end{array}$$

By the Perron method we would want to construct the solution

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

Notice that for any $j \in \mathbb{N}$

$$w_j(x) = \max\left(-\frac{1}{j}\frac{1}{|x|^{n-2}}, -1\right) \in S_g(\Omega).$$

In particular, w_j is the supremum of two harmonic functions and is thus subharmonic. Also both $-\frac{1}{j}\frac{1}{|x|^{n-2}}$ and -1 are less than g on $\partial\Omega$. Since $w_j \in S_g(\Omega)$ we have

$$u(x) \geq w_j(x)$$

for all $j \in \mathbb{N}$. But $w_j(x) \to 0$ as $j \to \infty$ for every $x \in \Omega$. This implies that $u(x) \geq 0$. But the maximum principle implies that $u(x) \leq \sup_{x\in\partial\Omega} g(x) = 0$. That is $u(x) = 0$. So $\lim_{x\to 0} u(x) = 0 \neq -1 = g(0)$ and therefore there is no barrier at $\xi = 0$.

In general we have a barrier at ξ if the complement on Ω is "large" close to ξ. In the above example the complement of Ω near the origin consists of just one point and that is why we do not have a barrier at the origin.

11.3 Existence of Solutions to the Dirichlet Problem.

We can now prove our first general existence Theorem for the Dirichlet problem.

Theorem 11.3. *Let Ω be a bounded domain that satisfies the exterior cone condition. Moreover, assume that $f \in C^\alpha_c(\mathbb{R}^n)$ and $g \in C(\partial\Omega)$. Then there exists a unique solution to*

$$\begin{array}{ll} \Delta u(x) = f(x) & \textit{in } \Omega \\ u(x) = g(x) & \textit{on } \partial\Omega. \end{array} \tag{11.13}$$

Proof: We know that, if N is the Newtonian potential then

$$v(x) = \int_{\mathbb{R}^n} N(x-y) f(y) dy$$

solves $\Delta v(x) = f(x)$ in $\mathbb{R}^n$. It is therefore enough to show that there exist a solution to

$$\begin{array}{ll} \Delta w(x) = 0 & \text{in } \Omega \\ w(x) = \tilde{g}(x) = g(x) - v(x) & \text{on } \partial\Omega, \end{array} \tag{11.14}$$

since then $u(x) = v(x) + w(x)$ would be a solution to (11.13).

By the Perron process we can find a harmonic function

$$w(x) = \sup_{h \in S_{\tilde{g}}(\Omega)} h(x) \tag{11.15}$$

and since every point in $\partial\Omega$ is regular it follows from Theorem 11.2 that $\lim_{x\to\xi} w(x) = \tilde{g}(\xi)$ for any $\xi \in \partial\Omega$. It follows that the function defined by (11.15) solves the boundary value problem (11.14).

Uniqueness is an easy consequence of the maximum principle. In particular if u^1 and u^2 are two solutions to (11.13) then $u^1(x) - u^2(x)$ is a harmonic function with zero boundary data in Ω. From the maximum principle we can deduce that $u^1(x) - u^2(x) = 0$, that is $u^1 = u^2$. $\square$

Chapter 12

Exercises:

Exercise 1. Let $g \in C(\partial\Omega)$ where Ω is a bounded domain. Define the Perron solution

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

Assume that u is sub-harmonic and prove that $\Delta u(x) = 0$ in Ω.

(Hint: Consider the harmonic replacement $\tilde{u}$ in some ball $B_r(x) \subset \Omega$. Use comparison to conclude that $\tilde{u} \geq u$, how does that relate to the definition of u?)

Remark: The above proof is much simpler than the one we gave during the lectures. The reason that we did not use that proof is that in order to show that u is sub-harmonic one need to show that u is integrable. That is to show that the supremum of an uncountable family of integrable functions is integrable. That requires measure theory (and also a slightly different definition of subharmonicity) which we do not assume for this course.

Exercise 2. Let $\Omega = B_1(0) \setminus \{x \in \mathbb{R}^3;\ x_1 = x_2 = 0\}$ be the unit ball in $\mathbb{R}^3$ minus the x_3-axis. Show that the origin is not a regular point with respect to Ω.

(Hint: How did we prove that the origin was not regular with respect to the punctured disk $B_1(0) \setminus \{0\}$ in $\mathbb{R}^2$? What is the relation between the punctured disk in $\mathbb{R}^2$ and Ω?)

Exercise 3. We say that $u \in C(\Omega)$ is a viscosity solution to $\Delta u(x) = 0$ in Ω if for any second order polynomial $p(x)$ the following holds:

1. if $u(x) - p(x)$ has a local maximum at x^0 then $\Delta p(x) \leq 0$ and
2. if $u(x) - p(x)$ has a local minimum at x^0 then $\Delta p(x) \geq 0$.

Prove that if $u \in C^2(\Omega)$ is harmonic then u is a viscosity solution to $\Delta u(x) = 0$. Prove that if $u \in C^2(\Omega)$ is a viscosity solution to $\Delta u(x) = 0$ then $u(x)$ is harmonic.

(HINT: Assume that 1 or 2 holds at a point $x^0 \in \Omega$ what is the second order Taylor expansion at x^0?)

Exercise 4. Assume that $u \in C^2(\Omega)$ is a solution to the following partial differential equation

$$\begin{array}{ll} \Delta u(x) = u(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega. \end{array}$$

Prove that $u(x) = 0$ in Ω.

Exercise 5. Prove that any convex function is subharmonic.

Chapter 13

Variable coefficients.

So far we have been able to show existence for solutions to the Dirichlet problem for Laplace equation. It is of some interest to generalize that result to more general equations. We will consider the following general elliptic second order PDE,

$$\begin{array}{ll} \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array} \tag{13.1}$$

where Ω is some bounded domain, $g \in C(\partial\Omega)$, $a_{ij}(x)$, $b_i(x)$, $c(x)$ and $f(x)$ are given functions.

Equation (13.1) is to general for us to be able to say anything specific about the solution $u(x)$. We need to impose some conditions on $a_{ij}(x)$, $b_i(x)$ and $c(x)$ to assure that the solutions are "well behaved".

A powerful tool we used in the solution of the Laplace equation was the maximum principle. To assure that solutions $u(x)$ to (13.1) satisfy the maximum principle we make the following definition.

Definition 13.1. *We say that an partial differential, equation defined a domain Ω,*

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x)$$

is strictly elliptic in Ω if there exists a constant $\lambda > 0$ such that for all $x \in \Omega$

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for any vector $\xi = (\xi_1, \xi_2, ..., \xi_n)$.

Remark: If we let A be the matrix with coefficients $a_{ij}(x)$ then the ellipticity condition say that

$$\begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & & a_{nn}(x) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \geq \lambda |\xi|^2.$$

This is the same as demanding that all (generalized) eigenvalues[1] of A are greater than λ.

One might ask what ellipticity has to do with the maximum principle. A simple example will suffice to show that ellipticity is related to the maximum principle.

Example: Let Ω be a bounded domain, $\epsilon > 0$ and $u(x) \in C^2(\Omega) \cap C(\Omega)$ be a solution to

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) = \epsilon \quad \text{in } \Omega.$$

Assume furthermore, for simplicity, that $a_{ii}(x) = a_i(x)$ and $a_{ij}(x) = 0$ for $i \neq j$, $a_{ii}(x) \geq 1$ (that is the PDE is elliptic with $\lambda = 1$) and that $c(x) \leq 0$. Then $u(x)$ does not have any non-negative interior maximum.

This is quite obvious. We argue by contradiction and assume that $u(x)$ has an interior non-negative maximum at $x^0 \in \Omega$. Then $\nabla u(x^0) = 0$ and $\frac{\partial^2 u(x^0)}{\partial x_i^2} \leq 0$. We can thus calculate

$$0 < \epsilon = \underbrace{\sum_{i=1}^{n} a_{ii}(x^0) \frac{\partial^2 u(x^0)}{\partial x_i^2}}_{\leq 0} + \underbrace{\sum_{i=1}^{n} b_i(x^0) \frac{\partial u(x^0)}{\partial x_i}}_{=0 \text{ since } \nabla u = 0} + \underbrace{c(x^0)u(x^0)}_{\leq 0} \leq 0,$$

where we used that $a_{ii}(x^0) \geq 1$, and $c(x) \leq 0$ by assumption and that $u(x^0) \geq 0$ since x^0 is the non-negative maximum. Clearly this is a contradiction. In particular, elliptic PDE with $c(x) \leq 0$ seems to satisfy a maximum principle.

Remark on different kinds of PDE: We will only study elliptic PDE in this course. However, there are other classes of important PDE that appears in the applied sciences. Besides elliptic the most important classes of PDE are *parabolic* and *hyperbolic*.

The heat equation,

$$\Delta u(x,t) - \frac{\partial u(x,t)}{\partial t} = 0,$$

is the archetypical *parabolic* equation. A parabolic equation is, more or less, an elliptic equation minus a time derivative.

[1]Since $\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \frac{\partial^2 u(x)}{\partial x_i \partial x_j}$ for a C^2 function there is no loss of generality to assume that A is diagonalizable and thus that the n eigenvalues exists.

The third important class of equations is represented by the *wave equation*

$$\Delta u(x,t) - \frac{\partial^2 u(x,t)}{\partial t^2} = 0.$$

The wave equation is the basic representative of the *hyperbolic* PDE.

Of the three classes of PDE one can say that elliptic and parabolic are the most similar. Most of the results for elliptic PDE also exist for parabolic PDE. For instance the maximum principle (suitably interpreted) and the regularity theory that we develop also exist for parabolic PDE. However, one needs to formulate the problems and results slightly different for parabolic PDE since the PDE has a time variable t. We will not discuss parabolic or hyperbolic equations in this course.

13.1 The maximum Principle for Elliptic PDE.

For simplicity we will write, for any $u \in C^2(\Omega)$

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) \quad \text{in } \Omega, \tag{13.2}$$

where L is an elliptic operator, $a_{ij}(x), b_i(x), c(x) \in C(\Omega)$.

Lemma 13.1. [THE WEAK MAXIMUM PRINCIPLE.] *Suppose that $u \in C^2(\Omega)$, where Ω is a bounded domain, and $Lu(x) = f(x)$ where $f(x) \in C(\Omega)$. Assume that*

1. *$c(x) \le 0$ and $f(x) > 0$ or*
2. *$c(x) < 0$ and $f(x) \ge 0$.*

The $u(x)$ does not achieve a positive local maximum in Ω.

In particular, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ then

$$\sup_{\Omega} u(x) = \sup_{\partial\Omega} u(x).$$

Proof: The proof is very similar to the example in the previous section. We argue by contradiction and assume that $u(x^0) > 0$ and that x^0 is a local maximum for $u(x)$. Then

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} =$$

$$= \underbrace{-c(x)u(x) + f(x)}_{>0 \text{ at } x^0}.$$

Since $u(x^0)$ is a local maximum we can conclude that $\nabla u(x^0) = 0$ and $D^2 u(x^0)$ is a non-positive matrix.

In particular,

$$0 < \sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} + \underbrace{\sum_{i=1}^{n} b_i(x^0)\frac{\partial u(x^0)}{\partial x_i}}_{=0} = \sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} \tag{13.3}$$

If we can show that the right hand side in (13.3) is non positive we get the desired contradiction.

Since the matrix $A(x^0) = [a_{ij}(x^0)]_{ij}$ is strictly positive by ellipticity it has a square root $\sqrt{A}$. Also $-D^2u_\epsilon(x^0)$ is non-negative so it has a square root $\sqrt{-D^2u(x^0)}$. Now we notice that

$$\sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x^0)}{\partial x_i \partial x_j} = \mathrm{trace}\left(A \cdot D^2u(x^0)\right) =$$

$$= -\mathrm{trace}\left(A \cdot \left(-D^2u(x^0)\right)\right) = -\mathrm{trace}\left(\sqrt{A}\sqrt{A}\sqrt{-D^2u(x^0)}\sqrt{-D^2u(x^0)}\right) =$$

$$= -\mathrm{trace}\left(\left(\sqrt{A}\sqrt{-D^2u(x^0)}\right)^T \sqrt{A}\sqrt{-D^2u(x^0)}\right) \leq 0,$$

where we have used linear algebra freely and that the last inequality follows from $\mathrm{trace}(C^T \cdot C) = \sum_{i,j=1}^n (c_{ij})^2 \geq 0$ for any matrix C. This finishes the proof. $\square$

Corollary 13.1. [The Comparison Principle.] *Let Ω be a bounded domain and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$\begin{array}{ll} Lu(x) \geq Lv(x) & \text{in } \Omega \\ u(x) \leq v(x) & \text{on } \partial\Omega. \end{array}$$

Then, if $c(x) \leq 0$, it follows that $u(x) \leq v(x)$ in Ω.

Proof: The proof is simple, and follows directly from Lemma 13.1 if $Lu(x) > Lv(x)$ since then $L(u-v) > 0$ and can not archive a positive maximum.

We will modify the function $u - v$ by a function w to obtain the strict inequality and then use Lemma 13.1 to prove the Corollary.

To that end we define

$$w(x) = e^{Nr^2} - e^{N|x|^2},$$

where r is chosen large enough that $w(x) \geq 0$ in Ω and N is to be determined later. Notice that

$$Lw(x) = \sum_{i,j=1}^{n} a_{ij}(x)\left(-2N\delta_{ij} - 4N^2 x_i x_j\right)e^{N|x|^2} + \tag{13.4}$$

$$+\underbrace{\sum_{i=1}^{n} b_i(x)\left(-2Nx_i\right)e^{N|x|^2}}_{\leq 2Ne^{N|x|^2}|x|\sup_\Omega|b(x)|} + \underbrace{c(x)(e^{Nr^2} - e^{N|x|^2})}_{\leq 0} \leq$$

$$\leq \sum_{i,j=1}^{n} a_{ij}(x)\left(-2N\delta_{ij} - 4N^2 x_i x_j\right) e^{N|x|^2} + 2Ne^{N|x|^2}|x| \sup_{\Omega} |b(x)|,$$

where we used that $c(x) \leq 0$ and $w \geq 0$ in Ω. We need to estimate

$$\sum_{i,j=1}^{n} a_{ij}(x)\left(-2N\delta_{ij} - 4N^2 x_i x_j\right) = -2N\sum_{i=1}^{n} a_{ii}(x) - 4N^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j \leq$$
$$\leq -2N\lambda - 4N^2\lambda|x|^2, \tag{13.5}$$

since the first sum is just the trace of A and the second sum can be estimated from below by $\lambda|x|^2$ by the definition of ellipticity with $\xi = x$.

Using (13.5) in the estimate (13.4) we can conclude that

$$Lw(x) \leq -4N^2\lambda\left(\frac{1}{2N} + |x|^2 - \frac{|x|\sup_\Omega |b(x)|}{2N\lambda}\right) e^{N|x|^2} = \tag{13.6}$$

$$= -4N^2\lambda\left(\underbrace{\left(|x| - \frac{B}{4N\lambda}\right)^2}_{\geq 0} + \frac{1}{2N} - \frac{1}{N^2}\left(\frac{B}{4\lambda}\right)^2\right) e^{N|x|^2} < 0,$$

where $B = \sup_\Omega |b(x)|$ and the last inequality follows if N is large enough.

In particular $Lw(x) < 0$, so $w(x)$ is a super-solution.

Now consider

$$h_\epsilon(x) = u(x) - v(x) - \epsilon w(x).$$

Then

$$\begin{array}{ll} Lh_\epsilon(x) > 0 & \text{in } \Omega \\ h_\epsilon(x) \leq -\epsilon w(x) & \text{on } \partial\Omega. \end{array}$$

We may conclude, from Lemma 13.1, that h_ϵ can not obtain an interior maximum. Thus, for any $\epsilon > 0$,

$$\sup_\Omega \left(u(x) - v(x) - \epsilon w(x)\right) \leq \epsilon \sup_{\partial\Omega}(-w).$$

If we let $\epsilon \to 0$ this implies that

$$\sup_\Omega \left(u(x) - v(x)\right) \leq 0 \Rightarrow u(x) \leq v(x).$$

$\square$

Corollary 13.2. *Let Ω be a bounded domain and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$\begin{array}{ll} Lu(x) = Lv(x) & \textit{in } \Omega \\ u(x) = v(x) & \textit{on } \partial\Omega. \end{array}$$

Then, if $c(x) \leq 0$, it follows that $u(x) = v(x)$ in Ω.

Proof: By the previous Corollary it follows that $u(x) \leq v(x)$ and $v(x) \leq u(x)$ in Ω. $\square$

Chapter 14

Apriori estimates.

We know that the solutions to

$$\begin{array}{lr} Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array} \tag{14.1}$$

are unique, if they exist. The difficult part is to prove existence. That will take considerable effort. We will start lay the foundations of the existence theory in this chapter. In the next section we will prove sketch a strategy of how to solve the problem. In particular, we will try to motivate the need for apriori estimates. Then we will prove the estimates for the Laplace equation. At the end of the chapter we will prove existence in a very basic case and use that basic case as a springboard for a continued discussion of the strategy.

14.1 Discussion.

We need to find an approach to analyze a very difficult equation. We are in particular interested in showing existence of solutions. One way to approach the problem is to first consider operators L that somehow are close to the Laplace equation - which we can solve. Let us consider

$$L_t u(x) = \Delta u(x) + t\left(L - \Delta\right) u(x),$$

then $L_0 = \Delta$ and $L_1 = L$ so, at least intuitively, $L_t \approx \Delta$ for small t and for $t = 1$ we are back at the general case. If we assume that, for every small t, there exists a solution $u_t(x)$ to the following equation

$$\begin{array}{ll} L_t u_t(x) = f(x) & \text{in } \Omega \\ u_t(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

Then for t small we would expect $u_t(x) \approx u^0(x) + tu^1(x)$ for some functions $u^0(x)$ and $u^1(x)$. What equations would we have to solve to calculate u^0 and

u^1? If we set $t = 0$ we get, since $L_0 \cdot = \Delta \cdot$,

$$\begin{array}{ll} L_0 u_0(x) = \Delta u^0(x) f(x) & \text{in } \Omega \\ u^0(x) = g(x) & \text{on } \partial\Omega, \end{array}$$

which is fine since we know how to solve the Dirichlet problem for the Laplacian. However to calculate $u^1(x)$ we would need to solve, and here I am rather informal,

$$f(x) = L_t u_t(x) \approx \Delta(u^0 + tu^1) + t\,(L - \Delta)\,(u^0 + tu^1) \approx \underbrace{\Delta u^0}_{=f(x)} + t\left(\Delta u^1 + (L - \Delta)\, u^0\right),$$

where we have disregarded terms of order t^2. We see that we need to solve

$$\Delta u^1(x) = (\Delta - L)\, u^0(x), \tag{14.2}$$

Equation (14.2) is in principle fine since we can solve the Dirichlet problem and the right hand side is well defined. But we have only shown that $u^0 \in C^2(\Omega)$ so the right hand side of (14.2) is, as far as we know, only continuous. But we need the right hand side to be C^α to solve (14.2).

In general, we will need to improve our regularity results so that $\Delta u(x) = f(x) \in C^\alpha$ implies that $u \in C^{2,\alpha}$. We will prove this in the next section and also show that these estimates are strong enough to show existence in some simple cases.

14.2 Interior Aproiri Estimates for the Laplacian.

Sinc eour aim in this section is to estimate $\left|D^2u(x) - D^2u(y)\right|$ where $\Delta u(x) = f(x)$ we need to have a better understanding of the Newtonian kernel which we will provide in the next lemma.

Lemma 14.1. *Let $x, y \in \mathbb{R}^n$, $|x - y| = r$ and*

$$N_{ij}(x) = \frac{\partial^2 N(x)}{\partial x_i \partial x_j}$$

be the second derivatives of the Newtonian kernel. Then,

$$|N_{ij}(x - \xi) - N_{ij}(y - \xi)| \le \frac{C|x - y|}{|x - \xi|^{n+1}}$$

for any $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$.

Proof: Fix a $\xi \in \mathbb{R}^n \setminus B_{2r}(x)$. Then $N(z - \xi) \in C^\infty(\mathbb{R}^n \setminus \{z = \xi\})$. In particular, $N(z - \xi) \in C^\infty(B_{3r/2}(x))$ so we may calculate

$$|N_{ij}(x - \xi) - N_{ij}(y - \xi)| = \left|\int_0^1 (x - y) \cdot \nabla N_{ij}(sx + (1 - s)y - \xi)ds\right| \le$$

$$\leq |x-y| \sup_{z\in B_r(x)} |\nabla N_{ij}(z-\xi)|\,. \tag{14.3}$$

Next we notice that

$$\sup_{z\in B_r(x)} |\nabla N_{ij}(z-\xi)| \leq \sup_{z\in B_{|\xi-x|+r}(0)\setminus B_{|\xi-x|-r}} |\nabla N_{ij}(z)|\,.$$

But since

$$N(z)=N(x)=\begin{cases} -\frac{1}{2\pi}\ln(|x|) & \text{for } n=2 \\ -\frac{1}{(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{for } n\neq 2, \end{cases}$$

It follows that

$$\sup_{z\in B_{|\xi-x|+r}(0)\setminus B_{|\xi-x|-r}} |\nabla N_{ij}(z)| \leq \frac{C_n}{(|\xi-x|-r)^{n+1}}, \tag{14.4}$$

but if $\xi\in\mathbb{R}^n\setminus B_{2r}(x)$ it clearly follows that

$$|\xi-x|-r\geq \frac{1}{2}|\xi-x|$$

from which we may conclude that

$$\sup_{z\in B_{|\xi-x|+r}(0)\setminus B_{|\xi-x|-r}} |\nabla N_{ij}(z)| \leq \frac{C_n 2^n}{(|\xi-x|)^{n+1}}.$$

Using this last inequality together with (14.4) and (14.3) will result in

$$|N_{ij}(x-\xi)-N_{ij}(y-\xi)|\leq \frac{C_n 2^n|x-y|}{(|\xi-x|)^{n+1}}$$

which is the conclusion of the Lemma up to the naming of a constant. □

Theorem 14.1. *Let $f(x)\in C_c^\alpha(B_{2R}(0))$ for some $0<\alpha<1$ and define*

$$u(x)=\int_{\mathbb{R}^n} N(x-\xi)f(\xi)d\xi$$

then for any $x,y\in B_R(0)$, $x\neq y$, the following inequality holds

$$\frac{\left|\frac{\partial^2 u(x)}{\partial x_i\partial x_j}-\frac{\partial^2 u(y)}{\partial x_i\partial x_j}\right|}{|x-y|^\alpha}\leq C_{\alpha,n}\left([f]_{C^\alpha(B_{2R}(0))}+\frac{|x-y|^{1-\alpha}\sup_{B_R(0)}|f(x)|}{R}\right). \tag{14.5}$$

In particular, $u\in C^{2,\alpha}(B_R(0))$ and

$$[D^2u]_{C^\alpha(B_R(0)}\leq C_{\alpha,n}\left([f]_{C^\alpha(B_{2R}(0))}+\frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right),$$

where $C_{\alpha,n}$ only depends on the dimension and α.

