Selected Topics in PDE.

John Andersson johnan@kth.se

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Contents

Preface to the course.

In this course we will try to understand the main aspects of the theory of partial differential equations (PDE). PDE theory is a vast subject with many different approaches and subfields. We can not cover everything in one course. Some selection has to be made. In this course we will try to achieve the following:

1: We will try to stress that the foundation of theory of PDE is basic real analysis.

2: We will try to motivate the increasing levels of abstraction in the theory. Our starting point will be a difficult problem, to find a solution $u(x)$ to the equation

$$\begin{aligned} \Delta u(x) = f(x) & \quad \text{in some domain } D \\ u(x) = g(x) & \quad \text{on the boundary } \partial D. \end{aligned} \tag{1}$$

We will use the tools we have from analysis to attack the problem. But with the tools at hand we will not be able to solve the problem in its full generality. Instead we are going to simplify the problem to something that we can solve and then add more and more (and more abstract) theory in order to solve the problem in its full generality. And at every step of the way we will try to motivate the theory and why we move into the more abstract areas of mathematics.

3: We will try to introduce and motivate a priori estimates in the theory of PDE. A priori estimates are one of the most important, most technical and most difficult to understand part of PDE theory. Often it is not mentioned on the masters level. But due to its importance we will introduce it and try to understand its significance.

4: We will try to show some of the different aspects of PDE theory. In particular, in the later parts of the course, we will discuss some functional analysis and viscosity solutions approaches in the course.

For various reasons this course will be based on lecture notes written by myself. I guess that only a madman would conceive to write a book in parallel with giving a course on that book. Writing the course material has some advantages. In particular for me! I get the course that I want and a course that covers the material that I think is important. The specific contents of the course will be whatever I put in these notes. It also have some disadvantages to write the course book in parallel to teaching it. Writing is time consuming and I am a terrible bad writer even under the best circumstances. Besides my lack of suitability as a course book author, the manuscript will inevitable contain many many typos.

I therefore feel that I should suggest some supporting literature that might be red in parallel to these notes.

One of the standard PDE texts today is Lawrence C. Evans *Partial Differential Equations* published by the American Mathematical society. Evans' book is an excellent introduction to PDE theory and it covers much material that we will not have the chance to discuss. In particular Evans' book covers elliptic, parabolic and hyperbolic equations of any order as well as variational calculus and Sobolev spaces. We will focus on second order elliptic equations - but we will go further than Evans' book in some respects. Evans' book could be seen as a complement to my notes.

The next book that can be seen as a complement to this course is D. Gilbarg and N.S. Trudingers *Elliptic Partial Differential Equations of Second Order* published by Springer. Gilbarg and Trudinger's book is an excellent PDE book that covers much regularity theory. However, Gilbarg-Trudinger's treatment of the topic is very terse and I don't think that it is suitable for a masters course. As a complement to this course it is however a great book. It also covers much more material than we will be able to cover in a term. One of my hopes is that you will be able to easily understand the first six chapters of Gilbarg-Trudinger after finishing this course.

Since the course will assume that you have a good understanding of basic analysis I would also recommend that you have an analysis book at hand. Something at the level of Walter Rudin's *Principles of Mathematical Analysis* published by McGraw-Hill Higher Education.

The course will be defined by my notes and no other course literature is necessary.

Chapter 1

The Laplace Equation, some Heuristics.

One of the most fundamental partial differential equations, and also one of the must studied object in mathematics is the Laplace equation:

Solving the laplace equation means to find a function $u(x)$ such that for every $x \in D$, where D is a given open set,

$$\Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = 0. \tag{1.1}$$

Equation (1.1) appears in many applications. For instance (1.1) models the steady state heat distribution in the set D. In applications it is often necessary not only to find just any solution to $\Delta u(x) = 0$ but a specific solution that attains certain values on ∂D (say the temperature on the boundary of of the domain).

Notation: *We will denote open sets in $\mathbb{R}^n$ by D. By ∂D we mean the boundary of D, that is $\partial D = \overline{D} \setminus D$. An open connected set will be called a domain.*

Since we will always consider domains in $\mathbb{R}^n$ we will use $x = (x_1, x_2, ..., x_n)$ to denote a vector in $\mathbb{R}^n$ with coordinates $(x_1, x_2, ..., x_n)$.