Proof: We have already shown that $u(x) \in C^2$ and that

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_{2R}(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_{2R}(0)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi).$$

We will use this representation to prove (14.5). We set $r = |x-y|$ and calculate

$$\left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right| \leq$$

$$\leq \left| \int_{B_{2R}(0)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_{2R}(0)} N_i(x-\xi) \nu_j dA(\xi) - \right.$$

$$\left. - \int_{B_{2R}(0)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi + f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi) \nu_j dA(\xi) \right| \leq$$

$$\leq \left| \int_{B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi \right| + \left| \int_{B_{2r}(x)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi \right| +$$

$$+ \left| \int_{B_{2R}(0) \setminus B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi + \int_{B_{2R}(0) \setminus B_{2r}(x)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi + \right.$$

$$\left. + f(x) \int_{\partial B_{2R}(0)} N_i(x-\xi) \nu_j dA(\xi) - f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi) \nu_j dA(\xi) \right|$$

$$\leq \left| \int_{B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi \right| + \left| \int_{B_{2r}(y)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi \right| +$$

$$+ \left| \int_{B_{2R}(0) \setminus B_{2r}(x)} \left(N_{ij}(x-\xi) - N_{ij}(y-\xi)\right) \left(f(\xi) - f(x)\right) d\xi \right| + \tag{14.6}$$

$$+ \left| \int_{B_{2R}(0) \setminus B_{2r}(x)} N_{ij}(y-\xi) \left(f(y) - f(x)\right) d\xi + \right.$$

$$\left. - f(x) \int_{\partial B_{2R}(0)} N_i(x-\xi) \nu_j dA(\xi) + f(y) \int_{\partial B_{2R}(0)} N_i(y-\xi) \nu_j dA(\xi) \right|.$$

We will estimate the terms in turn. First we use that $|f(\xi) - f(x)| \leq [f]_{C^\alpha} |\xi - x|^\alpha$ to conclude that

$$\left| \int_{B_{2r}(x)} N_{ij}(x-\xi) \left(f(\xi) - f(x)\right) d\xi \right| \leq [f]_{C^\alpha} \int_{B_{2r}(x)} |N_{ij}(x-\xi)| \, |\xi - x|^\alpha d\xi \leq$$

$$\leq C[f]_{C^\alpha} \int_{B_{2r}(x)} |\xi - x|^{\alpha - n} d\xi \leq \frac{C[f]_{C^\alpha}}{\alpha} (2r)^\alpha \leq C_\alpha [f]_{C^\alpha} r^\alpha$$

where C_α only depend on α, and n. Similarly we may estimate

$$\left| \int_{B_{2r}(x)} N_{ij}(y-\xi) \left(f(\xi) - f(y)\right) d\xi \right| \leq C_\alpha [f]_{C^\alpha} r^\alpha.$$

Next we use Lemma 14.1 to estimate

$$\left|\int_{B_{2R}(0)\setminus B_{2r}(x)} \left(N_{ij}(x-\xi) - N_{ij}(y-\xi)\right)\left(f(\xi) - f(x)\right) d\xi\right| \leq$$

$$\leq [f]_{C^\alpha} \int_{B_{2R}(0)\setminus B_{2r}(x)} \frac{C|x-y|}{|x-\xi|^{n-1}} |\xi - x|^\alpha d\xi \leq$$

$$\leq C[f]_{C^\alpha}|x-y| \int_{B_{2R}(0)\setminus B_{2r}(x)} \frac{C}{|x-\xi|^{n+1-\alpha}} d\xi \leq$$

$$\leq C[f]_{C^\alpha} r \left(\frac{1}{(2r)^{1-\alpha}} - \frac{1}{(2R)^{1-\alpha}}\right) \leq C[f]_{C^\alpha} r^\alpha + C[f]_{C^\alpha} \frac{r}{R^{1-\alpha}}.$$

To estimate the final integral in (14.6) we do an integration by parts in the first term and use the triangle inequality as follows

$$\left| \int_{B_{2R}(0)\setminus B_{2r}(x)} N_{ij}(y-\xi)\left(f(y) - f(x)\right) d\xi - \right.$$

$$\left. -f(x)\int_{\partial B_{2R}(0)} N_i(x-\xi)\nu_j dA(\xi) + f(y)\int_{\partial B_{2R}(0)} N_i(y-\xi)\nu_j dA(\xi)\right| =$$

$$\left| - \int_{\partial B_{2R}(0)} N_i(y-\xi)\nu_j \left(f(y) - f(x)\right) d\xi - \int_{\partial B_{2r}(x)} N_i(y-\xi)\nu_j \left(f(y) - f(x)\right) d\xi \right.$$

$$\left. -f(x)\int_{\partial B_{2R}(0)} N_i(x-\xi)\nu_j dA(\xi) + f(y)\int_{\partial B_{2R}(0)} N_i(y-\xi)\nu_j dA(\xi)\right| \leq$$

$$\leq \left| \int_{\partial B_{2r}(x)} N_i(y-\xi)\nu_j \underbrace{\left(f(y) - f(x)\right)}_{\leq [f]_{C^\alpha} r^\alpha} d\xi \right| +$$

$$+|f(x)| \int_{\partial B_{2R}(x)} \underbrace{|N_i(x-\xi) - N_i(y-\xi)|}_{\leq \frac{C|x-y|}{R^n} \text{ on } \partial B_{2R}} dA(\xi) \leq$$

$$\leq C[f]_{C^\alpha} r^\alpha + \frac{C|f(x)|r}{R},$$

notice that we get out an extra minus when we integrte by parts in the first equality since ξ hs a minus in the argument of $N_{ij}(y-\xi)$.

Collecting the terms we arrive at

$$\left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j}\right| \leq C\left([f]_{C^\alpha}\left(|x-y|^\alpha + \frac{|x-y|}{R^{1-\alpha}}\right) + \frac{|f(x)||x-y|}{R}\right) \leq$$

$$\leq C\left([f]_{C^\alpha}|x-y|^\alpha + \frac{|x-y| \sup_{B_R(0)}|f(x)|}{R}\right),$$

dividing both sides by $|x-y|^\alpha$ and taking the supremum over all $x, y \in B_R(0)$ gives the desired estimate. □

Observe that the above Theorem only estimates the second derivatives in $B_R(0)$ - that is away from the boundary. For further applications we will however need the estimate close to the boundary.

Proposition 14.1. *Let Ω be a domain and assume that $u(x)$ is a solution to*

$$\Delta u(x) = f(x) \qquad \text{in } \Omega$$

assume furthermore that $|u| \leq M$ in Ω and that $f \in C^\alpha_{loc}(\Omega)$ and that for any compact set $K \subset \Omega$ the function $f(x)$ satisfies the following estimate

$$\sup_{x\in K} |f(x)| \leq \frac{C_{0,f}}{dist(K, \partial\Omega)^2} \tag{14.7}$$

and

$$|f(x) - f(y)| \leq \frac{C_{\alpha,f}|x-y|^\alpha}{dist(K, \partial\Omega)^{2+\alpha}} \tag{14.8}$$

then there exists a constant $C_{n,\alpha}$ depending only on α and the dimension n such that

$$\sup_{x\in K} |D^2 u(x)| \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|}{dist(K, \partial\Omega)^2} \tag{14.9}$$

and

$$\sup_{x,y\in K} \frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \leq C_{n,\alpha} \frac{C_{0,f} + C_{\alpha,f} + \sup_\Omega |u|}{dist(K, \partial\Omega)^{2+\alpha}} \tag{14.10}$$

Proof: We will begin by showing (14.9). The proof is not that difficult - even though the result is very technical.

Part 1: *The inequality (14.9) holds.*

We fix a compact set $K \subset \Omega$. Since K is compact and Ω open the distance $\text{dist}(K, \partial\Omega) > 0$, we define $d = \frac{\text{dist}(K,\partial\Omega)}{4} > 0$. Let $x^0 \in K$ be an arbitrary point then $\text{dist}x^0, \partial\Omega \geq 4d$ and the following function

$$v(x) = u(dx + x^0)$$

is well defined in $B_4(0)$. The chain rule implies that

$$\Delta v(x) = d^2 \Delta u(dx + x^0) = d^2 f(dx + x^0) \equiv g(x) \qquad \text{in } B_4(0),$$

where we define $g(x)$ in the last step.

We see that $v(x)$ and $g(x)$ satisfies

$$\sup_{x\in B_4(0)} |v(x)| = \sup_{x\in B_{4d}(x^0)} |u(x)| \leq \sup_{x\in\Omega} |u(x)|,$$

$$\sup_{x\in B_3(0)} |g(x)| = d^2 \sup_{x\in B_{3d}(x^0)} |f(x)| \leq C_{0,f},$$

where we have used (14.7) in the last inequality as well as $\mathrm{dist}(B_{3d}(x^0), \partial\Omega) \geq d$.

Furthermore, we may estimate for $x, y \in B_3(0)$

$$\frac{|g(x)-g(y)|}{|x-y|^\alpha} = d^2 \frac{|f(dx+x^0)-f(dy+x^0)|}{|x-y|^\alpha} =$$

$$\left\{ \begin{array}{l} \text{substitute} \\ \tilde{x} = dx + x^0,\ \tilde{y} = dy + x^0 \end{array} \right\} = d^{2+\alpha} \frac{|f(\tilde{x})-f(\tilde{y})|}{|\tilde{x}-\tilde{y}|^\alpha} \leq$$

$$\leq d^{2+\alpha} C_{\alpha,f} \underbrace{\mathrm{dist}(B_{3d}(x^0), \partial\Omega)^{-(2+\alpha)}}_{\leq d^{-(2+\alpha)}} \leq C_{\alpha,f},$$

where we again used that $\mathrm{dist}(B_{3d}(x^0), \partial\Omega) \geq d$ in the last inequality.

We have thus shown that $v(x)$ solves the following Dirichlet problem

$$\begin{array}{ll} \Delta v(x) = g(x) & \text{in } B_2(0) \\ v(x) = u(dx + x^0) & \text{on } \partial B_2(0), \end{array}$$

where $v(x)$ and $g(x)$ are is bounded by $\sup_\Omega |u|$ and $C_{0,f}$ respectively and $[g]_{C^\alpha(B_3(0))} \leq C_{\alpha,f}$.

Next we let $\varphi(x) \in C_c^\infty(B_3(0))$ be such that $\varphi = 1$ in $B_2(0)$ and $|\nabla\varphi| \leq 2$.[1] We also define

$$w(x) = \int_{\mathbb{R}^n} N(x-y) g(y)\varphi(y) dy,$$

where $N(x-y)$ is the Newtonian kernel, notice that the integral is well defined since $g(y)\varphi(y) = 0$ outside of $B_3(0)$. Clearly,

$$\sup_{B_2(0)} |w(x)| \leq \sup_{B_3(0)} |g(x)| \int_{B_2(0)} N(x-y) dy \leq C_n \sup_{B_3(0)} |g(x)| = C_n C_{0,f},$$

where the constant only depend on the dimension. Furthermore, by the estimates in Theorem 1 in the first set of notes[2] we know that

$$\left| \frac{\partial^2 w(0)}{\partial x_i \partial x_j} \right| =$$

$$= \left| \int_{B_3(0)} \frac{\partial^2 N(y)}{\partial x_i \partial x_j} \left(g(y)\varphi(y) - g(0) \right) dy - g(0) \int_{\partial B_3(x)} \frac{\partial N(y)}{\partial x_i} \nu_j(\xi) dA(y) \right|$$

$$\leq \left| \int_{B_3(0)} \frac{\partial^2 N(y)}{\partial x_i \partial x_j} \left(g(y)\varphi(y) - g(0) \right) dy \right| +$$

[1] That this is possible is easy to see geometrically, or one could define $\varphi(x) = \int_{B_{5/2}(0)} \phi_{1/4}(x-y) dy$ where $\phi_{1/4}$ is the standard mollifier.

[2] See step 3 of that proof.

$$+|g(0)|\underbrace{\left|\int_{\partial B_3(x)}\frac{\partial N(y)}{\partial x_i}\nu_j(\xi)dA(y)\right|}_{\substack{\leq 1 \text{ by direct}\\ \text{calculation}}}$$

$$\leq (C_{\alpha,f}+C_{0,f})\int_{B_3(0)}\frac{1}{|y|^{n-\alpha}}dy + C_{0,f} \leq C_n\left(C_{\alpha,f}+C_{0,f}\right)$$

We can conclude that

$$\begin{array}{ll} \Delta w(x) = g(x) & \text{in } B_2(0) \\ |D^2 w(0)| \leq C_n\left(C_{\alpha,f}+C_{0,f}\right) & \text{and} \\ \sup_{B_2(0)}|w(x)| \leq C_n C_{0,f}. & \end{array}$$

This in turn implies that $h(x) = v(x) - w(x)$ satisfies

$$\Delta h(x) = 0 \quad \text{in } B_2(0)$$

$$\sup_{B_2(0)}|h(x)| \leq \sup_{B_2(0)}|v(x)| + \sup_{B_2(0)}|w(x)| \leq \sup_{\Omega}|u(x)| + C_n C_{0,f}.$$

In particular, we can conclude from our interior regularity for harmonic functions that

$$|D^2 h(0)| \leq \frac{n^3 2^{2n+4}}{\omega_n 2^{n+2}}\|h\|_{L^1(B_2(0))} \leq C_n\left(\sup_{\Omega}|u(x)| + C_{0,f}\right).$$

We may conclude that

$$\left|D^2 v(0)\right| = \left|D^2\left(v(0) - w(0) + w(0)\right)\right| =$$

$$= \left|D^2\left(h(0) + w(0)\right)\right| \leq C_{n,\alpha}\left(\sup_{\Omega}|u(x)| + C_{0,f} + C_{\alpha,f}\right).$$

But

$$D^2 v(0) = d^2 D^2 u(x^0)$$

which implies

$$\left|D^2 u(x^0)\right| \leq \frac{C_{n,\alpha}\left(\sup_{\Omega}|u(x)| + C_{0,f} + C_{\alpha,f}\right)}{d^2} =$$

$$= \frac{16 C_{n,\alpha}\left(\sup_{\Omega}|u(x)| + C_{0,f} + C_{\alpha,f}\right)}{\operatorname{dist}(K,\partial\Omega)^2}.$$

This proves part 1.

Part 2: *The inequality (14.10) holds.*

We use the same set-up as in part 1 and let $K \subset \Omega$ be a compact set and $x^0, y^0 \in K$ be arbitrary points. First we notice that if $|x^0 - y^0| \geq d$ then the estimate immediately follows, indeed:

$$\frac{|D^2 u(x^0) - D^2 u(y^0)|}{|x^0 - y^0|^\alpha} \leq \frac{|D^2 u(x^0)| + |D^2 u(y^0)|}{|x^0 - y^0|^\alpha} \leq$$

$$\leq \frac{2^{1+2\alpha}C_{n,\alpha}\left(\sup_\Omega|u(x)|+C_{0,f}+C_{\alpha,f}\right)}{\mathrm{dist}(K,\partial\Omega)^2|x^0-y^0|^\alpha} \leq \frac{2C_{n,\alpha}\left(\sup_\Omega|u(x)|+C_{0,f}+C_{\alpha,f}\right)}{\mathrm{dist}(K,\partial\Omega)^{2+\alpha}},$$

which is the desired estimate. Therefore we may assume, without loss of generality, that $|x^0-y^0|<d$.

We define $v(x)$ as in part 1 of this proof. Then, with $z^0=\frac{y^0-x^0}{d}\in B_1(0)$

$$\frac{\left|D^2v(0)-D^2v(z^0)\right|}{|0-z^0|^\alpha} = d^{2+\alpha}\frac{|D^2u(x^0)-D^2u(y^0)|}{|x^0-y^0|^\alpha}.$$

Therefore it is enough to show that

$$\frac{\left|D^2v(0)-D^2v(z^0)\right|}{|z^0|^\alpha} \leq C_{n,\alpha}\left(C_{0,f}+C_{\alpha,f}+\sup_\Omega|u|\right).$$

If we define $w(x)$ and $h(x)$ as in Part 1 of this proof then it follows from Theorem 14.1, in particular from (14.1), that

$$\frac{\left|\frac{\partial^2 w(0)}{\partial x_i\partial x_j}-\frac{\partial^2 w(z^0)}{\partial x_i\partial x_j}\right|}{|z^0|^\alpha} \leq C_{\alpha,n}\left(C_{\alpha,f}+C_{0,f}\right),$$

where we have used that $|z^0|<1$, that $[g]_{C^\alpha(B_2(0))}\leq C_{\alpha,f}$ and $\sup_{B_2(0)}|g|\leq C_{0,f}$.

Next we estimate

$$\frac{\left|\frac{\partial^2 h(0)}{\partial x_i\partial x_j}-\frac{\partial^2 h(z^0)}{\partial x_i\partial x_j}\right|}{|z^0|^\alpha} \leq \frac{\sup_{x\in B_1(0)}|D^3h(x)||z_0|}{|z^0|^\alpha} \leq \sup_{x\in B_1(0)}|D^3h(x)|$$

where we used the mean value theorem for the derivative to conclude that $\frac{\partial^2 h(0)}{\partial x_i\partial x_j}-\frac{\partial^2 h(z^0)}{\partial x_i\partial x_j}=z^0\cdot\nabla\frac{\partial^2 h(\xi)}{\partial x_i\partial x_j}$ for some ξ on the line from the origin to z^0. But since $h(x)$ is harmonic in $B_2(0)$ it follows that

$$\sup_{B_1(0)}|D^3h(x)|\leq C_n\|h\|_{L^1(B_2(0))}\leq C_n\sup_{B_2(0)}|h(x)|\leq C_n\left(\sup_\Omega|u(x)|+C_{0,f}\right).$$

We can thus conclude that

$$\frac{\left|D^2v(0)-D^2v(z^0)\right|}{|z^0|^\alpha} \leq \frac{\left|D^2w(0)-D^2w(z^0)\right|}{|z^0|^\alpha}+$$

$$+\frac{\left|D^2h(0)-D^2h(z^0)\right|}{|z^0|^\alpha} \leq C_{n,\alpha}\left(C_{0,f}+C_{\alpha,f}+\sup_\Omega|u|\right).$$

This finishes the proof. □

14.2.1 An application.

Before we consider the general case of an elliptic PDE we will consider a simpler perturbation result with a PDE that is some sense is close to the Laplace equation. We will improve on the following result significantly later in the course.

In this section we will assume that

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \qquad \text{in } B_1(0), \tag{14.11}$$

where a_{ij} satisfies the ellipticity condition and the following conditions

$$\|a_{ij}\|_{C^\alpha(B_1(0))} \leq \epsilon \qquad \text{for } i,j = 1,2,...,n \text{ and } i \neq j \tag{14.12}$$

and

$$\|a_{ii} - 1\|_{C^\alpha(B_1(0))} \leq \epsilon \qquad \text{for } i = 1,2,...,n. \tag{14.13}$$

The conditions (14.12) and (14.13) means that the partial differential operator is close to Laplace in some sense. In particular, if $\epsilon = 0$ then $L = \Delta$.

Lemma 14.2. *Let $f(x) \in C^\alpha(B_1(0))$ and $g(x) \in C(\partial B_1(0))$. Assume furthermore that L is as in (14.11) and that $a_{ij}(x)$ satisfies (14.12)-(14.13). Assume furthermore that there exists a $\delta > 0$ such that*

$$a_{ij}(x) = 0 \qquad \text{if } x \in B_1(0) \setminus B_{1-\delta}(0) \text{ and } i \neq j,$$

and

$$a_{ii}(x) = 1 \qquad \text{if } x \in B_1(0) \setminus B_{1-\delta}(0),$$

that is $L\cdot = \Delta\cdot$ in $B_1(0) \setminus B_{1-\delta}(0)$.

Then there exists an $\epsilon_\delta > 0$ (depending on $\delta > 0$ as well as f, g, a_{ij} and Ω) such that if $\epsilon < \epsilon_\delta$ then there exists a unique solution to

$$\begin{aligned} Lu(x) = f(x) & \quad \text{in } \Omega \\ u(x) = g(x) & \quad \text{on } \partial\Omega. \end{aligned}$$

Proof: Even though we have made many preparations the proof is quite complicated. We will prove the Lemma by constructing a convergent sequence of approximating solutions starting with the Dirichlet problem.

Observe that we can find a solution, $u^0(x)$, to

$$\begin{aligned} \Delta u^0(x) = f(x) & \quad \text{in } \Omega \\ u^0(x) = g(x) & \quad \text{on } \partial\Omega. \end{aligned}$$

We will inductively define $u^k(x)$, for $k = 1,2,...$, as the solution to

$$\begin{aligned} \Delta u^k(x) = \Delta u^{k-1} - Lu^{k-1}(x) + f(x) & \quad \text{in } \Omega \\ u^k(x) = g(x) & \quad \text{on } \partial\Omega. \end{aligned}$$

Since we are going to work with the differences $u - u^{k-1}$ for most of the proof we define $w^k(x) = u^k(x) - u^{k-1}(x)$ for $k \geq 1$ and $w^0(x) = u^0(x)$.

Then

$$\Delta w^k(x) = \Delta u^{k-1}(x) - Lu^{k-1}(x) - \Delta u^{k-2}(x) + Lu^{k-2}(x) = \tag{14.14}$$
$$= (\Delta - L)\left(w^{k-1}(x)\right).$$

But on each compact set $K \subset \Omega$ we have, by Proposition 14.1 in particular (14.9), that

$$\sup_{x\in K} |D^2 w^k(x)| \leq \tag{14.15}$$
$$\leq C_{n,\alpha} \frac{d^2 \sup_{x\in K} |(\Delta - L)w^{k-1}| + d^{2+\alpha}\left[(\Delta - L)(w^{k-1})\right]_{C^\alpha(K)} + \sup_\Omega |w^k|}{d^2}$$

where $d = \mathrm{dist}(K, \partial\Omega)$ and similarly

$$\left[D^2 w^k(x)(x))\right]_{C^\alpha(K)} \leq \tag{14.16}$$
$$\leq C_{n,\alpha} \frac{d^2 \sup_{x\in K} |(\Delta - L)w^{k-1}| + d^{2+\alpha}[(\Delta - L)w^{k-1}]_{C^\alpha(K)} + \sup_{B_1(0)} |w^k|}{d^{2+\alpha}}.$$

But clearly, with the notation $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$,

$$\sup_{x\in K} |(\Delta - L)w^{k-1}| \leq$$
$$\underbrace{\sup_{i,j=1,\ldots,n} \left(\sup_{x\in K} |a_{ij}(x) - \delta_{ij}|\right)}_{\leq\epsilon} \sup_{x\in K} |D^2 w^{k-1}(x)| \leq$$
$$\leq \epsilon \sup_{x\in K} |w^{k-1}(x)|$$

and similarly[3]

$$\left[(\Delta - L)(w^{k-1})\right]_{C^\alpha(K)} \leq \epsilon \left[D^2 w^{k-1}\right]_{C^\alpha(K)}.$$

In conclusion we get form (14.15) and (14.16) that

$$\sup_{x\in K} |D^2 w^k(x)| \leq \tag{14.17}$$
$$\leq C_{n,\alpha}\epsilon \left(\sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) + \frac{\sup_{B_1(0)} |w^k|}{d^2}$$

and

$$\left[|D^2 w^k(x)\right]_{C^\alpha(K)} \leq \tag{14.18}$$
$$\leq \frac{C_{n,\alpha}\epsilon}{d^\alpha} \left(\sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) + \frac{\sup_{B_1(0)} |w^k|}{d^{2+\alpha}}$$

[3] Here we are cheating a little. To be exact, we are skipping some details. The assertion is justified but it uses some results that we will cover later.

We need to estimate $\sup_\Omega |w^k|$. Notice that $w^k(x) = u^k - u^{k-1} = 0$ on $\partial B_1(0)$ and from (14.14) we get that

$$\left|\Delta w^k\right| = \left|(\Delta - L)\, w^{k-1}(x)\right| \le$$
$$\le \epsilon \sup_{x\in B_{1-\delta}(0)} \left|D^2 w^{k-1}(x)\right|.$$

Therefore, by the comparison principle we can deduce that

$$-B(x) \le w^k(x) \le B(x) \tag{14.19}$$

where

$$B(x) = \frac{\epsilon \sup_{x\in B_{1-\delta}(0)} \left|D^2 w^{k-1}(x)\right|}{2n}(1-|x|^2).$$

In particular $B(x) = w^k(x) = 0$ on $\partial B_1(0)$ and $-\Delta B(x) \ge \Delta w^k \ge \Delta B(x)$.