It is also of interest to solve the problem $\Delta u(x) = f(x)$ for a given function $f(x)$. We therefore formulate the Dirichlet problem:

$$\begin{array}{ll} \Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = f(x) & \text{in } D \subset \mathbb{R}^n \\ u(x) = g(x) & \text{on } \partial D. \end{array} \tag{1.2}$$

D is a given domain (open set in $\mathbb{R}^n$ that might equal $\mathbb{R}^n$) and $f(x)$ a given function defined in D and g a given function defined on ∂D. Later on we will have to make some assumptions on f, g and D.

Let us fix some notation before we begin to describe these equations.

Definition 1. *We say that a function $u(x)$ is harmonic in an open set D if*

$$\Delta u(x) = 0 \text{ in } D.$$

We call the operator $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ the Laplace operator or the laplacian.

We will call the problem of finding a solution to (1.2) the Dirichlet problem or at times the boundary value problem.

Our first goal will be to solve the equations (1.2).

1.1 A Naive Approach - and a motivation for the theory ahead.

Warning: *This section is an informal discussion to motivate the theory that we will develop later. Reading this should give you a feeling that you could, if given some time, come up with the main ideas yourself. There is nothing miraculous in mathematics - just ordinary humans carefully following their intuition and the mathematical method. Later we will give stringent arguments for the intuitive ideas presented in this section.*

We stand in front of a new, interesting and very difficult problem: given a domain D and two functions f (defined on D) and g (defined on ∂D) we want to find a function $u(x)$ defined on D such that

$$\begin{array}{ll} \Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = f(x) & \text{in } D \subset \mathbb{R}^n \\ u(x) = g(x) & \text{on } \partial D. \end{array} \tag{1.3}$$

One should remark that the problem is extremely difficult. We may, at least apriori, choose f (and g) in any way we want which means that for any of the infinitively many points $x^0 \in D$ we want to prescribe the value of $\Delta u(x^0)$ so we have infinitely many equations that we want to solve simultaneously at the same time as we want the solution to satisfy $u(x) = g(x)$ for every $x \in \partial D$.

How do we start? How do we approach a new problem? We need to play with it. Try something and see where it leads.

The easiest way to attack a new problem is to make it simpler! So let us consider the simpler problem[1]

$$\Delta u(x) = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.4}$$

This problem is still quite difficult. So let us simplify (1.4) further and look for solutions $u(x)$ that are radial, that is functions that only depend on $|x|$. This should make the problem solvable - since we know how to solve differential equations depending only on one variable.

[1] Remember that we are just playing with the problem now. At the end of this section we will see that this approach leads to a theory for the equation $\Delta u = f$ in $\mathbb{R}^n$. My point is that whenever we encounter a new problem in mathematics we need to find an "in". A way to approach the problem and tie it into formulas. The we need to see where those formulas take us. If we pay attention to the formulas, and if we are lucky, we will gain some understanding of the problem. Not always the understanding we set out to find.

Lemma 1. *If $\Delta u(x) = 0$ and $u(x)$ is radial: $u(x) = h(|x|)$. Then*

$$\frac{\partial h(r)}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial h(r)}{\partial r} = 0.$$

In particular

$$u(x) = \begin{cases} \frac{a}{|x|^{n-2}} + b & \text{if } n \geq 3 \\ a\ln(|x|) + b & \text{if } n = 2 \end{cases} \tag{1.5}$$

for some $a, b \in \mathbb{R}$.

Proof: If we set $r = |x|$ then we see that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \Rightarrow \frac{\partial}{\partial x_i} = \frac{x_i}{r}\frac{\partial}{\partial r} + \text{angular derivatives}$$

and

$$\frac{\partial^2}{\partial x_i^2} = \frac{1}{r}\frac{\partial}{\partial r} - \frac{x_i^2}{r^2}\frac{\partial}{\partial r} - \frac{x_i^2}{r^3}\frac{\partial^2}{\partial r^2} + \text{ angular part.}$$

In particular for a radial function $u(x) = h(r)$ we have

$$\frac{\partial h(r)}{\partial x_i} = h'(r)\frac{\partial r}{\partial x_i} = h'(r)\frac{x_i}{r}$$

and

$$\frac{\partial^2 h(r)}{\partial x_i^2} = h''(r)\frac{x_i^2}{r^2} + h'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right),$$

where we have used that $\sum_{i=1}^n x_i^2 = r^2$.

We may thus calculate

$$0 = \Delta u(x) = \Delta h(r) = \sum_{i=1}^n \frac{\partial^2 h(r)}{\partial x_i^2} = h''(r)\sum_{i=1}^n \frac{x_i^2}{r^2} + h'(r)\sum_{i=1}^n\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) =$$

$$= h''(r) + \frac{n-1}{r}h'(r).$$

We have thus shown that $h(r)$ satisfies the ordinary differential equation

$$0 = h''(r) + \frac{n-1}{r}h'(r) = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial h(r)}{\partial r}\right). \tag{1.6}$$

Multiplying (1.6) by r^{n-1} and integrating twice gives the desired result. $\square$

The solutions in (1.5) have a singularity, and are not differentiable, in $x = 0$. So Our radial solutions only solve $\Delta u(x) = 0$ in $\mathbb{R}^n \setminus \{x = 0\}$.

We would want to analyze the singularity at the origin. If you know anything about the theory of distributions you could consider $\Delta u(x)$ to be a distribution. Here we will use more elementary methods to analyze the singularity. The main difficulty with $u(x) = h(|x|)$ as defined in (1.5) is that u isn't differentiable at the origin. So let us approximate u by a two times differentiable function.

We may then analyze the approximation and use the information about the approximated function to say something about u.

In order to simplify things somewhat we will assume that $n=3$ and we will define an approximation to u according to

$$u_\delta(x) = \begin{cases} -\left(\frac{3}{8\pi\delta} - \frac{1}{8\pi\delta^3}|x|^2\right) & \text{if } |x| \leq \delta \\ -\left(\frac{1}{4\pi}\frac{1}{|x|}\right) & \text{if } |x| > \delta. \end{cases} \tag{1.7}$$

Here we have chosen $a = \frac{1}{4\pi}$ and $b=0$, the particular choice of a will be explained later. Moreover, we have chosen the coefficients u_δ so that u_δ is continuously differentiable.

Then

$$\Delta u_\delta(x) = \begin{cases} \frac{3}{4\pi\delta^3} & \text{if } |x| < \delta \\ 0 & \text{if } |x| > \delta. \end{cases}$$

Since we are only trying to gain an understanding of the problem we don't care so much about the value of Δu_δ on the set $\{|x|=\delta\}$[2] - the set where the second derivatives are not defined.

In order to simplify notation somewhat we will define the characteristic function of a set A according to

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{1.8}$$

Then

$$\Delta u_\delta = \frac{3}{4\pi\delta^3}\chi_{B_\delta(0)}(x).$$

We may also translate the function and solve, for an $x^0 \in \mathbb{R}^3$,

$$\Delta u_\delta(x - x^0) = \frac{3}{4\pi\delta^3}\chi_{B_\delta(x^0)}(x). \tag{1.9}$$

Notice that equation (1.3), with $D = \mathbb{R}^3$, is to find for each $x^0 \in \mathbb{R}^3$ a function $u(x)$ such that $\Delta u(x^0) = f(x^0)$. But $u_\delta(x - x^0)$ accomplishes almost that when δ is small.

In particular, given N points $x^1, x^2, ..., x^N \in \mathbb{R}^3$ and values $f(x^1), f(x^2), ..., f(x^N)$ such that $|x^i - x^j| > \delta$ for $i \neq j$ then the function

$$u(x) = \sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}u_\delta(x - x^j) \tag{1.10}$$

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[2] If you are really worried about it we could define u_δ to be equal to $\frac{n^2+2n-8}{32\pi\delta^{n-2}} - \frac{n^2-4}{16\pi\delta^n}|x|^2 + \frac{n^2-2n}{32\pi\delta^{n+2}}|x|^4$ for $|x| \leq \delta$. With that definition u_δ becomes two times continuously differentiable and all the analysis in the rest of this section would follow with minor changes. Also, if you know anything about weakly differentiable functions, you will realize that u_δ has weak second derivatives in L^∞ which justifies the following calculations.

will solve

$$\Delta u(x) = \Delta\left(\sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}u_\delta(x-x^j)\right) =^{3} \tag{1.11}$$

$$= \sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}\delta u_\delta(x-x^j) = \left\{ \begin{array}{c} \text{if} \\ x = x^i \end{array} \right\} = f(x^i).$$

That is, given any finite set of points $x^1, x^2, ..., x^N$ and values $f(x^1), f(x^2)..., f(x^N)$, we may find a function u, defined according to (1.10), such that $\Delta u(x^i) = f(x^i)$.