We may thus estimate

$$\sup_{B_1(0)} |w^k(x)| \le \epsilon \sup_{x\in B_{1-\delta}(0)} \left|D^2 w^{k-1}(x)\right|. \tag{14.20}$$

Using (14.20) in (14.17) and (14.18) we can conclude that

$$\begin{aligned}
&\sup_{x\in K} |D^2 w^k(x)| + d^\alpha \left[D^2 w^k(x)\right]_{C^\alpha(K)} \le \\
&\le C\epsilon \left(\frac{1+d^2}{d^2} \sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) \le \\
&\le \frac{1}{2}\left(\sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right)
\end{aligned} \tag{14.21}$$

where the last inequality follows if ϵ is small enough, say $\epsilon \le \frac{c\delta^2}{2}$ for some small c, and $d \ge \delta$.

Equation (14.21) is the heart of the proof since it shows that

$$\begin{aligned}
&\sup_{x\in K} |D^2 w^k(x)| + d^\alpha \left[D^2 w^k(x)\right]_{C^\alpha(K)} \le \\
&\le \frac{1}{2}\left(\sup_{x\in K} |D^2 w^{k-1}(x)| + d^\alpha \left[D^2 w^{k-1}\right]_{C^\alpha(K)}\right) \le \\
&\le \frac{1}{2^2}\left(\sup_{x\in K} |D^2 w^{k-2}(x)| + d^\alpha \left[D^2 w^{k-2}\right]_{C^\alpha(K)}\right) \le \\
&\le \cdots \le \frac{1}{2^k}\left(\sup_{x\in K} |D^2 w^{0}(x)| + d^\alpha \left[D^2 w^{0}\right]_{C^\alpha(K)}\right).
\end{aligned}$$

That is $D^2 u^k(x)$ forms a Cauchy sequence in $C^\alpha(K)$ for any $K \in B_{1-\delta}(0)$. So, by the Arzela-Ascoli Theorem, $u^k \to u$ uniformly on $B_{1-\delta}(0)$. And on

$B_1(0) \setminus B_{1-\delta}(0)$ it directly follows that u^k converges, at least for a subsequence, since $\Delta u^k(x) = f(x)$ in $B_1(0) \setminus B_{1-\delta}(0)$.

So $u^k \to u$ in $C^{2,\alpha}_{\text{loc}}(B_1(0))$. By the definition of u^k it als follows that

$$\Delta u^k = (\Delta - L)u^{k-1}(x) + f(x) \Rightarrow Lu^{k-1} = \underbrace{\Delta(u^{k-1}(x) - u^k(x))}_{\to 0 \text{ in } C^\alpha_{\text{loc}}} + f(x).$$

It follows that

$$Lu(x) = f(x) \qquad \text{in } B_1(0).$$

Using (14.19) together with

$$|w^k(x)| \leq B(x) \leq \frac{C(1-|x|^2)}{2^k}$$

we can conclude that

$$|u^k(x) - u^0(x)| \leq \sum_{i=1}^k |u^k - u^{k-1}| \leq \sum_{i=1}^k |w^k(x)| \leq C(1-|x|^2) \sum_{i=1}^k \frac{1}{2^i} \leq C(1-|x|^2)$$

which implies that for every $k = 1, 2, 3, ...$

$$u^0(x) - C(1-|x|^2) \leq u^k(x) \leq u^0(x) + C(1-|x|^2).$$

Since $u^0 \in C(\overline{\Omega})$ we can conclude that $u(x) \in C(\overline{\Omega})$.

Uniqueness of $u(x)$ follows by the maximum principle, in particular Corollary 13.2. $\square$

Remarks: There are several things to say about this Lemma.

1. First of all, the result is very unsatisfactory in several respects. The most obvious is that we assume that $a_{ij}(x) = \delta_{ij}$ in $B_1(0) \setminus B_{1-\delta}(0)$. But this assumption is necessary for us to estimate (14.21).

   That we need this assumption is because we our estimates of $D^2u(x)$ breaks down when x is close to the boundary. In particular the presence of the inverse of the distance to the boundary in the statement of Proposition 14.1.

   Therefore we need to develop a theory that better estimates the solutions close to the boundary, estimates without the $\text{dist}(K, \partial\Omega)^{-1}$ terms.

2. We need to develop some better terminology in order not to get lost in the technicalities. In particular we need to develop the language of Banach spaces as well as some functional analysis.

3. It is also rather unsatisfactory that the proof only works for small ϵ. The theory only works for small ϵ because, and this is very important, we do not have a regularity theory for the general equation $Lu(x) = f(x)$. If we had such a theory we could, using the terminology of the beginning of this chapter, apply the same proof in Lemma 14.2 to find a solution to $L_{t+\epsilon}u(x) = f(x)$ if we could solve a solution to $L_t u(x) = f(x)$ - and then for $L_{t+2\epsilon}u(x) = f(x)$ etc.

In the next chapter we will continue to develop the regularity theory for elliptic PDE. Then we will see that the interior regularity theory actually implies boundary regularity. Once we have the regularity theory in place we will be able to show existence for the general equation $Lu(x) = f(x)$ - under some assumptions on L and on the boundary of Ω.

Chapter 15

Apriori interior estimates for constant coefficient PDE.

In the last chapter we saw that we can estimate $[D^2u]_{C^\alpha}$ for the solution to $\Delta u(x) = f(x)$ in terms of f and $\sup|u|$. And very importantly, we also saw that such estimates leads to existence of solutions for PDE with coefficients that are close, in $C^\alpha-$norm, to that are close to the coefficients of Δ (that is $a_{ij}(x) \approx \delta_{ij}$). We will use this knowledge to construct solutions to general variable coefficients PDE.

In particular, if we consider a general linear PDE with variable coefficients:

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \quad \text{in } \Omega, \tag{15.1}$$

where $a_{ij}(x), b_i(x)$ and $c(x) \in C^\alpha$. Then if we consider a small enough ball $B_r(x^0) \subset \Omega$ then

$$a_{ij}(x) \approx a_{ij}(x^0),\ b_i(x) \approx b_i(x^0) \text{ and } c(x) \approx c(x^0) \qquad \text{in } B_r(x^0).$$

This means that in the small ball $B_r(x^0)$ we will have that $L\cdot$ is close to a PDE with constant coefficients:

$$Lu(x) \approx \sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x^0)\frac{\partial u(x)}{\partial x_i} + c(x^0)u(x) \approx f(x) \quad \text{in } B_r(x^0). \tag{15.2}$$

One usually say that a PDE like (15.2) has *frozen coefficients* and the method we will use is often called *freezing of the coefficients*.

Thus if we understand constant coefficient PDE better then we should be able to better understand a variable coefficient equation. The method is quite subtle, and it is not at all clear at this point that freezing of the coefficients will yield any useful results. However, in this chapter we will prove a simple regularity result for constant coefficient equations. In the next chapter we will show that we can actually freeze the coefficients to get a good regularity theory for variable coefficient equations.

Before reading the rest of this chapter it is advisable to read the appendixes on Banach spaces and interpolation inequalities.

Proposition 15.1. *Assume that Ω is a bounded domain and that $u(x) \in C^2(\Omega)$ solves the following constant coefficient PDE*

$$\sum_{ij=1}^{n} a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) \qquad \text{in } \Omega$$

where a_{ij} are constants satisfying the following ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \tag{15.3}$$

for some constants $\Lambda, \lambda > 0$ and all $\xi \in \mathbb{R}^n$.

Then, for any $0 < \alpha < 1$ there exists a constant $C = C(\lambda, \Lambda, n, \alpha)$ such that

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \leq C \left(\|u\|_{C(\Omega)} + \|f\|_{C^{\alpha}_{int,(2)}(\Omega)} \right).$$

Proof: The proof is very simple. We will show that a change of variables transforms $u(x)$ into a harmonic function $v(x)$ and the estimates for $u(x)$ follows from the corresponding estimates for harmonic functions. We will do the proof in several steps - some of them we will only sketch.

Step 1: *We may change variables to transform $u(x)$ into a harmonic function.*

Since the matrix $A = [a_{ij}]$ is symmetric we may write it as

$$A = O^T D O,$$

where O is an orthogonal matrix (with rows consisting of the eigenvectors of A) and D is the diagonal matrix with the eigenvalues of A along the diagonal. Using that A is elliptic, (15.3), we know that the eigenvalues of A are bounded from above and below by Λ and $\lambda > 0$ and we may thus take the square root of D. Now define $P = \sqrt{D} O$, then it follows that $A = P^T P$. Expressed in terms of components:

$$a_{jk} = \sum_{i=1}^{n} p_{ij} p_{ik}.$$

So if we define

$$v(x) = u(Px)$$

then

$$\sum_{i=1}^{n} \frac{\partial^2 v(x)}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} p_{ij} \frac{\partial u(Px)}{\partial x_j} \right) =$$

$$= \sum_{i,j,k=1}^{n} p_{ij} p_{ik} \frac{\partial^2 u(Px)}{\partial x_j \partial x_k} = \sum_{j,k=1}^{n} \underbrace{\left(\sum_{i=1}^{n} p_{ij} P_{ik} \right)}_{=a_{jk}} \frac{\partial^2 u(Px)}{\partial x_j \partial x_k} =$$

$$= \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u(Px)}{\partial x_j \partial x_k} = f(Px).$$

Thus it follows that $\Delta v(x) = f(Px)$.

It follows from Proposition 1 (Part 5 of these notes, also reformulated in Proposition 17.1 in the appendix) that

$$\|v\|_{C^{2,\alpha}_{\text{int}}(\Omega)} \leq C \left(\|u\|_{C(\Omega)} + \|f(P\cdot)\|^{(2)}_{C^{\alpha}_{\text{int},(2)}(\Omega)} \right). \tag{15.4}$$

Step 2: *Bound of* $|\nabla u(x)|$ *on compact sets.*

Since P is an orthogonal matrix times a diagonal matrix with diagonal elements in $[\sqrt{\lambda}, \sqrt{\Lambda}]$ it follows that P is invertible. We may therefore write

$$u(x) = v(P^{-1}x).$$

In particular,

$$\nabla u(x) = P^{-1} \cdot \nabla v(P^{-1}x),$$

But since all eigenvalues of P^{-1} lay in the interval $[\Lambda^{-1/2}, \lambda^{-1/2}]$ it follows that

$$|\nabla u(x)| \leq \frac{1}{\sqrt{\lambda}} |\nabla v(P^{-1/2}x)|. \tag{15.5}$$

Now for any compact set $K \subset \Omega$ we have that

$$P(K) = \{Px;\ x \in K\} \subset P(\Omega) = \{Px;\ x \in \Omega\},$$

and if

$$\text{dist}(K, \partial\Omega) = d \quad \text{ then } \text{dist}(P(K), \partial P(\Omega)) \geq \sqrt{\lambda} d. \tag{15.6}$$

In particular for any $x \in K \subset \Omega$ it follows (15.4), (15.5) and (15.6) that

$$|\nabla u(x)| \leq \frac{C}{\lambda} \frac{\left(\|u\|_{C(\Omega)} + \|f(P\cdot)\|^{(2)}_{C^{\alpha}_{\text{int},(2)}(\Omega)} \right)}{\text{dist}(K, \partial\Omega)}. \tag{15.7}$$

Step 3: *Estimates for* $D^2 u(x)$ *and* $[D^2 u]_{C^\alpha(K)}$.

This works exactly the same as in step 1. That is we may write $D^2 u$ and $[D^2 u]_{C^\alpha}$ in terms of v and use (15.4). $\square$

Chapter 16

Apriori interior estimates for PDE with variable coefficients.

We are now ready to prove interior apriori estimates[1] for equations with variable coefficients. We will prove the following estimate

$$\|u\|_{C^{2,\alpha}_{\text{int}}(\Omega)} \leq C\left(\|f\|_{C^{\alpha}_{\text{int},(2)}} + \|u\|_{C(\Omega)}\right)$$

where $C = C(n, \alpha, \Omega, \lambda, \Lambda)$ and on the coefficients in the equation:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \tag{16.1}$$

We have already seen all the ideas that we are going to use. Our method of proof will be to freeze the coefficients. In particular, if the coefficients of the equation are close to constant, say that $|a_{ij}(x) - a_{ij}(x^0)| \leq \epsilon$ for some small $\epsilon > 0$ in a ball $B_r(x^0)$ then we may write equation (16.1) as

$$\sum_{i,j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) - \sum_{i,j=1}^{n} \underbrace{\left(a_{ij}(x) - a_{ij}(x^0)\right)}_{\leq \epsilon}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \tag{16.2}$$

$$-\sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} - c(x)u(x) \quad \text{ in } B_r(x^0)$$

[1]An apriori estimate is an estimate for an equation that is made before we know that solutions exist. Typically one assumes that there exist a solution in some Banach space, say $C^{2,\alpha}_{\text{int}}(\Omega)$, and then proves that there is a bound on the norm of that space that does not depend on the solution.

We can view this as a constant coefficient equation (with right hand side depending on u) and apply Proposition 15.1 and derive that

$$\|u\|_{C^{2,\alpha}_{\text{int}}(B_r(x^0))} \le C\left(\|u\|_{C(B_r(x^0))} + \|F\|_{C^{\alpha}_{\text{int},(2)}(B_r(x^0))}\right),$$

where $F(x)$ is the right hand side in (16.2). Now $\|F\|_{C^{\alpha}_{\text{int},(2)}(B_r(x^0))}$ will depend on u. But since we multiply the second derivatives of the $u-$term by something of order ϵ in (16.2) the dependence will not be significant if ϵ is small enough.

Therefore we can estimate the $C^{2,\alpha}_{\text{int}}(B_r(x^0))$ (or even the norm in Ω) if $a_{ij}(x) \approx a_{ij}(x^0)$. But, and here is the second main idea[2], if the coefficients are continuous then $|a_{ij}(x) - a_{ij}(x^0)| \le \epsilon$ in $B_r(x^0)$ for any x^0 if $r > 0$ is small enough. And since we can cover any compact set $K \subset \Omega$ by finitely many balls $B_r(x)$ it is enough to do prove the regularity in a small ball.

We are now ready to state and prove the Theorem.

Theorem 16.1. *Let $u \in C^{2,\alpha}_{int}(\Omega)$, where Ω is a bounded domain and $\alpha \in (0,1)$, be a solution to*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \quad \text{in } \Omega.$$

Assume furthermore that $a_{ij}(x), f(x) \in C^{\alpha}(\Omega)$ and that $a_{ij}(x)$ satisfy the ellipticity condition $\lambda|\xi|^2 \le \sum_{ij} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$. Then there exists a constant $C = C(n,\alpha,\Omega,\lambda,\Lambda,a_{ij})$ such that

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \le C\left(\|f\|_{C^{\alpha}_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)}\right).$$

Proof: Let $K \subset \Omega$ be a compact set. We need to show that

$$\sum_{j=0}^{2}\left(\text{dist}\,(K,\partial\Omega)^j \sup_{x\in K}|D^j u(x)|\right) + \text{dist}\,(K,\partial\Omega)^{2+\alpha} \sup_{x,y\in K}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\alpha}} \le$$

$$\le C\left(\|f\|_{C^{\alpha}_{\text{int},(2)}(\Omega)} + \|u\|_{C(\Omega)}\right).$$

But by the interpolation inequality (Proposition 18.1 in the appendix.) it is enough to show that

$$\text{dist}\,(K,\partial\Omega)^{2+\alpha} \sup_{x,y\in K}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\alpha}} \le C\left(\|f\|_{C^{\alpha}_{\text{int},(2)}} + \|u\|_{C(\Omega)}\right). \quad (16.3)$$

We will prove the Theorem in three steps. First we will cover K by balls $B_{\delta_K}(x^k)$ in a very specific way, then we will prove (16.3) for an ball $B_\delta(x^k)$. In the final step we will show that it is enough to prove the Theorem for the balls $B_{\delta_K}(x^k)$ in order to prove the Theorem.

Step 1: *Let $K \subset \Omega$ be a compact set and $\epsilon > 0$ be a fixed constant (to be determined later) depending only on the coefficients of L. Then we may cover K by a balls $B_{\delta_K}(x^k)$. Where the balls $B_{\delta_K}(x^k)$ may be chosen to satisfy*

[2] Freezing of the coefficients was the first

1. $B_{4\delta_K}(x^k) \subset \Omega$,
2. $|a_{ij}(x) - a_{ij}(x^k)| < \epsilon$ *in* $B_{\delta_K}(x^k)$,
3. $\delta_K \geq \frac{\operatorname{dist}(K,\partial\Omega)}{4}$ if $\operatorname{dist}(K,\partial\Omega)$ is small enough.

Since $\|a_{ij}\|_{C^\alpha(\Omega)} < \infty$ there is a $\mu_\epsilon > 0$ such that for any $x \in \Omega$ we have $|a_{ij}(x) - a_{ij}(y)| < \epsilon$ for every $y \in B_{\mu_\epsilon}(x)$. Now let us denote

$$d_K = \frac{\operatorname{dist}(K,\partial\Omega)}{4}$$

and $\delta = \min(d_K, \mu_\epsilon)$. Then obviously $K \subset \cup_{x\in K} B_\delta(x)$. Since K is compact we can find a finite sub-cover $B_\delta(x^k)$ as described in step 1.

Step 2: *The following estimate holds*

$$\delta_K^2 \|D^2 u\|_{C^\alpha_{\mathrm{int}}(B_{2\delta_K}(x^k))} \leq C_L \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} \right),$$

where C_L depend on the coefficients a_{ij}, b_i and c through their $C^\alpha(\Omega)-$norm and the ellipticity constants λ, Λ and also on the dimension n.

Here we use the freezing of the coefficients argument and write, in the ball $B_{2\delta_K}(x^k)$

$$\sum_{i,j=1}^n a_{ij}(x^0) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) - \sum_{i,j=1}^n \left(a_{ij}(x) - a_{ij}(x^0) \right) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} -$$

$$- \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} - c(x) u(x) = F(x),$$

where $F(x)$ is defined by the last inequality.

Viewing this a s a constant coefficient PDE we may use Proposition 15.1 to deduce that

$$\delta_K^2 \|D^2 u\|_{C^\alpha_{\mathrm{int}}(B_{2\delta_K}(x^k))} \leq C \left(\|u\|_{C(\Omega)} + \|F\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} \right) \leq$$

$$\leq C \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} + \left\| \sum_{i,j=1}^n \left(a_{ij}(x) - a_{ij}(x^0) \right) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} \right) +$$

$$+ C \left(\left\| \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} \right\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c(x) u(x)\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} \right) \leq$$

$$\leq C \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)} \right) +$$

$$+\sum_{i,j=1}^{n} \underbrace{\left\|a_{ij}(x)-a_{ij}(x^0)\right\|_{C(B_{2\delta_K})}}_{<\epsilon} \left\|\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right\|_{C_{\mathrm{int},(2)}(B_{2\delta_K})} + \tag{16.4}$$

$$+\sum_{i,j=1}^{n} \underbrace{\left\|a_{ij}(x)-a_{ij}(x^0)\right\|_{C(B_{2\delta_K})}}_{<\epsilon} \left[\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right]_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} +$$

$$+\sum_{i,j=1}^{n} \left[a_{ij}(x)-a_{ij}(x^0)\right]_{C^\alpha(B_{2\delta_K})} \left\|\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right\|_{C_{\mathrm{int},(2)}(B_{2\delta_K})} +$$

$$+C\left(\sum_{i=1}^{n} \|b_i(x)\|_{C^\alpha(\Omega)} \left\|\sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i}\right\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c\|_{C(\Omega)} \|u(x)\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})}\right) \leq$$

$$< C\left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\mathrm{int},(2)}(\Omega)}\right) +$$

$$+C\left(\sum_{i=1}^{n} \|b_i(x)\|_{C(\Omega)} \left\|\sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i}\right\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c\|_{C(\Omega)} \|u(x)\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})}\right) +$$

$$+C_L\delta_K^2 \left\|D^2 u\right\|_{C_{\mathrm{int}}(B_{2\delta_K})} + \underbrace{C\epsilon\delta_K^2 \left[D^2 u\right]_{C_{\mathrm{int}}(B_{2\delta_K})}}_{\leq \frac{\delta_K^2}{4}[D^2u]_{C_{\mathrm{int}}(B_{2\delta_K})}},$$

where the constant C_L depend on the coefficients a_{ij}, b_i and c through their $C^\alpha(\Omega)-$norm and the ellipticity constants λ, Λ and also on the dimension n. We have also used that $[\cdot]_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} \leq C\delta_K^2[\cdot]_{C^\alpha_{\mathrm{int}}(B_{2\delta_K})}$ and the final "underbrace" holds if ϵ is small enough.

Using Propoisition 18.1 in the appendix we can deduce that

$$C_L\delta_K^2 \left\|D^2 u\right\|_{C_{\mathrm{int}}(B_{2\delta_K})} \leq$$

$$\leq C_L C_\epsilon \delta_K^2 \|D^2 u\|_{C_{\mathrm{int}}(B_{2\delta_K})} + \epsilon C_L \delta_K^2 \left[D^2 u\right]_{C^\alpha_{\mathrm{int}}(B_{2\delta_K})} \leq \tag{16.5}$$

$$\leq C_L C_\epsilon \delta_K^2 \|D^2 u\|_{C_{\mathrm{int}}(B_{2\delta_K})} + \frac{\delta_K^2}{4} \left[D^2 u\right]_{C^\alpha_{\mathrm{int}}(B_{2\delta_K})},$$

where the last inequality holds if ϵ is small enough.

We may also use the the interpolation inequality to estimate the lower order terms:

$$C\left(\sum_{i=1}^{n} \|b_i(x)\|_{C^\alpha(\Omega)} \left\|\sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i}\right\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})} + \|c\|_{C(\Omega)} \|u(x)\|_{C^\alpha_{\mathrm{int},(2)}(B_{2\delta_K})}\right) \leq \tag{16.6}$$

$$\leq C_L C_\epsilon \delta_K^2 \|D^2 u\|_{C_{\mathrm{int}}(B_{2\delta_K})} + \frac{\delta_K^2}{4} \left[D^2 u\right]_{C^\alpha_{\mathrm{int}}(B_{2\delta_K})}.$$

Using (16.5) and (16.6) in (16.4) we can deduce that, for a somewhat larger C_L,

$$\begin{aligned} \delta_K^2 \|D^2u\|_{C^\alpha_{\text{int}}(B_{2\delta_K}(x^k))} \leq \\ \leq C_L \left(\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)} \right) + \\ + \frac{3\delta_K^2}{4} \|D^2u\|_{C^\alpha_{\text{int}}(B_{2\delta_K}(x^k))}. \end{aligned} \tag{16.7}$$

Rearranging terms in (16.7) implies the statement in step 2.

Step 3: *Proof of the Theorem.*

Since the balls $B_{\delta_K}(x^k)$ cover K it follows directly from step 2 and that $\delta_K \geq \frac{\text{dist}(K,\partial\Omega)}{4}$ that

$$\sup_K |D^2u(x)| \leq C_{a_{ij}} \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^2}.$$

Moreover, for any two $x, y \in K$ such that $|x-y| > \frac{\text{dist}(K,\partial\Omega)}{8}$ it follows that $x, y \in B_{\delta_k}(x^k)$ for some ball and thus

$$\frac{\left|D^2u(x) - D^2u(y)\right|}{|x-y|^\alpha} \leq \frac{2}{|x-y|^\alpha} \sup_K |D^2u(x)| \leq C \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^{2+\alpha}}.$$

So we only need to estimate $\frac{\left|D^2u(x)-D^2u(y)\right|}{|x-y|^\alpha}$ for $|x-y| \leq \frac{\text{dist}(K,\partial\Omega)}{8} \leq \frac{\min_k(\delta_K)}{2}$. But if $|x-y| \leq \frac{\min_k(\delta_K)}{2}$ then there exists a ball $B_{\delta_K}(x^k)$ such that $x, y \in B_{\delta_K}(x^k)$ so we may use step 2 again and conclude that

$$\frac{\left|D^2u(x) - D^2u(y)\right|}{|x-y|^\alpha} \leq C \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^{2+\alpha}}.$$

Thus it follows that

$$\text{dist}(K,\partial\Omega)^2 \|D^2u\|_{C^\alpha_{\text{int}}(\Omega)} \leq C \frac{\|u\|_{C(\Omega)} + \|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}}{\text{dist}(K,\partial\Omega)^{2+\alpha}},$$

where $C = C(n, \alpha, \Omega, \lambda, \Lambda, a_{ij})$. The Theorem follows by the interpolation inequality Proposition 18.1. $\square$

Chapter 17

Barnach Spaces.