This opens up for many possibilities. Could we consider a dense set of points $\{x^j\}_{j=1}^{\infty}$ and find a solution u^N to (1.11) for the points $\{x^j\}_{j=1}^{N}$ and values $\{f(x^j)\}_{j=1}^{N}$? Then let $N \to \infty$ and hope that $u = \lim_{N\to\infty} u^N$ solves $\Delta u(x) = f(x)$ for any $x \in \mathbb{R}^3$? This might work, but will use a different approach.

For that we need to notice that

$$\int_D g(x)dx \approx \sum_j \left(\text{volume}(\Omega_j)g(x^j)\right),$$

if $g(x)$ is continuous and Ω_j are a collection of disjoint sets such that $D \subset \cup_j \overline{\Omega}_j$ and the diameter of Ω_j is small. If we compare that to (1.10) we see that

$$u(x) = \sum_{j=1}^{N} f(x^j)\frac{4\pi\delta^3}{3}u_\delta(x-x^j) = \sum_{j=1}^{N} f(x^j)\text{volume}(B_\delta(x^j))u_\delta(x-x^j) \approx$$

$$\approx \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy,$$

we are very informal here and we can not claim that we have proved anything. But let us, still very informally, make the following conjecture:

An informal conjecture: *Let $f(x)$ be a continuous function and $\delta > 0$ a small real number. Then*

$$u^\delta(x) = \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy$$

should be an approximate solution to

$$\Delta u^\delta(x) = f(x).$$

Let us try to, still very informally, see if the conjecture makes sense. We make the following calculation

$$|\Delta u^\delta(x) - f(x)| = \left|\Delta \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy - f(x)\right| =$$

[3]Here we use an important property for the laplace equation, that it is linear. We will say more about the linearity later.

$$= \left\{ \begin{array}{l} \text{differentiation under} \\ \text{the integral} \end{array} \right\} =$$

$$= \left| \int_{\mathbb{R}^3} f(y)\Delta u_\delta(x-y)dy - f(x) \right| = \left| \frac{3}{4\pi\delta^3} \int_{\mathbb{R}^3} f(y)\chi_{B_\delta(x)}(y)dy - f(x) \right| =$$

$$= \left| \frac{3}{4\pi\delta^3} \int_{B_\delta(x)} f(y)dy - f(x) \right| \leq \frac{3}{4\pi\delta^3} \int_{B_\delta(x)} |f(y) - f(x)|\, dy \leq$$

$$\leq \sup_{y \in B_\delta(x)} |f(x) - f(y)|,$$

where we used that $\int_{B_\delta(x)} dy = \frac{4\pi\delta^3}{3}$ in the last step and that $\int_{\mathbb{R}^3} \chi_A g(x)dx = \int_A g(x)dx$ at the end of the third line of the calculation.

If f is uniformly continuous[4] then for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that

$$\sup_{y \in B_\delta(x)} |f(x) - f(y)| \leq \epsilon \qquad \text{for all } x \in \mathbb{R}^3.$$

That is if we choose $\delta = \delta_\epsilon$ then we have, at least informally, shown that

$$|\Delta u^\delta(x) - f(x)| < \epsilon.$$

It appears that

$$u(x) = \lim_{\delta \to 0} u^\delta(x) = \lim_{\delta \to 0} \int_{\mathbb{R}^3} f(y)u_\delta(x-y)dy = \left\{ \text{ informally } \right\} =$$

$$= \int_{\mathbb{R}^3} f(y)\frac{1}{4\pi|x-y|}dy$$

solves $\Delta u(x) = f(x)$ in $\mathbb{R}^3$.

We may thus make the following new conjecture

Another informal conjecture: *Let $f(x)$ be a uniformly continuous function defined in $\mathbb{R}^3$. Then*

$$u(x) = \int_{\mathbb{R}^3} f(y)\frac{1}{4\pi|x-y|}dy \tag{1.12}$$

solves $\Delta u(x) = f(x)$.

We will consider the informal conjecture as a working hypothesis to motivate the formal theory we develop later. One can already see that we need more assumptions on $f(x)$ in order for the conjecture to make sense. For instance, we need some assumption on $f(x)$ to assure that the integral in (1.12) is well defined.

[4]In our informal conjecture we just assumed continuity, but after some calculations we see that uniform continuity is a more natural assumption. The point in playing with mathematics is that we have the opportunity to see what assumptions we will need in the theorems we prove. At this point we have not proved anything. But we need to understand the problem before we can get down to the work of writing a proof.