We will need some notation from functional analysis in order to simplify the exposition somewhat. The point of this appendix is not to cover functional analysis, which is a very large area of mathematics. But just to remind ourselves of some basic notions. We start with the following definition.

Definition 17.1. *We say that a set A is a linear space over $\mathbb{R}$ if*

1. *A is a commutative group. That is there is an operation "$+$" defined on $A \times A \mapsto A$ such that*
   (a) *For any $u, v, w \in A$ the following holds: $u + v = v + u$ (addition is commutative), $(u + v) + w = u + (v + w)$ (addition is associative).*
   (b) *There exists an element $0 \in A$ such that for all $u \in A$ we have $u + 0 = u$.*
   (c) *For every $u \in A$ there exists an element $v \in A$ such that $u + v = 0$, we usually denote $v = -u$.*
2. *There is an operation (multiplication) defined on $\mathbb{R} \times A \mapsto A$ such that*
   (a) *For all $a, b \in \mathbb{R}$ and $u, v \in A$ we have $a \cdot (u + v) = a \cdot u + a \cdot v$ and $(a + b) \cdot u = a \cdot u + b \cdot u$.*
   (b) *For all $a, b \in \mathbb{R}$ and $u \in A$ we have $(ab) \cdot u = a \cdot (b \cdot u)$.*

Examples: **1:** The most obvious example is if $A = \mathbb{R}^n$ and "$+$" is normal vector addition and "$\cdot$" is normal multiplication by a real number.

2: Another example that will be much more important to us is if A is a set of functions, say the set of functions with two continuous derivatives on Ω. Clearly all the above assumptions are satisfied for twice continuously differentiable functions if we interpret "$+$" and "$\cdot$" as the normal operations.

Many linear spaces satisfies another important structure: that we can measure distances. Distances allow us to talk about convergence and to do analysis. We will only be interested in spaces where we have a norm.

Definition 17.2. *A norm $\|\cdot\|$ on a linear space A is a function from $A \mapsto \mathbb{R}$ such that the following axioms are satisfied:*

1. *For any $u \in A$ we have $\|u\| \geq 0$ with equality if and only if $u = 0$ (The Positivity Axiom).*
2. *For any $u, v \in A$ we have $\|u+v\| \leq \|u\| + \|v\|$ (The Triangle Inequality).*
3. *For any $u \in A$ and $a \in A$ we have $\|a \cdot u\| = |a|\|u\|$ (The Homogeneity Axiom).*

If a linear space A has a norm we say that A is a normed linear space, or just a normed space.

Examples: 1: The linear space $\mathbb{R}^n$ is a normed space with norm $\|(u_1, ..., u_n)\| = (u_1^2 + u_2^2 + ... + u_n^2)^{1/2}$.

2: The set of continuous functions on $[0, 1]$ is a normed space under the norm

$$\|u\| = \int_0^1 |u(x)|dx.$$

3: If we define

$$\|u\|_{C^2(\Omega)} = \sup_{x\in\Omega} |u(x)| + \sup_{x\in\Omega} |\nabla u(x)| + \sup_{x\in\Omega} |D^2 u(x)|, \tag{17.1}$$

Then the set of two times continuously differentiable functions $u(x)$ on Ω for which $\|u\|_{C^2(\Omega)}$ is finite forms a normed space: $C^2(\Omega)$. Notice that $\frac{1}{x} \notin C^2(0, 1)$ even though $\frac{1}{x}$ is continuous with continuous derivatives on $(0, 1)$.

The final property that we need in our function-spaces is completeness.

Definition 17.3. *Let A be a normed linear space. Then we say that A is complete if every Cauchy sequence $u^j \in A$ converges in A.*

Remember that we say that $u^j \in A$ is a Cauchy sequence if there for every $\epsilon > 0$ exists a N_ϵ such that $\|u^j - u^k\| < \epsilon$ for all $j, k > N_\epsilon$. So if A is complete and u^j is a Cauchy sequence in A then there should exist an element $u^0 \in A$ such that $\lim_{j\to\infty} \|u^j - u^0\| = 0$.

Examples: 1: It is an easy consequence of the the Bolzano-Weierstrass theorem that $\mathbb{R}^n$ is complete. In particular, every Cauchy sequence is bounded. Therefore the Bolzano-Weierstrass theorem implies that it has a convergent subsequence. That the Cauchy condition implies that the entire sequence converges to the same limit is easy to see.

2: The space of continuous functions on $[0, 1]$ with norm $\|u\| = \int_0^1 |u(x)|dx$ is not complete. For instance if

$$u^j(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{j} \\ \frac{j}{2}\left(x - \left(\frac{1}{2} - \frac{1}{j}\right)\right) & \text{if } \frac{1}{2} - \frac{1}{j} < x < \frac{1}{2} + \frac{1}{j} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{j} \leq x \leq 1 \end{cases}$$

then u^j is continuous and forms a Cauchy sequence. However the limit is clearly

$$u^0(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

But u^0 is not continuous and therefore not in the space of continuous functions on $[0,1]$. Therefore that space is not complete.

However, if we consider the space $C([0,1])$ of continuous functions with norm

$$\|u\|_{C([0,1])} = \sup_{x\in[0,1]} |u(x)|$$

then we get a complete space. This since the limit $\lim_{j\to\infty} u^j(x)$ is uniform and continuity is preserved under uniform limits.

It is important to notice that the properties of the space is dependent on the norm. Continuous functions with an integral are not complete, but continuous spaces with a supremum norm are complete.

3: The space $C^2(\Omega)$ with norm defined by the supremum as in (17.1) is also a complete space.

Clearly, in order to do analysis on a linear space it is desirable that the linear space is complete. We therefore make the following definition.

Definition 17.4. *We call a complete linear space is a Banach space.*

17.1 Banach spaces and PDE.

Banach spaces helps us to formulate questions in PDE in a new way.

The initial way to view a PDE is to view it point-wise. That is, for the Laplace equation for instance, we think of a solution as twice differentiable function $u(x)$ that should satisfy $\sum_{i=1}^n \frac{\partial^2 u(x)}{\partial x_i^2} = f(x)$ at every point $x \in \Omega$. There is nothing wrong with this viewpoint, and as a matter of fact everything we do in Banach spaces will depend on results we derived by using this point of view. However, as the equations becomes more complicated it is reasonable to look for a simplified conceptualization of what a PDE is. By formulating a PDE as a problem in Banach spaces we are able to leave the point-wise viewpoint behind and consider the PDe as a mapping between Banach spaces.

Let us consider a function $u \in C^2(\Omega)$ and we let

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x)$$

be an elliptic partial differential operator with continuous coefficients.[1] Then for any $u \in C^2(\Omega)$ we clearly have that $Lu(x) \in C(\Omega)$.

[1] We already know that it is more natural to consider PDE with Hölder continuous coefficients. But it is enough to have continuous coefficients for us to introduce the next idea.

We can thus view the partial differential operator L as a map between Banach spaces: $L : C^2(\Omega) \mapsto C(\Omega)$. That is, for every $u \in C^2(\Omega)$ there exists an $f \in C(\Omega)$ such that $Lu(x) = f(x)$. Similarly, L maps the subset

$$C_g^2(\Omega) = \{u \in C^2(\Omega);\ u(x) = g(x) \text{ on } \partial\Omega\} \subset C^2(\Omega)$$

into $C(\Omega)$.

Solving the PDE

$$\begin{array}{ll} Lu(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array}$$

for a given $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ is therefore the same as finding an inverse L^{-1} of the mapping $L : C_g^2(\Omega) \mapsto C(\Omega)$. If such a mapping exists then the solution is given by $u(x) = L^{-1}f(x)$.

There are several reasons to change the re-conceptualize of a problem in mathematics. One reason is that changing the point of view might clarify a difficult concept, simplify statements or show that several problems have a similar underlying structure[2]. The most important reason to change the point of view on a subject is however that one might be able to use different techniques and prove new results in the new conceptualization.

In this section we will only reformulate some of our results in this new language and fix some notation. In later chapters we will prove some fixed point theorems[3] in Banach spaces that will help us to prove existence of solutions to PDE with variable coefficients.

Example: In Theorem 1 in Chapter 2 (in the first part of these lecture notes) we proved that if $f \in C_c^\alpha(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi)f(\xi)d\xi, \tag{17.2}$$

where $N(x)$ is the Newtonian kernel, solves $\Delta u(x) = f(x)$. Using the Liouville Theorem it is easy to see that the function $u(x)$ is the only solution to $\Delta u(x) = f(x)$ that tends to zero as $x \to \infty$.

If we consider Δ as an operator

$$\Delta :\ C_0^2(\mathbb{R}^n) = \{u \in C^2(\mathbb{R}^n);\ \lim_{x\to\infty} u(x) = 0\} \mapsto C(\mathbb{R}^n).$$

Then Theorem 1 actually shows that the inverse of the Laplacian, Δ^{-1}, is well defined on $C_c^\alpha(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ and given by (17.2).

Example: We know that Δ is does not have a well defined inverse from $C_c(\mathbb{R}^n)$ to $C^2(\mathbb{R}^n)$ since there are functions $u \notin C^2(\mathbb{R}^n)$ with $\Delta u \in C_c(\mathbb{R}^n)$, see exercise 3 in the first part of these notes.

[2]For instance, viewing a PDE as a mapping between linear spaces highlights a similarity between PDE and linear algebra that might not be so easy to see otherwise.

[3]Specifically, the contraction mapping principle that we will use to develop a technique called *the method of continuity*.

Based on the above two examples it is a reasonable question to ask between what spaces does Δ have an inverse? Or more generally, when does a variable coefficient PDE $L\cdot$ have an inverse. In the next section we will introduce some Banach spaces that we know are of importance in inverting PDE.

17.2 Some Banach spaces that are important for PDE.

We already know that the Hölder spaces $C^{k,\alpha}(\Omega)$ are important in PDE theory.

Definition 17.5. *Given a domain Ω and u a k-times continuously differentiable function on Ω we will use the notation, for $k \in \mathbb{N}$ and $\alpha \in [0,1]$,*

$$\|u\|_{C^{k,\alpha}(\Omega)} = \sum_{j=1}^{k} \sup_{x\in\Omega} |D^j u(x)| + \sup_{x,y\in\Omega} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}.$$

Furthermore, we let $C^{k,\alpha}(\Omega)$ denote the set of all two times differentiable functions for which $\|u\|_{C^{k,\alpha}(\Omega)} < \infty$.

When $\alpha = 0$ we will disregard α and the last term in the definition of $\|u\|_{C^{k,\alpha}(\Omega)}$ and write

$$\|u\|_{C^{k,\alpha}(\Omega)} = \|u\|_{C^{k}(\Omega)} = \sum_{j=1}^{k} \sup_{x\in\Omega} |D^j u(x)|,$$

and when $k = 0$ and $\alpha \in (0,1)$ we will write $\|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_{C^{\alpha}(\Omega)}$.

It is easy to that the space $C^{k,\alpha}(\Omega)$ is a Banach space.

Lemma 17.1. *The space $C^{k,\alpha}(\Omega)$ is a Banach space with the norm $\|u\|_{C^{k,\alpha}(\Omega)}$.*

Proof: It is trivial to verify that $C^{k,\alpha}(\Omega)$ is a linear space and that $\|u\|_{C^{k,\alpha}}(\Omega)$ is a norm. That $C^{k,\alpha}(\Omega)$ is complete follows by the Arzela-Ascoli Theorem. $\square$

It is quite often that we only need information about the Hölder continuity, we will therefore define the semi-norm[4]

$$[u]_{C^{\alpha}(\Omega)} = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

We have also seen that the $C^{k,\alpha}(\Omega)$ space is not always suitable for expressing our theorems. We will therefore use introduce the alternative norms $\|u\|_{C^{k,\alpha}_{\mathrm{int}}(\Omega)}$ and $\|u\|_{C^{k,\alpha}_{\mathrm{int},(l)}(\Omega)}$ that we use in our interior estimates.

[4]A semi-norm is satisfies all the requirements for a norm except that $\|u\| = 0 \Leftrightarrow u = 0$.

Definition 17.6. *For any $k-$times continuously differentiable function $u(x)$ defined on a domain Ω we denote by $\|u\|_{C^{k,\alpha}_{int}(\Omega)}$ the least constant Γ such that*

$$\sum_{j=0}^{k}\left(dist\,(K,\partial\Omega)^j \sup_{x\in K}|D^j u(x)|\right) + dist\,(K,\partial\Omega)^{k+\alpha}\sup_{x,y\in K}\frac{|D^k u(x)-D^k u(y)|}{|x-y|^\alpha}\leq\Gamma$$

for all compact sets $K\subset\Omega$.

More generally, we will define $\|u\|_{C^{k,\alpha}_{int,(l)}(\Omega)}$ to be the least constant Γ such that

$$\sum_{j=0}^{k}\left(dist\,(K,\partial\Omega)^{j+l} \sup_{x\in K}|D^j u(x)|\right) + dist\,(K,\partial\Omega)^{k+l+\alpha}\sup_{x,y\in K}\frac{|D^k u(x)-D^k u(y)|}{|x-y|^\alpha}\leq\Gamma$$

for all compact sets $K\subset\Omega$.

Furthermore we will denote by $C^{k,\alpha}_{int}(\Omega)$ and $C^{k,\alpha}_{int,(l)}(\Omega)$ the Banach spaces of $k-$times continuously differentiable functions for which the norms $\|u\|_{C^{k,\alpha}_{int}(\Omega)}$ and $\|u\|_{C^{k,\alpha}_{int,(l)}(\Omega)}$ are bounded.

It is easy to see that $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ are Banach spaces with their respective norms.

The norms of the spaces $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ controls the functions in the interior of Ω. In particular if $u\in C^{k,\alpha}_{\text{int},(l)}(\Omega)$ then $u\in C^{k,\alpha}(K)$ for any compact set $K\subset\Omega$. However, the norm $\|u\|_{C^{k,\alpha}(K)}$ will depend on the distance $\text{dist}(K,\partial\Omega)$ and in general functions in $C^{k,\alpha}_{\text{loc},(l)}(\Omega)$ will have infinite $C^{k,\alpha}(\Omega)$ norm. Some examples might clarify the situation.

Examples: **1.** Consider $u(x)=\sin\left(\ln\left(\frac{1}{x}\right)\right)$ defined on $(0,1/2)$. Clearly $u(x)$ is bounded and continuous so $u(x)\in C(0,1/2)$. However, $u\notin C^1(0,1/2)$ since $Du(x)=-\frac{1}{x}\cos\left(\ln\left(\frac{1}{x}\right)\right)$ which isn't bounded. But $u(x)\in C^1_{\text{int}}(0,1/2)$ since for any compact set $K=[\kappa,1/2-\kappa]\subset(0,1/2)$ we have

$$\sup_{x\in K}|u(x)|+\kappa\sup_{x\in K}|Du(x)|\leq 1+\kappa\sup_{x\in[\kappa,1/2-\kappa]}\left|\frac{1}{x}\cos\left(\ln\left(\frac{1}{x}\right)\right)\right|\leq 2.$$

Thus $\|u\|_{C^1_{\text{int}}(0,1/2)}=2$.

2: Let $u(x)=\frac{1}{1-x^2}$ be defined on $(-1,1)$. Then $u(x)$ is unbounded so $u\notin C^{k,\alpha}(-1,1)$ for any k or α.

However, $u\in C^{1,\alpha}_{\text{int},(1)}(-1,1)$ since for any compact set $K=[-1+\kappa,1-\kappa]$ we have

$$\|u\|_{C^{1,\alpha}_{\text{int},(1)}}=\kappa\sup_{x\in K}|u(x)|+\kappa^2\sup_{x\in K}|Du(x)|+\kappa^{2+\alpha}\sup_{x,y\in K}\frac{|Du(x)-Du(y)|}{|x-y|^\alpha}<\infty$$

where the upper bound is independent of $\kappa\in(0,1)$.

Observe that the norm on $C^{k,\alpha}_{\text{int},(l)}(-1,1)$ allows the function and its derivative to tend to infinity at the boundary of Ω. The parameter l determines how fast the function and its derivatives may go to infinity. For instance the above function $u \in C^{k,\alpha}_{\text{int},(l)}(-1,1)$ for any $l \geq 1$ but not for any $l < 1$.

It is important to realize that these norms, even though they appear to be artificial, they are natural. For instance we may formulate the interior regularity result for harmonic functions as:

Proposition 17.1. *Let Ω be a domain and assume that $u(x)$ is a solution to*

$$\Delta u(x) = f(x) \qquad \text{in } \Omega$$

assume furthermore that $|u| \leq M$ in Ω and that $f \in C^{\alpha}_{int,(2)}(\Omega)$ then there exists a constant $C_{n,\alpha}$ such that

$$\|u\|_{C^{2,\alpha}_{int}(\Omega)} \leq C_{n,\alpha}\left(\|f\|_{C^{\alpha}_{int,(2)}(\Omega)} + \|u\|_{C(\Omega)}\right). \tag{17.3}$$

The proof of Proposition 17.1 is a direct consequence of Proposition 1 in the 5th part of these lecture notes together with an interpolation inequality that we will prove in the next appendix. Notice that the norms $\|u\|_{C^{2,\alpha}_{\text{int}}(\Omega)}$ and $\|f\|_{C^{\alpha}_{\text{int},(2)}(\Omega)}$ appears in the statement - and that these norms makes the statement of the Proposition much more compact than the formulation of Proposition 1 in the fifth part of these notes. The norms are natural in the sense that (17.3) is optimal and we can not prove a stronger statement without adding further assumptions on the boundary data of u and on the geometry of Ω.

Remark on scaling: One heuristic way to see that (17.3) is natural is to consider the "scaling" of the estimate. Since $\Delta u(x)$ involves two derivatives it is natural that if $\Delta u = f$ then u should have two more derivatives than f. This explains that we have a $(2,\alpha)$ norm on the right hand side in (17.3) whereas the left hand side is only a Hölder $\alpha-$norm. Since we are not making any assumptions on the boundary data of u in Proposition 17.1 we can not expect the derivatives of u to be bounded - in particular if the boundary data is discontinuous at $x^0 \in \partial\Omega$ then u can not have any continuous extension to $\overline{\Omega}$. So the best estimate we can hope for is an estimate that allows $|\nabla u(x)|$ and $|D^2 u(x)|$ to tend to infinity as $x \to \partial\Omega$. This explains why we have the *"int"* in the $C^{2,\alpha}_{\text{int}}(\Omega)-$norm in (17.3).

The difference between the $C^{2,\alpha}(\Omega)$ and the $C^{2,\alpha}_{\text{int}}(\Omega)-$norm is that the latter norm allows

$$|\nabla u(x)| \approx \text{dist}(x, \partial\Omega)^{-1}, \tag{17.4}$$

$$|D^2 u(x)| \approx \text{dist}(x, \partial\Omega)^{-2} \tag{17.5}$$

and

$$\sup_{x,y\in K} \frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^{\alpha}} \approx \text{dist}(\{x,y\}, \partial\Omega)^{-2-\alpha} \tag{17.6}$$

whereas the $C^{2,\alpha}(\Omega)$-norm requires uniform bounds in the entire domain Ω. To see that the exponents -1, -2 and $-2-\alpha$ in (17.4), (17.5) and (17.6) are natural we rescale the equation. We use the estimate (17.4) as an illustration. Assume that $\text{dist}(x^0, \partial\Omega) = 2r$ then the function $v(x) = u(rx + x^0)$ will solve

$$\Delta v(x) = r^2 f(rx + x^0) \qquad \text{in } B_2(0),$$

since

$$\sum_{i=1}^{n} \frac{\partial^2 v(x)}{\partial x_i^2} = \sum_{i=1}^{n} \left(r^2 \frac{\partial^2 u(rx + x^0)}{\partial x_i^2} \right) = r^2 \Delta u(rx + x^0) = r^2 f(rx + x^0). \tag{17.7}$$

Since $\sup_{B_2(0)} |v| \le \sup_\Omega |u|$ we can conclude that $|\nabla v(0)|$ is bounded independently of r. But $|\nabla v(0)| = r|\nabla u(x^0)|$ and if $r|\nabla u(x^0)|$ is bounded independently of $r \approx \text{dist}(x^0, \partial\Omega)$ then $|\nabla u(x)| \approx \text{dist}(x, \partial\Omega)^{-1}$ which is what what (17.4) states. If you consider the proof of Proposition 1 (in part 5 of the lecture notes) again you will see that that is exactly how we prove the estimates.

Finally, we need to say something about the $l = 2$ in the $\|f\|_{C^\alpha_{\text{int},(2)}(\Omega)}$–norm of (17.3). But we see directly from the scaling in (17.7) that $l = 2$ is the optimal l since if $|f(x^0)| \approx \text{dist}(x^0, \partial\Omega)^{-2}$ (that is the growth of f allowed by the norm $\|f\|_{C^\alpha_{\text{int},(l)}(\Omega)}$ with $l = 2$) then the right hand side in (17.7) is bounded since $r \approx \text{dist}(x^0, \partial\Omega)$.

Further properties of the Hölder spaces: In addition to being a Banach space the Hölder spaces $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ also have a multiplication defined[5]: if $\phi(x), \varphi(x) \in C^{k,\alpha}(\Omega)$ then $\phi(x) \cdot \varphi(x) \in C^{k,\alpha}(\Omega)$ (and similarly for $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$).

We will only prove this for $k = 0$, the general case is an easy consequence of this and the product rule for the derivative.

Proposition 17.2. *Assume that $\phi(x), \varphi(x) \in C^\alpha(\Omega)$ then $\phi(x) \cdot \varphi(x) \in C^{k,\alpha}(\Omega)$ and*

$$[\phi \cdot \varphi]_{C^\alpha(\Omega)} \le \left(\|\phi\|_{C(\Omega)} [\varphi]_{C^\alpha(\Omega)} + \|\varphi\|_{C(\Omega)} [\phi]_{C^\alpha(\Omega)} \right). \tag{17.8}$$

Proof: The proof uses the same trick as the proof of the multiplication rule for differentiation. In particular, we may estimate

$$|\phi(x)\varphi(x) - \phi(y)\varphi(y)| = |(\phi(x)\varphi(x) - \phi(x)\varphi(y)) - (\phi(y)\varphi(y) - \phi(x)\varphi(y))| \le$$

$$\le |\phi(x)|\,|\varphi(x) - \varphi(y)| + |\varphi(y)|\,|\phi(y) - \phi(x)| \le \tag{17.9}$$

$$\le \|\phi(x)\|_{C(\Omega)} |\varphi(x) - \varphi(y)| + \|\varphi(y)\|_{C(\Omega)} |\phi(y) - \phi(x)|,$$

where the last inequality follows since $\|\phi(x)\|_{C(\Omega)} = \sup_{x \in \Omega} |\phi(x)|$ by definition.

[5]The technical term is that $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}_{\text{int}}(\Omega)$ and $C^{k,\alpha}_{\text{int},(l)}(\Omega)$ are *algebras* over $\mathbb{R}$.

If we divide both sides in (17.9) by $|x-y|^\alpha$ it follows that

$$\frac{|\phi(x)\varphi(x)-\phi(y)\varphi(y)|}{|x-y|^\alpha} \leq \|\phi(x)\|_{C(\Omega)} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha} + \|\varphi(y)\|_{C(\Omega)} \frac{|\phi(y)-\phi(x)|}{|x-y|^\alpha} \leq$$

$$\leq \left(\|\phi\|_{C(\Omega)}[\varphi]_{C^\alpha(\Omega)} + \|\varphi\|_{C(\Omega)}[\phi]_{C^\alpha(\Omega)}\right),$$

by the definition of $[\varphi]_{C^\alpha(\Omega)}$ and $[\phi]_{C^\alpha(\Omega)}$. Taking the supremum over $x, y \in \Omega$ yields the result. $\square$

Chapter 18

Interpolation inequalities

An interpolation inequality is exactly what it sounds like. Given two inequalities we might derive a third inequality that somehow lies between the other two. In this chapter we will show that if the second derivatives and the function value (zeroth order derivatives) of u is bounded, then the first derivatives are bounded as well. We will only prove the two simple interpolation inequalities that we need

Proposition 18.1. [Interpolation inequality] *Suppose that* $u \in C(\Omega)$ *then:*

1. *If* $D^2u \in C_{int,(2)}(\Omega)$ *then, for any* $\epsilon > 0$*, there exists a* C_ϵ *such that the following inequality holds*
$$\|\nabla u\|_{C_{int,(1)}} \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon \|D^2u\|_{C_{int,(2)}}. \tag{18.1}$$
2. *If* $[D^2u]_{C^\alpha_{int,(2)}(\Omega)}$ *is bounded then, for any* $\epsilon > 0$*, there exists a* C_ϵ *such that the following inequality holds*
$$\|D^2u\|_{C_{int,(2)}} \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon [D^2u]_{C^\alpha_{int,(2)}(\Omega)}. \tag{18.2}$$
3. *The same is true without the "int″ and* (l) *in the norms.*

Remark on the proposition. The proposition might seem to be very abstract (in particular if one is unused to the rather intricate definitions of the norms). But what it states is that it is enough to control $\|u\|_{C(\Omega)}$ and $\|D^2u\|_{C_{\text{int},(2)}}$ in order to control the norm

$$\|u\|_{C^2_{\text{int}}(\Omega)} = \|u\|_{C(\Omega)} + \|\nabla u\|_{C_{\text{int},(1)}} + \|u\|_{C(\Omega)} + \|D^2u\|_{C_{\text{int},(2)}}.$$

Similarly, $\|u\|_{C(\Omega)}$ and $[D^2u]_{C^\alpha_{\text{int},(2)}(\Omega)}$ controls the norm $\|u\|_{C^{2,\alpha}_{\text{int}}(\Omega)}$.

Proof: We will only prove the first two points since the third point is analogous.

To show (18.1) we let $x^0 \in \Omega$. We need to show that

$$\operatorname{dist}(x^0, \partial\Omega)|\nabla u(x^0)| \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon \|D^2 u\|_{C_{\mathrm{int},(2)}}. \tag{18.3}$$

If we can show (18.3) then (18.1) follows by taking the supremum over all $x^0 \in \Omega$.

If we let $4d = \operatorname{dist}(x^0, \partial\Omega)$ then

$$\sup_{B_d(x^0)} |D^2 u(x)| \leq \frac{C}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}}$$

and from Taylors Theorem we can conclude that, for any $0 \leq t \leq d$,

$$\inf_{B_t(x^0)} |\xi \cdot \nabla u(x)| \geq |\nabla u(x^0)| - \frac{Ct}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}}, \tag{18.4}$$

where $\xi = \frac{\nabla u(x^0)}{|\nabla u(x^0)|}$.

Now for any $y^1, y^2 \in B_d(x^0)$ such that $y^2 = y^1 + s\xi$ there exists, by the mean value theorem a $z \in B_d(x^0)$ on the line between y^1 and y^2 such that

$$2 \sup_{B_d(x^0)} |u(x)| \geq |u(y^1) - u(y^2)| = \underbrace{|(y^2 - y^1) \cdot \nabla u(z)|}_{=|s\xi \cdot \nabla u(z)|} \geq \tag{18.5}$$

$$\geq s|\nabla u(x^0)| - \frac{Cs^2}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}},$$

where we used (18.4) with $s = t$ in the last inequality.

Rearranging (18.5) and then multiply both sides by $\frac{\operatorname{dist}(x^0, \partial\Omega)}{s}$ we see that

$$\frac{2\operatorname{dist}(x^0, \partial\Omega)}{s} \|u\|_{C(\Omega)} + \frac{Cs\operatorname{dist}(x^0, \partial\Omega)}{d^2} \|D^2 u\|_{C_{\mathrm{int},(2)}} \geq \operatorname{dist}(x^0, \partial\Omega)|\nabla u(x^0)||\nabla u(x^0)|.$$

But $4d = \operatorname{dist}(x^0, \partial\Omega)$ and $s > 0$ is arbitrary so we can choose $s = c\epsilon d$ for an appropriate $c > 0$ and conclude that

$$\frac{C}{\epsilon} \|u\|_{C(\Omega)} + \epsilon \|D^2 u\|_{C_{\mathrm{int},(2)}} \geq \operatorname{dist}(x^0, \partial\Omega)|\nabla u(x^0)|.$$

This is exactly what we want to prove with $C_\epsilon = C/\epsilon$, (18.1) follows.

Next we prove (18.2).The proof is very similar to the proof of (18.1). However, we will need to use a second order Taylor expansion instead of a first order expansion. As before we fix an $x^0 \in \Omega$ and set $4d = \operatorname{dist}(x^0, \partial\Omega)$.

We aim to show that for $x^0 \in \Omega$

$$\operatorname{dist}(x^0, \partial\Omega)^2 |D^2 u(x^0)| \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon [D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)}. \tag{18.6}$$

Notice that it is enough to show that for all unit vectors η

$$\operatorname{dist}(x^0, \partial\Omega)^2 |D^2_\eta u(x^0)| \leq C_\epsilon \|u\|_{C(\Omega)} + \epsilon [D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)},$$

where $D_\eta = \eta \cdot \nabla$ is the directional derivative in the η direction. There is no loss of generality to assume that $\eta = e_1$, otherwise we may change basis for our coordinate system so that $\eta = e_1$.

Using a Taylor expansion we see that, for $y^0 = x^0 + se_1$ and $|s| \le d$,

$$\begin{aligned}
&\left| u(y^0) - \left(u(x^0) + \frac{\partial u(x^0)}{\partial x_1}(y_1^0 - x_1^0) + \frac{1}{2}\frac{\partial^2 u(x^0)}{\partial x_1^2}(y_1^0 - x_1^0)^2 \right) \right| = \\
&\qquad = \left| u(y^0) - \left(u(x^0) + \frac{\partial u(x^0)}{\partial x_1} s + \frac{1}{2}\frac{\partial^2 u(x^0)}{\partial x_1^2} s^2 \right) \right| \le \\
&\qquad \le C\frac{|s|^{2+\alpha}}{d^2}[D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)}.
\end{aligned} \tag{18.7}$$

Let us, for the sake of definiteness assume that $\frac{\partial^2 u(x^0)}{\partial x_1^2} \le 0$ then we may choose s such that $s\frac{\partial u(x^0)}{\partial x_1} \le 0$ and conclude from (18.7) that

$$u(y^0) - u(x^0) - \frac{1}{2}\frac{\partial^2 u(x^0)}{\partial x_1^2} s^2 \le C\frac{|s|^{2+\alpha}}{d^2}[D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)},$$

which implies that

$$\left| \frac{\partial^2 u(x^0)}{\partial x_1^2} \right| \le \frac{4}{s^2}\|u\|_{C(\Omega)} + C\frac{|s|^\alpha}{d^2}[D^2 u]_{C^\alpha_{\mathrm{int},(2)}(\Omega)},$$

which gives (18.6) if we choose $|s|$ small enough and that $4d = \mathrm{dist}(x^0, \partial\Omega)$. $\square$

Chapter 19

An interlude - the Need for Boundary Estimates.

So far we have proved interior estimates, that is estimates for $\|u\|_{C^{2,\alpha}_{\text{int}}}$ if u solves an elliptic PDE. Unfortunately the interior estimates are not strong enough to prove existence of solutions since they allow the second derivatives to grow line $\text{dist}(x, \partial\Omega)^{-2}$.

In order to explain this let us review our strategy for finding solutions to the equation

$$\begin{array}{ll} \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega, \end{array} \tag{19.1}$$

let us for notational simplicity assume that $b_i = c = 0$. We write the equation

$$\sum_{i,j=1}^n a_{ij}(x^0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \underbrace{\sum_{i,j=1}^n \left(a_{ij}(x^0) - a_{ij}(x)\right)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + f(x)}_{=F(x)}, \tag{19.2}$$

where we assume that $|a_{ij}(x^0) - a_{ij}(x)| < \epsilon$. Notice that if $u \in C^{2,\alpha}_{\text{int}}(\Omega)$ then the right hand side in (19.2) may grow like $|F(x)| \approx \frac{\epsilon}{\text{dist}(x,\partial\Omega)^2}$ as we approach the bundary $\partial\Omega$.

In order to show that the boundary values are obtained in (19.1) we would need to construct a barrier $w(x)$ at each boundary point $x^0 \in \Omega$. A barrier was a super-solution to the equation that satisfied $w(x^0) = 0$ and $w(x^0) > 0$ in $\overline{\Omega} \setminus \{x^0\}$. But a super-solution would have to satisfy

$$\sum_{i,j=1}^n a_{ij}(x^0)\frac{\partial^2 w(x)}{\partial x_i \partial x_j} \leq F(x).$$

And if $F(x) \approx -\frac{\epsilon}{\text{dist}(x,\partial\Omega)^2}$ then it is easy to see that we can not find a barrier in general. The easiest way to see this is to consider the one dimensional problem $\Omega = (0,1)$ and $F(x) = -\frac{\epsilon}{\text{dist}(x,\partial\Omega)^2}$ and $a_{11}(x^0) = 1$. Then the equation for the barrier reduces to

$$\begin{array}{ll} \frac{d^2w(x)}{dx^2} \leq -\frac{\epsilon}{x^2} & \text{in } (0,1) \\ w(0) = 0 & \text{and} \\ w(x) > 0 & \text{in } (0,1). \end{array}$$

But integrating this differential equation leads to $w(x) = \epsilon \ln(x) + ax + b$ for some constants $a, b \in \mathbb{R}$ which clearly can not take the value $w(0) = 0$.

The problem is that the interior estimates allow the solution to grow to fast at the boundary (that is why they are called interior). Therefore we need to prove some estimates at the boundary of the domain. It is easy to see that we can not prove that the solution to (19.1) has bounded $C^{2,\alpha}$ norm without any assumptions on the boundary and on the domain.

Example: Let $\Omega = B_1^+(0) = \{x \in B_1(0)\ x_2 > 0\}$ be a domain in $\mathbb{R}^2$ and $u(x)$ be a solution to

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = |x_1|^\alpha & \text{on } \partial\Omega, \end{array}$$

for some $\alpha \in (0,1)$. Such a solution exists by the Perron method. However, if $\|u\|_{C^{2,\alpha}(\Omega)} \leq C$ then we would have that $\|u(x_1,0)\|_{C^{2,\alpha}(x_1 \in (-1,1))} \leq C$. But $u(x_1,0) = |x_1|^\alpha \notin C^{2,\alpha}$ which would lead to a contradiction. We may conclude that $u \notin C^{2,\alpha}(\Omega)$. As a matter of fact, this shows that the best we can hope for is that $u \in C^\alpha(\Omega)$. This shows that we must assume that the boundary data is in $C^{2,\alpha}$ to have any hope to show that $\|u\|_{C^{2,\alpha}(\Omega)}$ is bounded.

Example: Remember that the function $u(r,\phi) = r^\alpha \sin(\alpha\phi)$ solves the Dirichlet problem

$$\begin{array}{ll} \Delta u(r,\phi) = 0 & \text{in } \{r \in (0,\infty),\ \phi \in (0,\pi/\alpha) \\ u(r,\phi) = 0 & \text{for } \phi = 0 \text{ and } \phi = \frac{\pi}{\alpha}, \end{array}$$

for $\alpha \geq \frac{1}{2}$. Notice that if $\alpha \in [1/2,1)$ then $u \in C^\alpha \setminus C^1$. So we have harmonic functions with zero boundary data that are still not $C^{2,\alpha}$. The problem here is that the domain has a sharp corner at the origin. Apparently we need to assume something about the regularity of the domain in order to prove that the solutions are $C^{2,\alpha}(\Omega)$.

In the following chapters we will pursue estimates for the $C^{2,\alpha}-$norm for solutions to (19.1). The proofs will be quite similar to the proofs of the interior estimates. In particular, we will start to show boundary estimates for the Newtonian potential close to a part of the boundary where the boundary is assumed to be contained in a hyperplane. Then we will continue to investigate the Dirichlet problem for the laplace equation close to a boundary, again given by a hyperplane. Having those estimates at hand it is easy to show apriori estimates for solutions to the Dirichlet problem for variable coefficient PDE.

Chapter 20

Boundary regularity - The Laplace equation.

In this chapter we will investigate the boundary regularity properties for the Laplace equations close to a part of the boundary that is a hyperplane. The proof will be analogous to the interior regularity proof.

We begin by estimating the Newtonian potential in an upper half ball $B^+_{2R}(0)$. The proof consists of one major observation - that the boundary terms on the flat part of the boundary disappears in the estimate for the second derivatives for all second derivatives except $\frac{\partial^2 u(x)}{\partial x_n^2}$. But it is easy to estimate $\frac{\partial^2 u(x)}{\partial x_n^2}$ in terms of $\frac{\partial^2 u(x)}{\partial x_i^2}$, for $i = 1, 2, ..., n-1$ and $f(x)$. This since $\Delta u(x) = f(x)$ and thus $\frac{\partial^2 u(x)}{\partial x_n^2} = f(x) - \sum_{i=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_i^2}$.

Lemma 20.1. *Let $f(x) \in C^\alpha(B^+_{2R}(0))$ for some $0 < \alpha < 1$ and define*

$$u(x) = \int_{B^+_{2R}(0)} N(x-\xi) f(\xi) d\xi$$

then there exists a constant $C_{n,\alpha}$ depending only on n and α such that the following inequality holds

$$[D^2 u]_{C^\alpha(B^+_R(0))} \leq C_{n,\alpha} \left([f]_{C^\alpha(B^+_{2R}(0))} + \frac{\sup_{B_R(0)} |f(x)|}{R^\alpha} \right). \tag{20.1}$$

Proof: We have already shown, see Theorem 2.1, that $\frac{\partial^2 u(x)}{\partial x_i \partial x_j}$ has the following representation formula for $x \in B^+_{2R}(0)$

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B^+_{2R}(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x) \right) d\xi - \tag{20.2}$$

$$-f(x) \int_{\partial B^+_{2R}(0)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi).$$

Strictly we only proved this representation for the domain $B_{2R}(0)$ but the deduction for the upper half ball $B_{2R}^+(0)$ is exactly the same.

We will split the proof into two cases. The first case is very similar to the proof of Theorem 14.1 and we will only indicate the minor differences.

Case 1: *Estimates for* $\left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{C^\alpha(B_R^+(0))}$ *when* $i \neq n$ *or* $j \neq n$.

We may assume that $j \neq n$, if not then $i \neq n$ and we may use that $\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \frac{\partial^2 u(x)}{\partial x_j \partial x_i}$ to reduce to the case for $j \neq n$.

Observe that the normal $\nu = -e_n$ on $\partial B_{2R}^+(0) \cap \{x_n = 0\}$ and the boundary integral in (20.2) therefore reduces to

$$f(x)\int_{\partial B_{2R}^+(0)} \frac{\partial N(x-\xi)}{\partial x_i}\nu_j(\xi)dA(\xi) =$$

$$= f(x)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} \frac{\partial N(x-\xi)}{\partial x_i}\nu_j(\xi)dA(\xi).$$

Therefore, for $j \neq n$, the representation in (20.2) becomes

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_{2R}^+(0)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\left(f(\xi)-f(x)\right)d\xi - \tag{20.3}$$

$$-f(x)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} \frac{\partial N(x-\xi)}{\partial x_i}\nu_j(\xi)dA(\xi).$$

Notice that we do not integrate over the set $\{x_n = 0\}$ in (20.3). We may therefore form the difference

$$\left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right| =$$

$$= \left| \int_{B_{2R}^+(0)} N_{ij}(x-\xi)\left(f(\xi)-f(x)\right)d\xi - f(x)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} N_i(x-\xi)\nu_j dA(\xi) - \right.$$

$$\left. - \int_{B_{2R}^+(0)} N_{ij}(y-\xi)\left(f(\xi)-f(y)\right)d\xi + f(y)\int_{\partial B_{2R}(0)\cap\{x_n>0\}} N_i(y-\xi)\nu_j dA(\xi)\right|.$$

These are the integrals we estimated in the proof of Theorem 14.1 with the only difference that we now integrate over a smaller set $B_{2R}(0) \cap \{x_n > 0\}$ in place of $B_{2R}(0)$. But the estimates of Theorem 14.1 still works line for line in this case.

We may therefore conclude that, for $j \neq n$,

$$\left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{C^\alpha(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right). \tag{20.4}$$

Case 2: *Estimates for* $\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{C^\alpha(B_R^+(0))}$.

Since $\Delta u(x) = f(x)$ it follows that

$$\frac{\partial^2 u(x)}{\partial x_n^2} = f(x) - \sum_{j=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_j^2}.$$

In particular

$$\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{C^\alpha(B_R^+(0))} = \left[f(x) - \sum_{j=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_j^2}\right]_{C^\alpha(B_R^+(0))} \leq$$

$$\leq [f(x)]_{C^\alpha(B_R^+(0))} + \sum_{j=1}^{n-1} \left[\frac{\partial^2 u(x)}{\partial x_j^2}\right]_{C^\alpha(B_R^+(0))} \leq$$

$$\leq [f(x)]_{C^\alpha(B_R^+(0))} + (n-1)C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right),$$

where we used the triangle inequality in the first inequality and (20.4) in the last inequality.

If we redefine $C_{n,\alpha}$ to $1 + (n-1)C_{n,\alpha}$ we may conclude that

$$\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{C^\alpha(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}(0))} + \frac{\sup_{B_R(0)}|f(x)|}{R^\alpha}\right).$$

□

Corollary 20.1. *Let u be as in Lemma 20.1 then*

$$\|u\|_{C^{2,\alpha}(B_R^+(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B_{2R}^+(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f(x)\|_{C(B_{2R}^+(0))}\right).$$

Proof: By the interpolation inequality it is enough to show that

$$\|u\|_{C(B_{2R}^+(0))} \leq C_n R^2 \|f(x)\|_{C(B_{2R}^+(0))}.$$

By the definition of u we have

$$|u(x)| = \left|\int_{B_{2R}^+(0)} N(x-\xi)f(\xi)d\xi\right| \leq$$

$$\leq \|f\|_{C(B_{2R}^+(0))}\left|\int_{B_{2R}^+(0)} N(\xi)d\xi\right| \leq C_n R^2 \|f(x)\|_{C(B_{2R}^+(0))},$$

where we used the explicit formula for N in the last inequality. □

Next we estimate the solution to the Dirichlet problem up to the boundary in B^+_{4R} with zero boundary data on $x_n = 0$. The proof uses that we may reflect the potential solution from Lemma 20.1 in the hyperplane $\{x_n = 0\}$, just as we reflected the Newtonian kernel in order to find a Greens function in $\mathbb{R}^n_+$. This allows us to reduce the regularity problem to the case when $f(x) = 0$. An odd reflection in $x_n = 0$ to the solutions with $f(0) = 0$ reduces the boundary regularity case to an interior problem.

Proposition 20.1. *Assume that $u \in C^2(B^+_{4R})$ and that u solves*

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } B^+_{4R}(0) \\ u(x) = 0 & \text{on } B_{4R}(0) \cap \{x_n = 0\}, \end{array}$$

where $f \in C^\alpha(B^+_{4R}(0))$ for some $\alpha \in (0,1)$.

Then there exists a constant $C_{n,\alpha}$ depending only on n and α such that

$$[u]_{C^{2,\alpha}(B^+_R(0))} \leq \tag{20.5}$$

$$\leq C_{n,\alpha}\left([f]_{C^\alpha(B^0_{4R}(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

Proof: We will write $u(x) = v(x) + h(x)$ in the ball $B^+_{2R}(0)$ where

$$\begin{array}{ll} \Delta v(x) = f(x) & \text{in } B^+_{2R}(0) \\ v(x) = 0 & \text{on } B_{2R}(0) \cap \{x_n = 0\} \end{array}$$

and

$$\begin{array}{ll} \Delta h(x) = h(x) & \text{in } B^+_{2R}(0) \\ h(x) = 0 & \text{on } B_{2R}(0) \cap \{x_n = 0\} \\ h(x) = u(x) - v(x) & \text{on } \partial B_{2R}(0) \cap \{x_n > 0\}. \end{array} \tag{20.6}$$

We need to estimate the $C^{2,\alpha}(B^+_R(0))-$norms of $v(x)$ and $h(x)$ in turn.

Step 1: *Construction of and estimates for $v(x)$.*

We may define the reflection of $f(x)$ in $C^\alpha(B_{4R}(0))$ according to

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x_n \geq 0 \\ f(x_1, x_2, .., x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Then $\hat{f} \in C^\alpha(B_{4R}(0))$ and $\|\hat{f}\|_{C^\alpha(B_{4R}(0))} = \|f\|_{C^\alpha(B^+_{4R}(0))}$.

Now define

$$\hat{v}(x) = \int_{B_{4R}(0)} N(x-\xi)\hat{f}(\xi)d\xi \quad \text{for } x \in B_{4R}(0)$$

and

$$\check{v}(x) = \int_{B^+_{4R}(0)} N(x-\xi)f(\xi)d\xi \quad \text{for } x \in B^+_{4R}(0).$$

From Theorem 14.1 we derive that

$$[D^2\hat{v}]_{C^\alpha(B_2R(0)} \leq C_{\alpha,n}\left([f]_{C^\alpha(B_{4R}(0))} + \frac{\sup_{B_{4R}(0)}|f(x)|}{R^\alpha}\right),$$

and similarly from Lemma 20.1 we derive that

$$[D^2\check{v}]_{C^\alpha(B^+_{2R}(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B^+_{4R}(0))} + \frac{\sup_{B_{2R}(0)}|f(x)|}{R^\alpha}\right).$$

In particular we may conclude that $v(x) = 2\check{v}(x) - \hat{v}(x)$ satisfies the same estimate (possibly with a larger constant)

$$[D^2 v]_{C^\alpha(B^+_{2R}(0))} \leq C_{n,\alpha}\left([f]_{C^\alpha(B^+_{4R}(0))} + \frac{\sup_{B_{2R}(0)}|f(x)|}{R^\alpha}\right).$$

We claim that $v(x_1, x_2, ..., x_{n-1}, 0) = 0$. This follows easily from the symmetry of the Newtonian kernel:

$$N(x_1-\xi_1, ..., x_{n-1}-\xi_{n-1}, x_n-\xi_n) = N(x_1-\xi_1, x_2-\xi_2, ..., x_{n-1}-\xi_{n-1}, -x_n+\xi_n),$$

since $N(x-\xi)$ only depends on $|x-\xi|$.

Therefore, if $x_n = 0$,

$$\hat{v}(x) = \int_{B_{4R}(0)} N(x-\xi)\hat{f}(\xi)d\xi =$$

$$= \int_{B_{4R^+}(0)} N(x-\xi)\hat{f}(\xi)d\xi + \int_{B_{4R^-}(0)} N(x-\xi)\hat{f}(\xi)d\xi =$$

$$= 2\int_{B_{4R^+}(0)} N(x-\xi)\hat{f}(\xi)d\xi = 2\check{v}(x).$$

So if $x_n = 0$ then $v(x) = 2\check{v}(x) - \hat{v}(x) = 0$ as claimed.

Step 2: *Construction of and estimates for $h(x)$.*

The function $h(x) = u(x) - v(x)$ so we only need to estimate its $C^{2,\alpha}$ norm.

We will do that by considering the odd reflection of $h(x)$ - which we will show is harmonic in $B_{2R}(0)$ - together with interior estimates for harmonic functions. In particular that we may estimate the $C^3(B_R(0))-$norm of a harmonic functions by its $C(B_{2R})-$norm.