Moreover, in proving the conjecture (with whatever extra assumptions we need) we need to be much more formal than we have been so far. In doing mathematical research one needs to be able to take a leap in the dark and argue informally to set up a reasonable hypothesis. And then have the technical skill to turn that hypothesis into a stringent proof. So far we have, what I feel to be, a reasonable hypothesis for how a solution to $\Delta u(x) = f(x)$, for uniformly continuous $f(x)$, should look. In the next section we will prove this.

Observe that this is just a first step in the development of the theory for the laplace equation. Later on we will need to find methods to handle the boundary conditions, that is to find solutions in a domain $D \subset \mathbb{R}^n$ for which the boundary condition $u(x) = g(x)$ on ∂D is satisfied.

Chapter 2

The Laplace Equation in $\mathbb{R}^n$.

2.1 The fundamental solution.

In this chapter we will be much more stringent than in the previous chapter. In particular we will take care to prove every statement that we make and to clearly define our terms. Our goal is to prove that if $f(x)$ is an appropriate function in $\mathbb{R}^n$ then

$$u(x) = -\frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy \quad \text{if } n \geq 3 \tag{2.1}$$

or

$$u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy \quad \text{if } n = 2 \tag{2.2}$$

solves $\Delta u(x) = f(x)$. Here ω_n is the area of the unit sphere in $\mathbb{R}^n$.

It is clear that we need to choose $f(x)$ in a class of functions such that the integrals (2.1) and (2.2) are well defined. To that end we make the following definition.

Definition 2. *For a continuous function f we call the closure of all the points where f is not equal to zero the support of f. We denote the support of f by $spt(f) = \overline{\{x;\ f(x) \neq 0\}}$.*

We will denote by $C^k_{loc}(D)$ the set of all functions $f(x)$ defined on D that are two times continuously differentiable on every compact set $K \subset D$.

We denote by $C^k_c(D)$ the set of all functions in $C^k_{loc}(D)$ that has compact support. That is:

$$C^k_c(D) =$$
$$= \left\{ f \in C^k_{loc}(D);\ \textit{there exists a compact set } K \subset D s.t. f(x) = 0 \textit{ for all } x \notin K \right\}$$

In what follows we will often use the functions $-\frac{1}{(n-2)\omega_n}\frac{1}{|x|^{n-2}}$ and $-\frac{1}{2\pi}\ln(|x|)$ appearing in (2.1) and (2.2) so it is convenient to make the following definition.

Definition 3. *The function defined in $\mathbb{R}^n \setminus \{0\}$ defined by*

$$N(x) = \begin{cases} -\frac{1}{2\pi}\ln(|x|) & \text{for } n = 2 \\ -\frac{1}{(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{for } n \neq 2, \end{cases}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, will be called either the Newtonian potential or the fundamental solution of Laplace's equation.

With this definition we see that (2.1) and (2.2) reduces to

$$u(x) = \int_{\mathbb{R}^n} N(x-y)f(y)dy. \tag{2.3}$$

Notice that $N(x)$ is a radial function. That means that $N(x)$ depends only on $|x|$. The name fundamental solution is somewhat justified by the following Lemma (which is essentially covered in Lemma 1).

Lemma 2. *Let $N(x)$ be the fundamental solution to Laplace's equation then*

$$\Delta N(x) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Proof: This follows from the calculations of Lemma 1. $\square$

In order to prove that $\Delta u(x) = f(x)$, where u is defined in (2.3), we need to show that:

1. The function $u(x)$ is well defined. That is that the integral in (2.3) is convergent for each x.
2. That the second derivatives of $u(x)$ are well defined. This in order to make sense of $\Delta u(x) = \sum_{i=1}^n \frac{\partial^2 u(x)}{\partial x_i^2}$.
3. Show that $\Delta u(x) = f(x)$.

Let us briefly reflect on these steps in turn.

1: Since the singularity of $N(x-y)$ is integrable it should be enough to assume that $f(x)$ has compact support in order to assure that the integral in (2.3) is convergent.

2: This is a more subtle point. If we naively[1] differentiate twice under the integral sign in (2.3) we see that, if we for definiteness assume that $n \geq 3$,

$$\frac{\partial^2 u(x)}{\partial x_i^2} = \int_{\mathbb{R}^n} \left(\frac{1}{\omega_n}\frac{1}{|x-y|^n} - \frac{n}{\omega_n}\frac{|x_i - y_i|^2}{|x-y|^{n+2}} \right) f(y)dy.$$

We see that the formal expression of $\frac{\partial^2 u(x)}{\partial x_i^2}$ involves integration of the function $\frac{f(y)}{|x-y|^n}$ which is not locally integrable in $\mathbb{R}^n$ since

$$\int_{B_1(0)} \frac{1}{|y|^n} dy = \left\{ \begin{array}{c} \text{polar} \\ \text{coordinates} \end{array} \right\} = \omega_n \int_0^1 \frac{1}{r} dr = \infty.$$

[1]By naively I mean that we do not care to verify that it is justified at this point.

It is therefore far from certain that $\frac{\partial^2 u(x)}{\partial x_i^2}$ exists. As a matter of fact, we will have to add new assumption in order to assure that $u(x)$ is two times differentiable. (See also Exercise 3 at the end of the chapter.)

3: We already have good intuition that this should be true.

In view of the second point above it seems that we need to make some extra assumption of $f(x)$ in order to prove that $u(x)$ has second derivatives. Therefore we define the following class of functions.

Definition 4. *Let $u \in C(D)$ and $\alpha > 0$ then we say that $u \in C^{\alpha}(D)$ if*

$$\sup_{x,y\in D, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty.$$

We define the norm on $C^{\alpha}(D)$ to be

$$\|u\|_{C^{\alpha}(D)} = \sup_{x\in D}|u(x)| + \sup_{x,y\in D, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

If $u \in C^{\alpha}(D)$ then we say that u is Hölder continuous in D.

We say that that $u \in C^k(D)$ if u is $k-$times continuously differentiable on D and

$$\|u\|_{C^k(D)} = \sum_{j=0}^{k} \sup_{x\in D} |D^j u(x)| < \infty.$$

Moreover we say that that $u \in C^{k,\alpha}(D)$ if $u \in C^k(D)$ and for every multiindex[2] *β of length $|\beta| = k$*

$$\sup_{x,y\in D, x\neq y} \frac{|u_\beta(x)-u_\beta(y)|}{|x-y|^{\alpha}} < \infty,$$

where $u_\beta = \frac{\partial^{|\beta|}u}{\partial x^{\beta}}$.

We define the norm on $C^{k,\alpha}(D)$ according to

$$\|u\|_{C^{k,\alpha}(D)} = \|u\|_{C^k(D)} + \max_{|\beta|=k}\left(\sup_{x,y\in D, x\neq y} \frac{|u_\beta(x)-u_\beta(y)|}{|x-y|^{\alpha}}\right),$$

where the max *is taken over all multiindexes β of length $|\beta| = k$.*

We are now ready to formulate our main theorem.

Theorem 1. *Let $f \in C_c^{\alpha}(\mathbb{R}^n)$ for some $\alpha > 0$ and define*

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi. \tag{2.4}$$

Then $u(x) \in C^2_{loc}$ and satisfies

$$\Delta u(x) = f(x).$$

[2] See the appendix for an explanation of this notation.

Proof: We will only prove the Theorem for $n \geq 3$, the proof when $n = 2$ is the same except for very small changes. The proof is rather long so we will split it up into several smaller steps.

Step 1: *The function $u(x)$ in (2.4) is well defined.*

Proof of step 1: We need to show that the integral $\int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi$ is convergent for every x. Notice that the integral is generalized in two ways. First the integrand have a singularity at $x = \xi$, and secondly the domain of integration is not bounded. Therefore we need to show that

$$\lim_{\epsilon \to 0, R \to \infty} \int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \tag{2.5}$$

exists.

Since $f(\xi)$ is continuous by assumption and $\frac{1}{|x-\xi|^{n-2}}$ is continuous for $\xi \neq x$ it is clear that

$$\int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi$$

is well defined for each $R, \epsilon > 0$. Moreover, since $f(\xi)$ has compact support there exists an R_0 such that $f(\xi) = 0$ for every $\xi \notin B_{R_0}(x)$. This means that

$$\lim_{R \to \infty} \int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi = \int_{B_{R_0}(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi,$$

so the limit as $R \to \infty$ causes no difficulty.