We need to estimate $\|h\|_{C(B^+_{2R}(0))}$ which, by the maximum principle, is the same as estimating

$$\sup_{\partial B^+_{2R}(0)}|h(x)| = \sup_{\partial B^+_{2R}(0)}|u(x)-v(x)| \leq \sup_{\partial B^+_{2R}(0)}|u(x)| + \sup_{\partial B^+_{2R}(0)}|v(x)|.$$

The supremum of u appears in the right hand side of (20.5) so we only need to estimate $\sup_{\partial B^+_{2R}(0)}|v(x)|$. This is easily done as in the proof of Corollary 20.1. In particular,

$$|v(x)| \leq 2\left|\int_{B^+_{4R}(0)} N(x-\xi)f(\xi)d\xi\right| + \left|\int_{B_{4R}(0)} N(x-\xi)\hat{f}(\xi)d\xi\right| \leq$$

$$\leq 3\|f\|_{C(B^+_{4R}(0))}\int_{B_{4R}}|N(\xi)|\,d\xi \leq C_n\|f\|_{C(B^+_{4R}(0))}R^2.$$

Therefore

$$\|h\|_{C(B^+_{2R}(0)} \leq \|u\|_{C_{B^+_{2R}(0)}} + C_n\|f\|_{C(B^+_{4R}(0))}R^2.$$

Consider the odd reflection of h on $\partial B_{2R}(0)$:

$$\hat{h}(x) = \begin{cases} h(x) & \text{if } x_n \geq 0 \text{ and } x \in \partial B_{2R}(0) \\ -h(x) & \text{if } x_n < 0 \text{ and } x \in \partial B_{2R}(0). \end{cases}$$

Furthermore we let g solve the Dirichlet problem

$$\begin{array}{ll} \Delta g(x) = 0 & \text{in } B_{2R}(0) \\ g(x) = \hat{h}(x) & \text{on } \partial B_{2R}(0). \end{array} \tag{20.7}$$

Then, since $g(x)$ is uniquely determined by (20.7) and since $\hat{h}(x)$ is odd in x_n, it follows that $g(x)$ is an odd function in x_n. That is $g(x_1, x_2, ..., 0) = 0$ and therefore $g(x)$ solves (20.6). Uniqueness for the Dirichlet problem implies that $h(x) = g(x)$ in $B^+_{2R}(0)$.

Now, since $g(x)$ is harmonic in $B_{2R}(0)$ it follows that there exists a constant C_n such that

$$\|D^3h\|_{C(B^+_R(0))} = \|D^3g(x)\|_{C(B^+_R(0))} \leq \|D^3g(x)\|_{C(B_R(0))} \leq$$
$$\leq \frac{C_n}{R^3}\|g\|_{C(B_{2R})} = \frac{C_n}{R^3}\|h\|_{C(B_{2R})}.$$

An application of the mean-value Theorem from calculus implies that

$$[D^2h]_{C^\alpha(B^+_R(0))} \leq R^{1-\alpha}\|D^3h\|_{C(B^+_R(0))} \leq \frac{C_n}{R^{2+\alpha}}\|h\|_{C(B_{2R})} \leq$$
$$\leq C_n\left(\left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

In particular, we have shown that

$$[D^2u]_{C^\alpha(B^+_R(0))} \leq C_{n,\alpha}\left([h]_{C^\alpha(B^0_R(0))} + [h]_{C^\alpha(B^0_R(0))}\right) \leq$$
$$\leq C_{n,\alpha}\left([f]_{C^\alpha(B^0_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

□

Corollary 20.2. *Under the assumptions of Proposition 20.1 we have the estimate*

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \leq \tag{20.8}$$
$$\leq C_{n,\alpha}\left([f]_{C^\alpha(B^0_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

Proof: We only need to estimate $\|\nabla u\|_{C(B_R^+(0))}$ and $\|D^2 u\|_{C(B_R^+(0))}$. However, that can be done by an interpolation inequality. $\square$

We end this chapter with a proposition for constant coefficient PDE. The proof is, as it was for the interior case, based on a change of variables that reduces the PDE to the Laplacian.

Proposition 20.2. *Let $u(x)$ be a solution to the constant coefficient elliptic PDE*

$$\begin{array}{ll} \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) & \text{in } B_{2R}^+(0) \\ u(x) = 0 & \text{on } \partial\Omega \cap \{x_n = 0\}, \end{array} \tag{20.9}$$

where $a_{ij} = a_{ji}$ satisfies the ellipticity condition for all $\xi in \mathbb{R}^n$ and some $\lambda, \Lambda > 0$

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2.$$

Then there exists a constant $C_{\lambda,\Lambda,n,\alpha} > 0$ such that

$$\|u\|_{C^{2,\alpha}(B_R^+(0))} \leq \tag{20.10}$$

$$\leq C_{\lambda,\Lambda,n,\alpha} \left([f]_{C^\alpha(B_R^0(0))} + \left(\frac{1}{R^\alpha} + R^2 \right) \|f\|_{C(B_{4R}^+(0))} + \frac{1}{R^{2+\alpha}} \|u\|_{C(B_{2R}^+(0))} \right).$$

Proof: As in the proof of the interior estimates for constant coefficient PDEs we make the change of variables $v(x) = u(Px)$ where P is chosen such that $P^T A P = I$. Notice that the linear transformation P will map $\{x_n = 0\}$ unto a hyperplane that we may assume (possibly after a rotation of the coordinates) to be $\{x_n = 0\}$. We may thus apply Corollary 20.2 on $v(x)$ and then use $u(x) = v(P^{-1}x)$ to derive the desired estimates for u. For further details see the proof of Proposition 15.1 (Part 6 of these notes).

Chapter 21

Boundary Regularity - Variable Coefficient Equations.

In this chapter we prove apriori estimates up to the boundary for general linear variable coefficient PDE for $C^{2,\alpha}$ domains. We start by showing estimates for variable coefficient equations in upper half balls $B_{2R}(0)$ and then we show that general domains with $C^{2,\alpha}$ boundaries can be reduced to this case.

21.1 Boundary Regularity when the Boundary is a Hyperplane.

In this section we use a freezing of the coefficients argument, as in Theorem 16.1, to show that variable coefficient equations have $C^{2,\alpha}$ estimates up to the flat part of the boundary in an upper half ball.

Theorem 21.1. *Let $u \in C^{2,\alpha}(B^+_{2R}(0))$ be a solution, in $B^0_{2R}(0)$, to*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) \qquad (21.1)$$

$$u(x) = 0 \text{ on } \{x_n = 0\} \cap B_{2R}(0) \qquad (21.2)$$

Assume furthermore that $a_{ij}(x), b_i(x), c(x) \in C^\alpha(B^+_{2R})(0)$, that $a_{ij}(x) = a_{ji}(x)$, and that $a_{ij}(x)$ satisfy the following ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for some constants $0 < \lambda \le \Lambda$ and every $x \in B^+_{2R}(0)$ and all $\xi \in \mathbb{R}^n$.

Then there exists a constant $C = C(\lambda, \Lambda, n, a_{ij}, b_i, c)$ such that

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \le$$

$$\le C\left([f]_{C^\alpha(B^0_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

Proof: This proof mimics the proof of Theorem 16.1 (Theorem 1 in part 6). Therefore we will only indicate the minor differences. We may choose $\overline{B^+_{3R/2}(0)}$ as the compact set K and cover K by a finite number of balls $B_\delta(x^k)$ such that either $B_{4\delta}(x^k) \subset B^+_{3R/2}(0)$ or $B_{4\delta}(x^k) \cap B^+_{3R/2}(0) = B^+_{4\delta}(x^k)$. To estimate $\|D^2u\|_{C^\alpha_{\text{int},(2)}(x^k)}$ in the first case we may proceed exactly as in Theorem 16.1. In case $B_{4\delta}(x^k) \cap B^+_{3R/2}(0) = B^+_{4\delta}(x^k)$ we may apply the boundary estimates from the previous section in place of Proposition 15.1 (Prop 1 in part 6). $\square$

Corollary 21.1. *Let $u \in C^{2,\alpha}(B^+_{2R}(0))$ be a solution, in $B^0_{2R}(0)$, to*

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) \tag{21.3}$$

$$u(x) = g(x') \text{ on } \{x_n = 0\} \cap B_{2R}(0) \tag{21.4}$$

Assume furthermore that $f(x), a_{ij}(x), b_i(x), c(x) \in C^\alpha(B^+_{2R})(0)$, that $g \in C^{2,\alpha}(B'_{2R}(0))$, that $a_{ij}(x) = a_{ji}(x)$, and that $a_{ij}(x)$ satisfy the following ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2,$$

for some constants $0 < \lambda \le \Lambda$ and every $x \in B^+_{2R}(0)$ and all $\xi \in \mathbb{R}^n$.

Then there exists a constant $C = C(\lambda, \Lambda, n, a_{ij}, b_i, c)$ such that

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \le$$

$$\le C\left(\|g\|_{C^{2,\alpha}(B'_{2R}(0))} + [f]_{C^\alpha(B^+_R(0))} + \left(\frac{1}{R^\alpha} + R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right).$$

Proof: We may define $v(x) = u(x) - g(x')$. Then $Lv(x) = Lu(x) - Lg(x') = f(x) - Lg(x')$. We may thus define $\hat{f} = f(x) - Lg(x') \in C^\alpha(B^+_{2R}(0))$. Clearly

$$\|\hat{f}\|_{C^\alpha(B^+_{2R}(0))} \le C\left(\|g\|_{C^{2,\alpha}(B'_{2R}(0))} + \|f\|_{C^\alpha(B^+_{2R}(0))}\right)$$

and

$$\|u\|_{C^{2,\alpha}(B^+_R(0))} \le \|v\|_{C^{2,\alpha}(B^+_R(0))} + \|g\|_{C^{2,\alpha}(B^+_R(0))}. \tag{21.5}$$

We may apply the previous proposition on v with $\hat{f}$ in place of f and then estimate $\|u\|_{C^{2,\alpha}(B^+_R(0))}$ by (21.5). $\square$

21.2 Boundary regularity for $C^{2,\alpha}$ boundaries.

Now we change our perspective to domains with boundaries that are locally given by the graph of a $C^{2,\alpha}$-function - which we will call $C^{2,\alpha}$-domains. The proofs are not that difficult since we may make a change of variables and transform the $C^{2,\alpha}-$domains to domains with the boundary given by a hyperplane and then use the estimates from the previous chapter.

We will use the notation $x' = (x_1, x_2, ..., x_{n-1})$ and $\nabla' = (\partial_1, \partial_2, ..., \partial_{n-1})$ etc. We will also always assume that the PDE we study satisfy the standard ellipticity condition:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for some constants $\Lambda, \lambda > 0$.

The next Lemma makes the important reduction of the $C^{2\alpha}-$domain to a domain that locally has the boundary contained in a hyperplane $\{x_n = 0\}$ which allows us to use the theory from the previous chapter. The method is commonly refereed to as a *"straightening of the boundary argument"*.

Lemma 21.1. *Let $g(x') \in C^{2,\alpha}(B'_{2R}(0))$, $g(0) = |\nabla' g(0)| = 0$ and*

$$\Omega = B_{2R}(0) \cap \{x_n > g(x')\}.$$

Assume furthermore that $u(x)$ is a solution in Ω to

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \tag{21.6}$$

$$u(x) = 0 \text{ on } \{x_n = g(x')\} \cap B_{2R}(0) \tag{21.7}$$

where L satisfies the assumptions of Theorem 21.1.

Then there exists a constant $c(\lambda, \Lambda) > 0$ such that if $|\nabla g(x')| \leq c(\lambda, \Lambda)$ then $v(x) = u(x_1, x_2, ..., x_{n-1}, x_n - g(x'))$ satisfies an elliptic equation in $\{(x', x_n - g(x')) \in \Omega\}$

$$\tilde{L}v(x) = \sum_{i,j=1}^n \tilde{a}_{ij}(x)\frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n \tilde{b}_i(x)\frac{\partial v(x)}{\partial x_i} + \tilde{c}(x)v(x) = f(x', x_n - g(x')) \tag{21.8}$$

$$v(x) = 0 \text{ on } \{x_n = 0\}, \tag{21.9}$$

where $\tilde{a}_{ij}$, $\tilde{b}_i, \tilde{c} \in C^\alpha$ with $C^\alpha-$norms only depending on the corresponding norms for a_{ij}, b_i and c and the $C^{2,\alpha}-$norm of g. Furthermore, $\tilde{a}_{ij}$ satisfies the following ellipticity condition

$$\frac{\lambda}{2}|\xi|^2 \leq \sum_{i,j=1}^n \tilde{a}_{ij}(x)\xi_i\xi_j \leq 2\Lambda|\xi|^2, \tag{21.10}$$

for all $\xi \in \mathbb{R}^n$.

Proof: The proof is straight forward. We may write $u(x) = v(x', x_n + g(x'))$ and calculate

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial v(x', x_n + g(x'))}{\partial x_i} + \frac{\partial g(x')}{\partial x_i}\frac{\partial v(x', x_n + g(x'))}{\partial x_n} \text{ for } i \neq n,$$

and

$$\frac{\partial u(x)}{\partial x_n} = \frac{\partial v(x', x_n + g(x'))}{\partial x_n}.$$

Similarly we can express the second derivatives of u in terms of $v(x', x_n + g(x'))$ as follows, for $i, j \neq n$,

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} + \frac{\partial^2 g(x')}{\partial x_i \partial x_j}\frac{\partial v(x', x_n + g(x'))}{\partial x_n} +$$

$$+ \frac{\partial g(x')}{\partial x_i}\frac{\partial g(x')}{\partial x_j}\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2} + \frac{\partial g(x')}{\partial x_i}\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_j \partial x_n},$$

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_n} = \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_n} + \frac{\partial g(x')}{\partial x_i}\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2} \text{ for } i \neq n,$$

and

$$\frac{\partial^2 u(x)}{\partial x_n^2} = \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2}.$$

In particular we have that

$$\sum_{i,j=1}^{n} \underbrace{a_{ij}(x)}_{=\tilde{a}_{ij}(x', x_n - g(x'))} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n-1} a_{ij}(x)\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} +$$

$$+ \sum_{i=1}^{n-1} \underbrace{\left(a_{in}(x) + \sum_{j=1}^{n-1} \frac{\partial g(x')}{\partial x_j} a_{ij} \right)}_{=\tilde{a}_{in}(x', x_n - g(x'))} \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_n}$$

$$+ \underbrace{\left(a_{nn}(x) + 2\sum_{i=1}^{n-1} \frac{\partial g(x')}{\partial x_i} a_{in}(x) + \sum_{i,j=1}^{n-1} a_{ij}(x)\frac{\partial^2 g(x')}{\partial x_i \partial x_j} \right)}_{=\tilde{a}_{nn}(x', x_n - g(x'))} \frac{\partial^2 v(x', x_n + g(x'))}{\partial x_n^2} +$$

$$+ \sum_{i,j=1}^{n-1} \frac{\partial^2 g(x')}{\partial x_i \partial x_j} a_{ij}(x) \frac{\partial v(x', x_n + g(x'))}{\partial x_n},$$

where the underbraces indicate how we define $\tilde{a}_{ij}$.

We can also calculate

$$\sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} = \sum_{i=1}^{n-1} b_i(x)\frac{\partial v(x', x_n + g(x'))}{\partial x_i} +$$

$$+\left(b_n(x) + \sum_{i=1}^{n-1} b_i(x)\frac{\partial g(x')}{\partial x_i}\right)\frac{\partial v(x', x_n + g(x'))}{\partial x_n}.$$

Setting $\tilde{b}_{x', x_n + g(x')} = b_i(x)$,

$$\tilde{b}_n(x', x_n + g(x')) = b_n(x) + \sum_{i=1}^{n-1} b_i(x)\frac{\partial g(x')}{\partial x_i} + \frac{\partial^2 g(x')}{\partial x_i \partial x_j} a_{ij}(x', x_n + g(x')),$$

and $\tilde{c}(x', x_n - g(x')))$ we see that

$$f(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) =$$

$$= \sum_{i,j=1}^{n} \tilde{a}_{ij}(x', x_n + g(x'))\frac{\partial^2 v(x', x_n + g(x'))}{\partial x_i \partial x_j} +$$

$$+\sum_{i=1}^{n} \tilde{b}_i(x', x_n + g(x'))\frac{\partial v(x', x_n + g(x'))}{\partial x_i} + \tilde{c}(x', x_n + g(x'))v(x', x_n + g(x')).$$

Evaluating this at the point $(x', x_n - g(x'))$ gives

$$\tilde{L}v = \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \tilde{b}_i(x)\frac{\partial v(x)}{\partial x_i} + \tilde{c}(x)v(x) = f(x', x_n - g(x')) = \tilde{f}(x),$$

where the last equality defines $\tilde{f}(x)$.

Next we show that $\tilde{L}$ is elliptic. Observe that if write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

and

$$G = \begin{bmatrix} 1 & 0 & 0 & \cdots & \frac{\partial g}{\partial x_1} \\ 0 & 1 & 0 & \cdots & \frac{\partial g}{\partial x_2} \\ 0 & 0 & 1 & \cdots & \frac{\partial g}{\partial x_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 + \frac{\partial g}{\partial x_n} \end{bmatrix}$$

then we may write the matrix $\tilde{A} = [\tilde{a}_{ij}]_{i,j=1}^n$ as follows

$$\tilde{A} = G^T A G.$$

It follows in particular that for any vector $\xi = (\xi_1, ..., \xi_n)^T \in \mathbb{R}^n$ that:

$$\sum_{i,j=1}^n \tilde{a}_{ij}\xi_i\xi_j = (G\xi)^T \cdot A \cdot (G\xi) \geq \lambda \left|G\xi\right|^2 \tag{21.11}$$

Clearly $|G\xi|^2 \geq \frac{1}{2}|\xi|$ if $|\nabla' g(x')| < c$ for some constant c. Notice that (21.11) states that

$$\sum_{i,j=1}^n \tilde{a}_{ij}\xi_i\xi_j \geq \lambda \left|G\xi\right|^2 \geq \frac{\lambda}{2}|\xi|^2,$$

if $|\nabla' g(x')| < c$, which is the left inequality in (21.10). The right inequality (21.10) in is proved in an analogous way.

To verify that $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}, \tilde{f} \in C^\alpha$ it is enough to verify that

$$\sup_{x,y} \frac{|a_{ij}(x', x_n + g(x')) - a_{ij}(y', y_n + g(y'))|}{|x-y|^\alpha} \text{ only depends on } \|g\|_{C^{2,\alpha}} \text{ and } \|a_{ij}\|_{C^\alpha} \tag{21.12}$$

and similarly for b_i, c_i and f. This since $\tilde{a}_{ij}$ is defined by terms $a_{ij}(x', x_n+g(x'))$ multiplied by derivatives of g - which are clearly in C^α. So if $a_{ij}(x', x_n+g(x')) \in C^\alpha$ then $\tilde{a}_{ij} \in C^\alpha$ by Proposition 17.2 (Prop 3 Part 6). To prove (21.12) we notice that

$$\begin{aligned} &\sup_{x,y} \frac{|a_{ij}(x', x_n + g(x')) - a_{ij}(y', y_n + g(y'))|}{|x-y|^\alpha} = \\ &= \sup_{x,y} \frac{|a_{ij}(x) - a_{ij}(y)|}{|(x', x_n - g(x')) - (y', y_n - g(y'))|^\alpha} \leq \\ &\leq \left(\frac{|x-y|}{|(x', x_n - g(x')) - (y', y_n - g(y'))|} \right)^\alpha [a_{ij}]_{C^\alpha}. \end{aligned}$$

But $\frac{|x-y|}{|(x', x_n - g(x')) - (y', y_n - g(y'))|}| \leq C$ where C only depend on $\nabla' g$ and thus it follows that $\tilde{a}_{ij} \in C^\alpha$ with norm only depending on $\|f\|_{C^\alpha}$ and $\|g\|_{C^{2,\alpha}}$. $\square$

Next we apply the straightening of the boundary argument to show regularity in $C^{2,\alpha}-$domains. We also allow non-zero boundary data.

Proposition 21.1. *Let $g(x') \in C^{2,\alpha}(B'_{2R}(0))$, $g(0) = |\nabla' g(0)| = 0$ and $|\nabla g(x')| \leq c$, where $c > 0$ is as in Lemma 21.1. Also let $\Omega = B_{2R}(0) \cap \{x_n > g(x')\}$.*

Then any solution $u(x)$ in Ω to the following PDE

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \tag{21.13}$$

$$u(x) = h(x) \text{ on } \{x_n = g(x')\} \cap B_{2R}(0), \tag{21.14}$$

where L satisfies the assumptions of Theorem 21.1, will satisfy the estimate

$$\|u\|_{C^{2,\alpha}(B_R(0)\cap\Omega)} \leq \tag{21.15}$$

$$\leq C\left(\|h\|_{C^{2,\alpha}(\partial\Omega)} + [f]_{C^\alpha(B^+_R(0))} + \left(\frac{1}{R^\alpha}+R^2\right)\|f\|_{C(B^+_{4R}(0))} + \frac{1}{R^{2+\alpha}}\|u\|_{C(B^+_{2R}(0))}\right),$$

where $C = C(\lambda, \Lambda, n, a_{ij}, b_i, c, g)$.

Proof: We define $v(x) = u(x', x_n - g(x'))$. Lemma 21.1 implies that v solves an elliptic equation

$$\tilde{L}v(x) = f(x', x_n - g(x')) \text{ in } \{(x', x_n - g(x')) \in \Omega\} \tag{21.16}$$

$$v(x) = h(x', x_n - gx')) \text{ on } \{x_n = 0\}, \tag{21.17}$$

where the coefficients of $\tilde{L}$ only depend on the coefficients of L and on g.

Corollary 21.1 implies that v satisfies the right estimates. But $u(x) = v(x', x_n + g(x')$ so a simple application of the chain rule for differentiation will imply that u satisfies (21.15). □

21.3 Global regularity

We are now ready to glue the boundary and the interior regularity together to prove global regularity. To that end we define $C^{2,\alpha}-$domains as domains whose boundaries can be covered by balls of some fixed radius such that the boundary can be represented by a $C^{2,\alpha}$ graph in each ball. See the figure below.

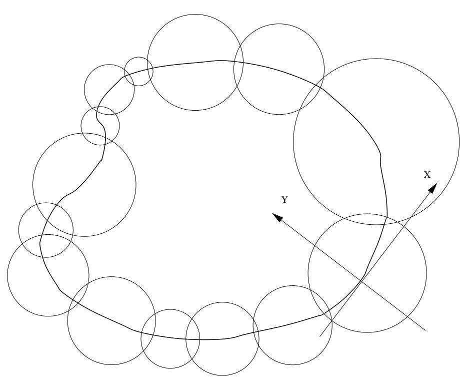

Figure 21.1: A $C^{2,\alpha}$-domain with the coordinate system for one ball $B_r(x^0)$ indicated.

Definition 21.1. *We say that a domain Ω is $C^{2,\alpha}$ if there exists an $r > 0$ such that for every $x^0 \in \partial\Omega$ there exists a coordinate system such that $B_r(x^0) \cap \partial\Omega$ is the graph of a $C^{2,\alpha}-$function in this coordinate system.*

We say that a constant C depend on a $C^{2,\alpha}$-domain Ω if C depend only on r and on the $C^{2,\alpha}-$norm of the boundary in the balls $B_r(x^0)$.

Remark: *The notion of a constant depending on a $C^{2,\alpha}-$domain is not very rigorously defined here. In particular, if Ω is a $C^{2,\alpha}$ domain with one $r>0$ then it is a $C^{2,\alpha}$ domain with respect to any smaller r. Also the $C^{2,\alpha}$-norm of the functions whose graph coincidences with $\partial\Omega\cap B_r(x^0)$ will depend on r and also on the coordinate system we choose to represent $\partial\Omega\cap B_r(x^0)$. This ambiguity will not affect our discussion here since we will not keep track on how constants depend on the domain. So in the rest of these notes one may think of each $C^{2,\alpha}$-domain having a specific $r>0$ attached to it and that we represent the boundary $\partial\Omega\cap B_r(x^0)=\{(x',g(x'));\ g\in C^{2,\alpha}\}$ where we choose the coordinate system in which we represent g so that $g(0)=|\nabla' g(0)|=0$.*

With this definition at hand it is easy to prove $C^{2,\alpha}$ estimates for solutions to linear PDE. We may cover a neighborhood of the boundary by balls such that we can apply the boundary regularity in each ball. The rest of the domain can be covered by a compact set K with a mixed distance to the boundary. Using the interior regularity results we can estimate the $C^{2,\alpha}-$norm of the solution in the compact set.

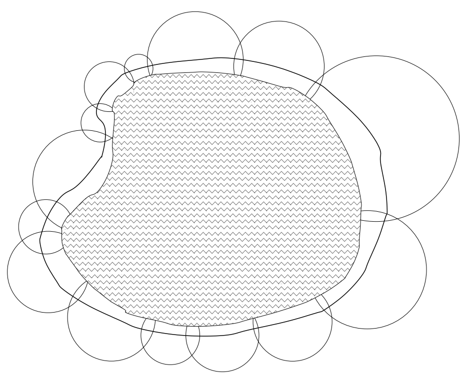

Figure 21.2: A $C^{2,\alpha}$-domain with a compact set (with the zig-zag pattern), where the solution is $C^{2,\alpha}$ by interior estimates, and a number of balls where the solution is $C^{2,\alpha}$ by the boundary estimates.

Theorem 21.2. *Assume that $u\in C^{2,\alpha}(\Omega)$, where Ω is a bounded domain, is a solution to*

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i}+c(c)u(x)=f(x)\ \text{in}\ \Omega \tag{21.18}$$

$$u(x)=g(x)\ \text{on}\ \partial\Omega. \tag{21.19}$$

Assume furthermore that $f(x),a_{ij}(x),b_i(x),c(x)\in C^{\alpha}(\Omega)$ and that $g(x)\in C^{2,\alpha}(\partial\Omega)$ and that Ω is a $C^{2,\alpha}-$domain and that $a_{ij}(x)$ satisfy the standard ellipticity assumption

$$\lambda|\xi|^2\le\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2.$$

Then there exists a constant C such that

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^{\alpha}(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)} + \|u\|_{C(\Omega)}\right), \tag{21.20}$$

here $C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$ where $\Lambda, \lambda > 0$ are the ellipticity constants of the PDE.

Proof: We will prove the Theorem in three simple steps.

Step 1: *Cover the domain.*

Since Ω is a $C^{2,\alpha}-$domain we may cover the boundary $\partial\Omega$ by balls $B_{r/4}(z)$, $z \in \partial\Omega$, where $\Omega \cap B_r(x)$ is given by the graph of some function $g \in C^{2,\alpha}$. We may also decrease r, if necessary, to assure that $|\nabla g| < c$ in B_r' where c is as in Lemma 21.1.

Let $K = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) \geq r/4\}$. Then K is compact and K together with the balls $B_{r/2}(z)$, $z \in \partial\Omega$, will cover Ω.

Step 2: *Local bounds for the norm.*

For any ball $y \in \Omega$ we will either have $B_{r/4}(y) \subset K$ or $B_{r/4}(y) \subset B_{r/2}(z)$ for some $z \in \partial\Omega$.

If $B_{r/y}(y) \subset K$ then Theorem 16.1 will imply that

$$\|u\|_{C^{2,\alpha}(B_{r/4}(y))} \leq \|u\|_{C^{2,\alpha}(K)} \leq \frac{C}{r^{2+\alpha}}\left(\|f\|_{C^{\alpha}(\Omega)} + \|u\|_{C(\Omega)}\right), \tag{21.21}$$

where we have used that $\mathrm{dist}(K, \partial\Omega) = r/4$ and thus $\mathrm{dist}(K, \partial\Omega)^{-(2+\alpha)} = 2^{-(2+\alpha)} r^{-(2+\alpha)}$ and that the factor $2^{-(2+\alpha)}$ may be included in the constant C.

And if $B_{r/4}(y) \subset B_{r/2}(z)$ then Proposition 21.1 will imply that

$$\|u\|_{C^{2,\alpha}(B_{r/4}(y))\cap\Omega} \leq C\left(\|f\|_{C^{\alpha}(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right). \tag{21.22}$$

Step 3: *Global estimates and the conclusion of the Theorem.*

Clearly (21.21) and (21.22) together implies that for any $x \in \Omega$

$$|\nabla u(x)| + |D^2 u(x)| \leq C\left(\|f\|_{C^{\alpha}(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right), \tag{21.23}$$

where $C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$ where we included the r dependence in the dependence on Ω.

Therefore we only need to show that

$$\frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^{\alpha}} \leq C\left(\|f\|_{C^{\alpha}(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right). \tag{21.24}$$

We will consider two (or three - depending on how you count) cases. Either $|x-y| < r/4$ or $|x-y| \geq r/4$. If $|x-y| < r/4$ and both $x, y \in K$ then (21.24) follows from (21.21) and if one of x or y, lets say y for definiteness, satisfies $y \notin K$ then there must exists a ball $B_{r/4}(z)$, $z \in \partial\Omega$ such that $y \in B_{r/4}(z)$. But then, since $|x-y| < r/4$, both $x, y \in B_{r/4}(y) \subset B_{r/2}(z)$ and (21.24) follows from (21.22). In any case, (21.24) follows if $|x-y| < r/4$.

If $|x-y| \geq r/4$ then

$$\frac{|D^2u(x)-D^2u(y)|}{|x-y|^\alpha} \leq \frac{4^\alpha}{r^\alpha}|D^2u(x)-D^2u(y)| \leq 2\frac{4^\alpha}{r^\alpha}\sup_{x\in\Omega}|D^2u(x)| \leq \tag{21.25}$$

$$\leq 2\frac{24^\alpha C}{r^\alpha}\left(\|f\|_{C^\alpha(\Omega)} + \|h\|_{C^{2,\alpha}} + \|u\|_{C(\Omega)}\right),$$

where we used (21.23) in the last inequality. Notice that the constant in the right hand side of (21.25) only depend on r and $n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega$. But r only depend on Ω so we may conclude that the constant in (21.25) will only depend on $n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c$ and Ω.

We have thus proved (21.24) which together with (21.23) implies the estimate (21.20). □

21.4 Barriers and an improved estimate.

We met barriers before when we proved that the solution we get from Perron's method attains the boundary data. In this section we will use barriers again to control the $\|u\|_{C(\Omega)}$−norm of solutions to

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) \quad \text{in } \Omega \tag{21.26}$$

when $c(x) \leq 0$. This will allow us to get rid of the dependence of the $\|u\|_{C(\Omega)}$−norm on the right hand side in the estimates of Theorem 21.2. This is not only more esthetically pleasing, but it also shows that the solution can be controlled by only using the given data.

We begin by slightly refine our definition of barriers with respect to equation

Definition 21.2. *Let Ω be a domain and $\xi \in \partial\Omega$. We say that w is an upper (lower) barrier at ξ for the partial differential equation (21.26) if*

1. *$w \in C(\overline{\Omega})$,*
2. *$w > 0$ in $\overline{\Omega} \setminus \{\xi\}$, $w(\xi) = 0$ and*
3. *w is a solution to $Lw(x) \leq f(x)$ in Ω ($Lw(x) \leq -f(x)$ for lower barriers).*

If Ω is a domain and there exist an upper and a lower barrier at ξ for the PDE (21.26) that ξ is a regular point.

Notice that if we can find a function satisfying the conditions for being an upper barrier and $Lw(x) \leq -\|f\|_{C(\Omega)}$ then $w(x)$ is both an upper and a lower barrier and we say that $w(x)$ is a barrier for (21.26).

Just as with the Laplace equation, the exterior ball condition implies the existence of a barrier.

Lemma 21.2. *Let Ω be a bounded domain and L be given by (21.26) with $c(x) \leq 0$ and $a_{ij}, b_i, c, f \in C(\Omega)$. Assume furthermore that $a_{ij}(x)$ satisfies the standard ellipticity condition*

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

for some $\lambda > 0$ and every $x, \xi \in \mathbb{R}^n$.

Then if $\xi \in \partial\Omega$ satisfies the exterior ball condition, that means that there exists a ball $B_r(x_\xi) \cap \Omega = \emptyset$ and $\overline{B_r(x_\xi)} \cap \overline{\Omega} = \{\xi\}$ see Definition 11.3, then there exists a barrier at ξ.

Proof: There is no loss of generality to assume that $B_r(0)$ touches $\partial\Omega$ at the point ξ - if not we may translate the coordinate system so that x_ξ becomes the origin. Mimicking the proof of Corollary 13.1 we define

$$w(x) = C_0\left(e^{-Nr^2} - e^{-N|x|^2}\right),$$

then $w(x) > 0$ for $|x| > r$ and $w(x) = 0$ for $|x| = r$ we also need to show that $Lw(x) \leq -\|f\|_{C(\Omega)}$. To that end we do a calculation similar[1] to (13.4)-(13.6) and conclude that for some small $c_0 > 0$ depending only on the diameter of Ω

$$Lw(x) \leq -C_0 N^2 r^2 \lambda e^{-Nr^2} \leq -c_0 C_0$$

if N is large enough (depending only on $\|b_i(x)\|_{C(\Omega)}$, λ and r). Choosing $C_0 = \frac{\|f\|_{C(\Omega)}}{c_0}$ implies that $Lw(x) \leq -\|f\|_{C(\Omega)}$ which finishes the proof. □

Corollary 21.2. *Let Ω be a bounded $C^{2,\alpha}-$domain and L be given by (21.26) with $c(x) \leq 0$ and $a_{ij}, b_i, c, f \in C(\Omega)$. Assume furthermore that $a_{ij}(x)$ satisfies the standard ellipticity condition*

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

for some $\lambda > 0$ and every $x, \xi \in \mathbb{R}^n$.

Then there exists a barrier at every point of $\partial\Omega$.

Proof: It is enough to notice that we may touch the boundary at any point $\xi \in \partial\Omega$ from the outside by some ball $B_r(x_\xi)$. This since $\partial\Omega$ is a closed and bounded set and thus compact and therefore the second derivatives of $\partial\Omega$ is bounded from above by some constant $1/r > 0$. From standard analysis we know that the radius of curvature of $\partial\Omega$ is greater than r. □

Using the barrier we may estimate $\|u\|_{C(\Omega)}$ in terms of $\|b_i(x)\|_{C(\Omega)}$, $\|f\|_{C(\Omega)}$ and r and formulate a sightly stronger version of the apriori estimates in Theorem 21.2 if we also assume that $c(x) \leq 0$ - which we need in order to use the comparison principle: Corollary 13.1.

[1]The only difference in this and the previous calculations is that we have an extra minus in the definition of the exponentials and that we multiply by a constant C_0. However, the calculation is still valid with minor changes.

Theorem 21.3. *Assume that $u \in C^2(\Omega)$, where Ω is a bounded domain, is a solution to*

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(c)u(x) = f(x) \text{ in } \Omega \tag{21.27}$$

$$u(x) = g(x) \text{ on } \partial\Omega. \tag{21.28}$$

Assume furthermore that $f(x), a_{ij}(x), b_i(x), c(x) \in C^\alpha(\Omega)$ and that $g(x) \in C^{2,\alpha}(\partial\Omega)$ and that Ω is a $C^{2,\alpha}-$domain and that $a_{ij}(x)$ satisfy the standard ellipticity assumption

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

Finally we assume that $c(x) \leq 0$.

Then there exists a constant C such that

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)}\right), \tag{21.29}$$

here $C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$ where $\Lambda, \lambda > 0$ are the ellipticity constants of the PDE.

Proof: From Theorem 21.2, in particular equation (21.20), we know that

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)} + \|u\|_{C(\Omega)}\right).$$

Therefore we only need to show that

$$\|u\|_{C(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)}\right), \tag{21.30}$$

as a matter of fact we will prove something slightly stronger:

$$\|u\|_{C(\Omega)} \leq C\left(\|f\|_{C^\alpha(\Omega)} + \|g\|_{C(\partial\Omega)}\right), \tag{21.31}$$

where C may depend on $n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c$, and Ω.

According to Corollary 21.2 if $B_r(x^0)$ touches Ω at some point then there exists a barrier

$$w(x) = \frac{\|f\|_{C(\Omega)}}{c_0}\left(e^{-Nr^2} - e^{-N|x-x^0|^2}\right),$$

where $N >> 1$ only depends on $\|b_i(x)\|_{C(\Omega)}$, λ and Ω. In particular, by the comparison principle Corollary 13.1 it follows that

$$-\|g\|_{C(\partial\Omega)} - w(x) \leq u(x) \leq \|g\|_{C(\Omega)} + w(x) \qquad \text{in } \Omega$$

since the inequalities holds on $\partial\Omega$. But that implies that

$$\|u\|_{C(\Omega)} \leq \|g\|_{C(\partial\Omega)} + \|w\|_{C(\Omega)}.$$

but the supremum norm $\|w\|_{C(\Omega)}$ will only depend on N and on the diameter of Ω and linearly on $\|f\|_{C(\Omega)}$. This implies (21.31) which finishes the proof of the Theorem. $\square$

Chapter 22

The Method of continuity.

With the strong regularity theory at hand it is not very difficult to prove existence of solutions. Before we show how thin is done let us prove a simple proposition: *the contraction mapping principle*. Before we do that we need to define what a contracting map is.

Definition 22.1. *We say that a mapping $L : B \mapsto B$ from a Banach space B to itself is a contraction, or that L is a contracting map, if there exists a constant $0 < \tau < 1$ such that*

$$\|Lu - Lv\|_B \leq \tau \|u - v\|_B$$

for all $u, v \in B$.

Proposition 22.1. [The Contraction Mapping Principle.] *Let L be a contracting map on the Banach space B then there exists a unique $u \in B$ such that*

$$Lu = u.$$

Proof: The proof is simple. Pick a random $u_0 \in B$, say $u_0 = 0$, and inductively define

$$u_{k+1} = Lu_k.$$

Then, by the definition of u_k and using that L is a contracting map,

$$\|u_{k+1} - u_k\|_B = \|Lu_k - Lu_{k-1}\|_B \leq$$

$$\leq \tau \|u_k - u_{k-1}\|_B \leq \tau \|Lu_{k-1} - Lu_{k-2}\|_B \leq$$

$$\leq \tau^2 \|u_{k-1} - u_{k-2}\|_B \leq ... \leq \tau^k \|u_1 - u_0\|_B.$$

In particular, using the triangle inequality we see that, for $k > l$,

$$\|u_k - u_l\|_B \leq \sum_{j=l}^{k-1} \|u_{j+1} - u_j\|_B \leq \sum_{j=l}^{k-1} \tau^j \|u_1 - u_0\|_B =$$

$$= \frac{\tau^l - \tau^k}{1-\tau}\|u_1 - u_0\|_B \le \tau^l \frac{\|u_1 - u_0\|_B}{1-\tau}.$$

In particular, u_k is a Cauchy sequence and thus converges to some $u_\infty \in B$.

Next we notice that any contracting map is (uniformly) continuous. To see this we notice that given an $\epsilon > 0$ we may choose $\delta = \frac{\epsilon}{\tau}$ then $\|u - v\|_B < \delta$ implies that $\|Lu - Lv\|_B < \epsilon$. Therefore

$$u_\infty = \lim_{k\to\infty} u_k = \lim_{k\to\infty} Lu_{k-1} = L \lim_{k\to\infty} u_{k-1} = Lu_\infty, \tag{22.1}$$

where we used the definition on u_∞ in the first equality, the definition of u_k in the second and continuity in the third equality. Equation (22.1) concludes the proof. □

The importance of the contraction mapping principle is that if we have two PDE $L_1u(x) = f(x)$ and $L_2u(x) = f(x)$ and:

1. We can show that $L_1u(x) = f(x)$ in Ω and $u(x) = g(x)$ on $\partial\Omega$ has a solution.
2. We have estimates for the operator L_1, that is if $L_1u(x) = F(x)$ then there exists a constant C such that $\|u\|_B \le C_{L_1}\|F\|_B$.
3. The operator $Lu(x) = L_1u(x) - L_2u(x)$ has small norm, that is if $\|Lu\|_B = \|L_1u(x) - L_2u(x)\|_B \le c\|u\|_B$ for a small enough constant c, that will depend on C_{L_1}.

Then $L_2u(x) = f(x)$ in Ω and $u(x) = g(x)$ on $\partial\Omega$ has a solution.

Example: *How the Contraction mapping halps to find solutions to PDE:* Consider the PDE

$$\begin{array}{ll} Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega, \end{array} \tag{22.2}$$

where Ω is a $C^{2,\alpha}-$domain, $a_{ij}, b_i, c, f \in C^\alpha(\Omega)$ and $g \in C^{2,\alpha}(\partial\Omega)$. Furthermore we assume that for some small $\epsilon > 0$ we have that $\|a_{ij}(x) - \delta_{ij}\|_{C^\alpha(\Omega)} < \epsilon$ where δ_{ij} is the Kronecker delta, $\|b_i(x)\|_{C^\alpha(\Omega)} < \epsilon$ and $\|c(x)\|_{C^\alpha(\Omega)} < \epsilon$.

We claim that the contracting mapping principle can be used to show that there exists a solution to (22.2).

We may then define the operator

$$F : \{u \in C^{2,\alpha}(\Omega)\ u(x) = g(x) \text{ on } \partial\Omega\} \mapsto \{u \in C^{2,\alpha}(\Omega)\ u(x) = g(x) \text{ on } \partial\Omega\},$$

so that given an $F(u(x))$ solves the following PDE

$$\begin{array}{ll} \Delta F(u(x)) = f(x) + (\Delta u(x) - Lu(x)) & \text{in } \Omega \\ F(u(x)) = g(x) & \text{on } \partial\Omega. \end{array} \tag{22.3}$$

That $F(v(x))$ is well defined follows from the existence of a unique solution to the Dirichlet problem for Laplace equation. Also, $v(x) \in C^{2,\alpha}(\Omega)$ it is clear that the right hand side in (22.3) is in $C^\alpha(\Omega)$ and thus by our apriori estimates for the Laplacian that $F(v(x)) \in C^{2,\alpha}(\Omega)$.

Next we claim, and this is the important step, that F is a contracting map if $\epsilon > 0$ is small enough. To see this we notice that for $u, v \in C^{2,\alpha}(\Omega)$, $u(x) = v(x) = g(x)$ on $\partial\Omega$ then

$$\|\Delta\left(F(u(x)) - F(v(x))\right)\|_{C^\alpha(\Omega)} = \|\Delta(u(x) - v(x)) - L(u(x) - v(x))\|_{C^\alpha(\Omega)} \leq$$
$$\leq C\epsilon\|u(x) - v(x)\|_{C^{2,\alpha}(\Omega)},$$

where we used that the difference between the coefficients of the Laplacian and L are less than ϵ in $C^\alpha-$norm.

By the estimates **add REF** for the Laplace equation we can conclude that

$$\|F(u) - F(v)\|_{C^{2,\alpha}(\Omega)} \leq C\epsilon\|u(x) - v(x)\|_{C^{2,\alpha}(\Omega)},$$

where we also used that $F(u) - F(x) = 0$ on $\partial\Omega$. So if $\epsilon \leq \frac{1}{2C}$, where C only depends on n, α then F is a contracting map. Thus, it exists a function $u(x)$ such that $u(x) = F(u(x))$. Substituting this u into (22.3) it follows that

$$\begin{array}{ll} \Delta u(x) = f(x) + (\Delta u(x) - Lu(x)) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

It follows that u is a solution to (22.2).

In the above example we only used the following three conditions: (1) we knew that we had solutions for the Laplacian, (2) that we have $C^{2,\alpha}$-estimates for the Laplacian and (3) that the coefficients of L where close enough to the coefficients of the Laplacian. This was enough to show that there exists a solution to $Lu(x) = f(x)$, therefore, the first condition is satisfied for L. We also have $C^{2,\alpha}-$estimates for L and therefore the second condition is satisfied. This implies that we can show existence of a solution to any elliptic PDE with C^α coefficients ϵ-close to the coefficients of L. We may thus continue iteratively and show that any elliptic PDE with $C^\alpha-$coefficients admits a solution.

To formalize the above argument we let

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \quad \text{in } \Omega \tag{22.4}$$

and we define the linear elliptic PDE $L_t u(x) = (1-t)\Delta u(x) + tLu(x)$. By the argument in the example above we may show that $L_t u(x) = f(x)$ has a solution for $t \in [0, \epsilon]$ for $\epsilon > 0$ small enough, depending only on the operator L, $\alpha \in (0,1)$, the domain Ω and the dimension. Applying the same argument (as in the example) with L_ϵ, for which we now know there exists a solution, we can show that $L_t u(x) = f(x)$ has a solution for $t \in [0, 2\epsilon]$. Iterating will lead to the existence of solutions for any $t \in [0, k\epsilon]$, for any $k \in \mathbb{N}$ as long as $L_t u(x)$ is elliptic. In particular, $L_1 u(x) = f(x)$ has a solution which is the same as saying that $Lu(x) = f(x)$ has a solution. This method of proof is usually called the method of continuity.

Proposition 22.2. [THE METHOD OF CONTINUITY.] *Let $L_0 : B \mapsto V$ and $L_1 : B \mapsto V$ be two linear operators between the Banach spaces B and V. Furthermore we denote $L_t = (1-t)L_0 + tL_1$, for $t \in [0,1]$. Also assume that there exists a constant C_0 such that the following estimate holds*

$$\|u-v\|_B \leq C_0 \|L_t(u-v)\|_V \qquad \textit{for every } t \in [0,1]. \tag{22.5}$$

and that the operators are bounded, that is there exists a constant C_1 such that

$$\|L_t u\|_V \leq C_1 \|u\|_B. \tag{22.6}$$

Then $L_t u = f$ has a solution for every $f \in V$ for every $t \in [0,1]$ if and only if there exists one $t_0 \in [0,1]$ such that $L_{t_0} u = f$ for every $f \in V$.

Proof: Clearly it is enough to show that if there exists one $t_0 \in [0,1]$ such that $L_{t_0} u = f$ then a solutions exists for every $t \in [0,1]$.

We assume that $L_{t_0} u = f$ has a solution for every $f \in V$ and define the map

$$F: \ u \mapsto L_{t_0}^{-1} \underbrace{(f + L_{t_0} u - L_{t_0+\epsilon} u)}_{\in V} \in B.$$

This map is well defined since $L_{t_0} u = f$ admits a solution for every $f \in V$ and, from (22.5) it follows that, the solution to $L_{t_0} u = f$ is unique. To see uniqueness we simply notice that if $u, v \in B$ are two solutions then $\|u-v\|_B \leq C\|L_{t_0}(u-v)\|_V = \|f-f\|_V = 0$ since the operator is linear.

We claim that F is a contraction if $|\epsilon|$ is small enough (depending only on C). To see this we consider $u, v \in B$ and calculate

$$\|F(u) - F(v)\|_B = \left\| \left(L_{t_0}^{-1} \left(f + L_{t_0} u - L_{t_0+\epsilon} u\right)\right) - \left(L_{t_0}^{-1} \left(f + L_{t_0} v - L_{t_0+\epsilon} v\right)\right) \right\|_B =$$

$$= \left\| L^{-1} \left(\epsilon L_0 (u-v) - \epsilon L_1 (u-v)\right) \right\|_B \leq 2|\epsilon| C_0 C_1 \|u-v\|_B, \tag{22.7}$$

where we used the triangle inequality and (22.5)-(22.6) in the last inequality and the definition of L_t in the second equality.

We see that if $\epsilon < \frac{1}{C}$ then F is a contraction. We may conclude that for every $|\epsilon| < \frac{1}{C}$ there exists a solution to $F(u) = u$, that is

$$L_{t_0} u = f - L_{t_0} u - L_{t_0+\epsilon} u \Rightarrow L_{t_0+\epsilon} u = f.$$

In particular $L_t u = f$ admits a solution for every $t \in [t_0 - 1/(2C), t_0 + 1/(2C)]$.

Repeating the argument for any $t_1 = t_0 \pm 1/(2C)$ we see that $L_t u = f$ admits a solution for every $t \in [t_0 - 2/(2C), t_0 + 2/(2C)]$. Inductively it follows that the $L_t u = f$ admits a solution for every $t \in [t_0 - k/(2C), t_0 + k/(2C)] \cap [0,1]$. The Proposition follows by choosing k large enough, say $k > 2C$. $\square$

Chapter 23

Existence theory for variable coefficients.

In the last chapter we saw that the method of continuity is what we need in order to show existence of our PDE. In this chapter we carry out this argument and prove existence for a general elliptic PDE with variable coefficients by the method of continuity.

Theorem 23.1. *Let Ω be a bounded $C^{2,\alpha}-$domain and $g \in C^{2,\alpha}(\partial\Omega)$. Also let*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) \quad \text{in } \Omega$$

where $a_{ij}(x), b_i(x), c(x), f(x) \in C^{\alpha}(\Omega)$, $c(x) \leq 0$ and $a_{ij}(x) = a_{ji}(x)$ satisfies the following ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for some $\Lambda, \lambda > 0$ and every $x, \xi \in \mathbb{R}^n$.

Then there exists a unique solution $u \in C^{2,\alpha}(\Omega)$ to the boundary value problem

$$\begin{aligned} Lu(x) &= f(x) && \text{in } \Omega \\ u(x) &= g(x) && \text{on } \partial\Omega \end{aligned}$$

for every $f(x) \in C^{\alpha}(\Omega)$ and $g(x) \in C^{2,\alpha}(\partial\Omega)$.

Proof: We will use the method of continuity, Proposition 22.2. We define $L_0 = \Delta$ and $L_1 = L$. Then there exists a solution $u_0 \in C^{2,\alpha}(\Omega)$ to

$$\begin{aligned} L_0 u_0(x) &= f(x) && \text{in } \Omega \\ u_0(x) &= g(x) && \text{on } \partial\Omega. \end{aligned}$$

So it is enough to show that $L_t = (1-t)L_0 + tL_1$ satisfies the estimate (22.5) with $B = C^{2,\alpha}(\Omega)$ and $V = C^\alpha(\Omega)$:

$$\|u-v\|_{C^{2,\alpha}(\Omega)} \leq C\|L_t(u-v)\|_{C^\alpha(\Omega)}, \tag{23.1}$$

for any functions $u, v \in C^{2,\alpha}(\Omega)$ such that $u(x) = v(x) = g(x)$ on $\partial\Omega$. Notice that Theorem 21.3 implies that

$$\|u-v\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|L_t(u-v)\|_{C^\alpha(\Omega)} + \underbrace{\|u-v\|_{C^{2,\alpha}(\partial\Omega)}}_{=0}\right) = C\|L_t(u-v)\|_{C^\alpha(\Omega)},$$

where $C = C(n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c, \Omega)$. But this is exactly (23.1).

Therefore the Theorem follows from the method of continuity. $\square$

Chapter 24

The Perron Method.

So far we have, in Theorem 23.1, shown that there exists solutions to the following elliptic PDE

$$\begin{array}{ll} Lu(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega, \end{array} \tag{24.1}$$

if the domain Ω and $g(x)$ both are $C^{2,\alpha}$ and the coefficients of the PDE are C^α, at least if $c(x) \leq 0$. Even though this is an amazing result (at least one must admit that it involves very complicated constructions...) it is somewhat unsatisfying in several ways. First of all it seems to be too strong to assume that $g \in C^{2,\alpha}(\partial\Omega)$ since we never have to differentiate $g(x)$. Second, the assumption that Ω is $C^{2,\alpha}$ also seems strong, in particular for many applications. Thirdly, we did not need so strong assumptions on the domain or the boundary data to show existence of solutions for the Laplacian case.

In this final chapter we will use Perron's method to prove that (24.1) has a solution even for continuous $g(x)$ and domains Ω that are regular (that is has a barrier at every point of the boundary).

Perron's method was to show that the solution $u(x)$ is given by

$$u(x) = \sup_{v \in S_g(\Omega)} v(x) \tag{24.2}$$

where

$$S_g(\Omega) = \{v \in C(\Omega);\ v(x) \text{ is a sub-solution to (24.1)},\ v(x) \leq g(x) \text{ on } \partial\Omega\}$$

There are some small technical problems that we need to handle in order to define $u(x)$ according to (24.2).

First, we need to be able to extend the concept of sub-solution to functions that are continuous - before we used the definition that $v(x)$ was a sub-solution if $v \in C^2(\Omega)$ $Lv(x) \geq 0$. This definition requires that $v \in C^2(\Omega)$ which would result in that the supremum of two sub-solutions might not be a sub-solution. We will use the comparison principle to define sub-solutions in this chapter.

Second, an important step in the proof of Perron's method was to make a harmonic replacement in $B_r(x^0) \subset \Omega$. That is to define

$$\tilde{v}(x) = \begin{cases} v(x) & \text{for } x \in \Omega \setminus B_r(x^0) \\ w(x) & \text{for } x \in B_r(x^0), \end{cases}$$

where $w(x)$ solves $Lw(x) = f(x)$ in $B_r(x^0)$ and $w(x) = v(x)$ on $\partial B_r(x^0)$. For this we need to be able to solve the Dirichlet problem for L in any ball $B_r(x^0)$ with continuous boundary data. We also need to show that the replacement, hereafter called the $L-$harmonic replacement, is a sub-solution.

Fortunately, we have developed so much PDE theory so these last things will be comparatively simple. We begin to show existence of solutions to (24.1) when $\Omega = B_r(0)$ and $g \in C(\partial\Omega)$.

Proposition 24.1. *Let*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) \quad \text{in } B_r(0)$$

where $a_{ij}(x), b_i(x), c(x), f(x) \in C^\alpha(\Omega)$, $c(x) \le 0$ and $a_{ij}(x) = a_{ji}(x)$ satisfies the following ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for some $\Lambda, \lambda > 0$ and every $x, \xi \in \mathbb{R}^n$.

Then there exists a unique solution $u \in C^{2,\alpha}_{int}(B_r(0)) \cap C(B_r(0))$ to the boundary value problem

$$\begin{aligned} Lu(x) &= f(x) \quad && \text{in } B_r(0) \\ u(x) &= g(x) && \text{on } \partial B_r(0) \end{aligned}$$

for every $f(x) \in C^\alpha(B_r(0))$ and $g(x) \in C(\partial B_r(0))$.

Proof: We may find a sequence $g_\epsilon \in C^{2,\alpha}(\partial B_r(0))$ such that $g_\epsilon(x) \to g(x)$ uniformly as $\epsilon \to 0$. To see this we could for instance define the spherical mollifier

$$\hat{\phi}_\epsilon(x) = c_\epsilon \phi_\epsilon(x),$$

where ϕ_ϵ is the standard mollifier, see section 2.3, and c_ϵ is chosen so that $\int_{\partial B_r(0)} \hat{\phi}_\epsilon(x-y)dA_{\partial B_r(0)}(x) = 1$ for any $y \in \partial B_r(0)$. Then

$$g_\epsilon(x) = \int_{\partial B_r(0)} \hat{\phi}_\epsilon(x-y)g(y)dA_{\partial B_r(0)}(y)$$

would satisfy $g_\epsilon \in C^{2,\alpha}(\partial B_r(0))$ such that $g_\epsilon(x) \to g(x)$ uniformly as $\epsilon \to 0$[1].

[1]The proof is exactly the same as the proof of Lemma 2.2.

From Theorem 23.1 it follows that there exists a solution $u_\epsilon(x)$ to

$$\begin{aligned} Lu_\epsilon(x) &= f(x) \quad \text{in } B_r(0) \\ u_\epsilon(x) &= g_\epsilon(x) \quad \text{on } \partial B_r(0). \end{aligned}$$

Also, from Theorem 16.1,

$$\|u_\epsilon\|_{C^{2,\alpha}_{\text{int}}(B_r(0))} \le C\left(\|f\|_{C^\alpha_{\text{int},(2)}(B_r(0))} + \|u_\epsilon\|_{C(B_r(0))}\right). \tag{24.3}$$

Replicating the argument (21.31) we see that

$$\|u_\epsilon\|_{C(B_r(0))} \le C\left(\|f\|_{C^\alpha(B_r(0))} + \|g_\epsilon\|_{C(\partial B_r(0))}\right) \tag{24.4}$$

where C may depend on $n, \alpha, \lambda, \Lambda, a_{ij}, b_i, c,$.

The two estimates (24.3) and (24.4) together implies

$$\|u_\epsilon\|_{C^{2,\alpha}_{\text{int}}(B_r(0))} \le C\left(\|f\|_{C^\alpha(B_r(0))} + \|g_\epsilon\|_{C(\partial B_r(0))}\right).$$

In particular, u_ϵ forms and equicontinuous family of functions in $B_r(0)$. So, by the Arzela-Ascoli Theorem, we may find a sub-sequence $u_{\epsilon_j}(x) \to u(x)$ in $C^2(B_r(0))$ where $u(x)$ solves

$$Lu_\epsilon(x) = f(x) \quad \text{in } B_r(0).$$

We only need to verify that $u(x) = g(x) = \lim_{\epsilon\to 0} g_\epsilon(x)$ on $\partial B_r(0)$. We will show that by showing that $u_\epsilon \to u$ uniformly on $\overline{B_r(0)}$ and thus, since u_ϵ is continuous for each ϵ, it follows that u is continuous on $\overline{B_r(0)}$.

Since $g_\epsilon \to g$ uniformly, it follows that $w_{\delta,\epsilon}(x) = u_\epsilon(x) - u_\delta(x)$ satisfies

$$\begin{aligned} Lw_{\epsilon,\delta}(x) &= 0 && \text{in } B_r(0) \\ w_{\epsilon,\delta}(x) &= g_\epsilon(x) - g_\delta(x) && \text{on } \partial B_r(0). \end{aligned}$$

In particular, by the maximum principle (Lemma 13.1) that

$$\sup_{B_r(0)} |w_{\epsilon,\delta}(x)| = \sup_{\partial B_r(0)} |g_\epsilon(x) - g_\delta(x)|.$$

Letting $\delta \to 0$ we see that

$$\sup_{B_r(0)} |w_{\epsilon,0}(x)| = \sup_{B_r(0)} |u_\epsilon(x) - u(x)| = \sup_{\partial B_r(0)} |g_\epsilon(x) - g(x)|$$

and it follows that $u_\epsilon \to u$ uniformly since $g_\epsilon \to g$ uniformly. That $u \in C(\overline{B_r(0)})$ follows directly from the fact that u is the uniform limit of a sequence of continuous functions. □

Next we need to extend the definition of sub-solution to include merely continuous functions. When we defined sub-solutions for the Laplace equation, Definition 7.2, we defined a sub-solution to be any function $v \in C^2(\Omega)$ such that $\Delta v(x) \ge 0$. However, for the Perron method we need that the $L-$harmonic replacement of a sub-solution is still a sub-solution. And in general the $L-$harmonic replacement is not $C^2(\Omega)$. We therefore define sub-solution in the following way.

Definition 24.1. *We say that $v(x) \in C(\Omega)$ is a sub-solution to (24.1) if for every ball $B_r(x^0) \subset \Omega$ we have that $v(x) \leq w(x)$ in $B_{r_0}(x^0)$ where $w(x)$ is the solution*[2] *to*

$$\begin{array}{ll} Lw(x) = f(x) & \textit{in } B_{r_0}(x^0) \\ w(x) = v(x) & \textit{on } \partial B_{r_0} r(x^0). \end{array} \qquad (24.5)$$

Remark:*Just as we defined sub-harmonic by using the sub-meanvalue property we define sub-solution by using the comparison principle (Corollary 13.1). In particular if $v(x) \in C^2(\Omega)$ then it follows from the comparison, principle that $Lv(x) \geq f(x)$. Thus the new definition that $v(x)$ should satisfy the comparison principle is equivalent to the definition that $Lv(x) \leq f(x)$ for $v \in C^2(\Omega)$. However, the new definition only need $v \in C(\Omega)$ which renders the new definition more flexible.*

With this definition it is easy to prove that taking the $L-$harmonic replacement preserves sub-solutions. The proof is almost line for line the same as the proof of Lemma 11.1.

Lemma 24.1. *Suppose that $v \in C(\Omega)$ is a sub-solution to (24.1) in Ω. Moreover, we assume that $\overline{B_{r_0}(x^0)} \subset \Omega$. If we define $\tilde{v}$ to by the L-harmonic replacement of $v(x)$ in $B_{r_0}(x^0)$:*

$$\tilde{v}(x) = \begin{cases} v(x) & \textit{if } x \in \Omega \setminus B_{r_0}(x^0) \\ w(x) & \textit{for } x \in B_{r_0}(x^0), \end{cases}$$

where $w(x)$ is the solution to (24.5).

Then $\tilde{v}$ is sub-harmonic in Ω.

Proof: We need to show that

$$v(x) \leq h(x) \qquad \text{in } B_r(y) \qquad (24.6)$$

for any ball $\overline{B_r(y)} \subset \Omega$, where $\tilde{h}(x)$ solves

$$\begin{array}{ll} L\tilde{h}(x) = f(x) & \text{in } B_r(y) \\ \tilde{h}(x) = \tilde{v}(x) & \text{on } \partial B_r r(y). \end{array}$$

If $B_r(y) \subset B_{r_0}(x^0)$ or $B_r(y) \subset \Omega \setminus B_{r_0}(x^0)$ then (24.6) is clear since $\tilde{v}$ is a solution, and thus equal to $\tilde{h}(x)$ in the first case, and a sub-solution, and thus $\tilde{v}(x) \leq \tilde{h}(x)$ by definition, in the second case. It is therefore enough to show (24.6) in the case when $B_r(y) \cap B_{r_0}(x^0) \neq \emptyset$ and $B_r(y) \setminus B_{r_0}(x^0) \neq \emptyset$. We fix such a ball and continue to prove the Lemma in the same way we proved Lemma 11.1.

Step 1: *In $B_r(y) \setminus B_{r_0}(x^0)$ we have that $\tilde{h} \geq \tilde{v}$.*

Let h solve

$$\begin{array}{ll} Lh = f(x) & \text{in } B_r(y) \\ h(x) = v(x) & \text{on } \partial B_r(y), \end{array}$$

[2]The existence of such solution is guaranteed by Proposition 24.1.

then, since $v(x)$ is a sub-solution it follows that $v \leq h$ in $B_r(y)$.

Since $v \leq \tilde{v}$, which again follows from the assumption that $v(x)$ is a sub-solution, we have that $h \leq \tilde{h}$ on $\partial B_r(y)$ and since both h and $\tilde{h}$ solves $L\cdot = f(x)$ in $B_r(y)$ it follows from the Comparison principle that $\tilde{h} \geq h$ in $B_r(y)$. That is $v \leq h \leq \tilde{h}$ in $B_r(y)$.

Using that $\tilde{v} = v$ in $B_r(y) \setminus B_{r_0}(x^0)$ the claim in step 1 follows.

Step 2: *We claim that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \cap B_{r_0}(x^0)$.*

By step 1 we know that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \setminus B_{r_0}(x^0)$. Since $\tilde{v}$ and $\tilde{h}$ are continuous functions it follows that $\tilde{h} \geq \tilde{v}$ on $\big(\partial B_{r_0}(x^0)\big) \cap B_r(y)$. On $\big(\partial B_r(y)\big) \cap B_{r_0}(x^0)$ we have that $\tilde{h} = \tilde{v}$ by the definition of $\tilde{h}$.

In particular, $L\tilde{v} = L\tilde{h} = f(x)$ in $B_{r_0}(x^0) \cap B_r(y)$ and $\tilde{v} \leq \tilde{h}$ on $\partial\big(B_{r_0}(x^0) \cap B_r(y)\big)$. It follows that $\tau(x) = \tilde{v} - \tilde{h}$ solves

$$\begin{array}{ll} L\tau(x) = 0 & \text{in } B_{r_0}(x^0) \cap B_r(y) \\ \tau(x) \leq 0 & \text{on } \partial\big(B_{r_0}(x^0) \cap B_r(y)\big). \end{array}$$

By the maximum principle, Lemma 13.1, $\tau(x) \leq 0$ in $B_{r_0}(x^0) \cap B_r(y)$, that is $\tilde{v} \leq \tilde{h}$ in $B_{r_0}(x^0) \cap B_r(y)$.

This proves that $\tilde{v}(x)$ satisfies (24.6) for any ball $B_r(y)$ and thus concludes the proof. $\square$

After these preliminary considerations we are now ready to prove existence by Perron's method. We will prove it in a slightly different way than we did for the Laplacian. Partly, because I am bored and partly since we have not proved a strong maximum principle for general elliptic equations.

Theorem 24.1. [The Perron Method.] *Let Ω be a bounded domain and $g \in C(\partial\Omega)$. Define*

$$u(x) = \sup_{v \in S_g(\Omega)} v(x),$$

where

$$S_g(\Omega) = \{v \in C(\Omega);\ v(x) \text{ is a sub-solution to } (24.1),\ v(x) \leq g(x) \text{ on } \partial\Omega\},$$

and the coefficients of L satisfy the following assumptions $a_{ij}(x), b_i(x), c(x), f(x) \in C^\alpha(\Omega)$, $c(x) \leq 0$ and $a_{ij}(x) = a_{ji}(x)$ satisfies the following ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for some $\Lambda, \lambda > 0$ and every $x, \xi \in \mathbb{R}^n$.

Then

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) \quad \text{in } \Omega.$$

Furthermore, if $\xi \in \partial\Omega$ is a regular point then $\lim_{x\to\xi} u(x) = g(\xi)$.

Proof: We will prove the Theorem in several steps.

Step 1: *S_g is non-empty and bounded from above and therefore $u(x)$ is well defined.*

In the proof of Theorem 21.3 we constructed a barrier $w(x)$ such that $L(-w(x)) \geq f(x)$ in Ω and $-w(x) \leq g(x)$ on $\partial\Omega$. Thus $-w(x) \in S_g(\Omega)$ which proves that $S_g(\Omega) \neq$. Also, $w(x)$ is a super-solution that satisfies $w(x) \geq g(x)$ which implies that $w(x) \geq v(x)$ for any $v(x) \in S_g(\Omega)$. It follows that $S_g(\Omega)$ is bounded from above.

Step 2: *$u(x)$ is continuous.*

We will argue by contradiction and assume that

$$\lim_{x^j \to x^0} u(x^j) = \limsup_{x \to x^0} u(x) > \liminf_{x \to x^0} u(x) = \lim_{y^j \to x^0} u(y^j)$$

for some $x^0 \in \Omega$. We may thus find two points x^J and y^J such that

$$u(x^J) - u(y^J) > \kappa > 0 \tag{24.7}$$

and $|x^J - y^J|$ is smaller than a quantity that we will choose later.

Since Ω is open there exists a ball $B_{4r}(x^0) \subset \Omega$. We may therefore, by increasing J if necessary assume that $B_{2r}(X^J) \subset \Omega$ and $y^J \in B_r(X^J)$. Now let $v^j(x) \in S_g(\Omega)$ be a sequence such that $v^j(x^J) \to u(x^J)$. We may assume that v^j is bounded, if not $\sup\big(v^j(x), -w(x)\big) \in S_g(\Omega)$ will be a bounded function with the same properties as v^j. We also define the $L-$harmonic replacement of $v^j(x)$ in $B_{2r}(x^J)$ to be $\tilde{v}^j$. Then $L\tilde{v}^j = f(x)$ in $B_{2r}(x^J)$ and $\tilde{v}^j$ is bounded. It follows from Theorem 16.1 that $\|\tilde{v}^j\|_{C^{2,\alpha}(B_r(x^J))}$ is uniformly bounded. We may conclude that

$$\sup_{x \in B_r(x^J)} |\nabla \tilde{v}^j(x)| \leq C_0 \tag{24.8}$$

for some constant C_0 independent of j.

Since $x^J, y^J \in B_r(X^J)$ we can use the the mean value theorem to estimate

$$\tilde{v}^j(y^J) - \tilde{v}^j(x^J) \leq C_0|x^J - y^J|, \tag{24.9}$$

where C_0 is the constant in (24.8). But we may choose $|x^J - y^J|$ as small as we want by choosing J large enough. Choosing J so large that $|x^J - y^J| < \frac{\kappa}{2C_0}$ in (24.9) together with (24.7) will give the following contradiction:

$$\frac{\kappa}{2} > C_0|x^J - y^J| \geq \lim_{j\to\infty} \big(\tilde{v}^j(x^J) - \tilde{v}^j(y^J)\big) \geq u(x^J) - u(y^J) > \kappa,$$

where we also used that $u(y^J) \geq \lim_{j\to\infty} \tilde{v}^j(y^J)$.

Step 3: *$u(x)$ is a sub-solution.*

We need to show that if $\overline{B_r(y)} \subset \Omega$ and

$$\begin{aligned} Lh &= f(x) && \text{in } B_r(y) \\ h(x) &= u(x) && \text{on } \partial B_r(y), \end{aligned}$$

then $u(x) \leq h(x)$ in $B_r(y)$. Again we will argue by contradiction. Therefore we assume that there exists a point $z \in B_r(y)$ and a $\kappa > 0$ such that $u(z) = h(z)+\kappa$.

Let $v^j(x) \in S_g(\Omega)$ be such that $v^j(z) \to u(z)$ and denote by $\tilde{v}^j$ the L-harmonic replacement of v^j in $B_r(y)$. Then $\tilde{v}^j \in S_g(\Omega)$ by Lemma 24.1 and $v^j(z) \leq \tilde{v}^j(z)$ since v^j a sub-solution. We can conclude that $\tilde{v}^j(z) \to u(z)$.

Observe that $\tilde{v}^j(x) \leq u(x) = h(x)$ on $\partial B_r(y)$ and since $L\tilde{v}^j(x) = f(x) = Lh(x)$ in $B_r(y)$ it follows from the comparison principle. In particular, $\tilde{v}^j(z) \to u(z)$ and $\tilde{v}^j(z) \leq h(z) = u(z) - \kappa$ which clearly is a contradiction.

Step 4: *$u(x)$ is a solution.*

Since $u(x)$ is a sub-solution it follows that $u \in S_g(\Omega)$. Also any L-harmonic replacement $\tilde{u}$ of u will be a sub-solution by Lemma 24.1. Since $\tilde{u} \in S_g(\Omega)$ it follows that $u(x) = \sup_{v \in S_g(\Omega)} v(x) \geq \tilde{u}(x)$ and since $u(x)$ is a sub-solution it follows that $u(x) \leq \tilde{u}(x)$. We can conclude that $u(x) = \tilde{u}(x)$ for any L-harmonic replacement $\tilde{u}$ of u. Thus $Lu(x) = f(x)$ for any ball $\overline{B_r(y)} \subset \Omega$. It follows that $u(x)$ is a solution.

Step 5: *If $\xi \in \partial\Omega$ is a regular point then $\lim_{x\to\xi} u(x) = g(\xi)$.*

If $\xi \in \partial\Omega$ is a regular point then there exists a barrier at ξ by definition. The proof that the existence of a barrier implies that $\lim_{x\to\xi} u(x) = g(\xi)$ was done in Theorem 11.2. $\square$

We may end these notes by stating a general existence result.

Corollary 24.1. *Let Ω be a bounded domain that satisfies the exterior ball property at every point $\xi \in \partial\Omega$. Then there exists a unique solution $u \in C^{2,\alpha}_{int}(\Omega)$ to the Dirichlet problem:*

$$\begin{array}{ll} \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array}$$

where $a_{ij}(x), b_i(x), c(x), f(x) \in C^\alpha(\Omega)$, $g(x) \in C(\partial\Omega)$, $c(x) \leq 0$ and $a_{ij}(x) = a_{ji}(x)$ satisfies the following ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for some $\Lambda, \lambda > 0$ and every $x, \xi \in \mathbb{R}^n$.

Proof: The existence of a solution follows from Theorem 11.2 and that the exterior ball condition implies that the boundary point is regular follows from Lemma 21.2. $\square$