So we only need to consider the limit as $\epsilon \to 0$. To that end we show that $\int_{B_R(x) \setminus B_\epsilon(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi$ is Cauchy in ϵ. That is, for each $\mu > 0$ there exists a $\delta_\mu > 0$ such that

$$\left| \int_{B_R(x) \setminus B_{\epsilon_2}(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi - \int_{B_R(x) \setminus B_{\epsilon_1}(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \right| < \mu \tag{2.6}$$

for every $0 < \epsilon_1 \leq \epsilon_2 < \delta_\mu$.

We may rewrite the left hand side in (2.6) as

$$\left| \int_{B_{\epsilon_2}(x) \setminus B_{\epsilon_1}(x)} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \right| \leq$$

$$\leq \sup_{B_{\epsilon_2}(x)} |f(\xi)| \int_{B_{\epsilon_2}(x) \setminus B_{\epsilon_1}(x)} \frac{1}{|x-\xi|^{n-2}} d\xi =$$

$$= \left\{ \begin{array}{l} \text{polar} \\ \text{coordinates} \end{array} \right\} = \omega_n \sup_{B_{\epsilon_2}(x)} |f(\xi)| \int_{\epsilon_1}^{\epsilon_2} r dr \leq \frac{\omega_n \epsilon_2^2}{2} \sup_{B_{\epsilon_2}(x)} |f(\xi)|.$$

Clearly (2.6) follows with $\delta_\mu = \frac{1}{2}\sqrt{\frac{2\mu}{\omega_n}}$. It follows that $u(x)$ is well defined.

Step 2: *The function $u(x)$ is $C^1_{\text{loc}}(\mathbb{R}^n)$ and*

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi.$$

Proof of step 2: First we notice, using a similar argument as in step 1, that

$$w_i(x) = \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \tag{2.7}$$

is well defined for every $i = 1, 2, ..., n$. We aim to show that $w_i(x) = \frac{\partial u(x)}{\partial x_i}$ (which is what we would expect by differentiating under the integral sign). To prove this we define $u_\epsilon(x)$ according to

$$u_\epsilon(x) = -\frac{1}{\omega_n(n-2)} \int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-2}} \eta_\epsilon(|x-\xi|) d\xi = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) \eta_\epsilon(|x-\xi|) d\xi$$

where $\eta_\epsilon(|x|) \in C^\infty(\mathbb{R}^n)$ is an increasing function such that $\eta'_\epsilon(|x|) \le C/\epsilon$ and satisfies[3]

$$\eta_\epsilon(|x|) = \left\{ \begin{array}{ll} 0 & \text{if } |x| < \epsilon \\ 1 & \text{if } |x| > 2\epsilon. \end{array} \right\}$$

The reason we define u_ϵ in this way is that we have no singularity in the integral in the definition of u_ϵ. This means that we may manipulate u_ϵ with more ease than u. In particular, since the integrand in the definition of u_ϵ is in $C^1_c(\mathbb{R}^n)$ and we integrate over a compact set (since $f = 0$ outside a compact set) with respect to x we may use Theorem 3 in the appendix and differentiate under the integral sign and conclude that $u_\epsilon \in C^1_{\text{loc}}(\mathbb{R}^n)$.

Clearly $u_\epsilon \to u$ uniformly since

$$|u(x) - u_\epsilon(x)| = \left| \frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-2}} d\xi \right| \le$$

$$\le \frac{\sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)|}{(n-2)\omega_n} \left| \int_{B_{2\epsilon}(x)} \frac{1}{|x-\xi|^{n-2}} d\xi = \right| \le$$

$$\frac{\sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)|}{(n-2)} \epsilon^2.$$

If we can show that $\frac{\partial u_\epsilon(x)}{\partial x_i} \to w_i(x)$ uniformly it follows that u is the uniform limit of a sequence of C^1 functions whose derivatives converge uniformly to $w_i(x)$. It follows, from Theorem 2 in the appendix, that $u(x) \in C^1_{\text{loc}}(\mathbb{R}^n)$ and that $\frac{\partial u(x)}{\partial x_i} = w_i(x)$ and the proof is done.

It remains to show that $\frac{\partial u_\epsilon(x)}{\partial x_i} \to w_i(x)$ uniformly. To that end we estimate

$$\left| \frac{\partial u_\epsilon(x)}{\partial x_i} - w_i(x) \right| =$$

[3] Such an η_ϵ exists. Take for instance η_ϵ, as in Lemma 3 in the appendix, where $\eta(x) = \chi_{\mathbb{R}^n \setminus B_{2\epsilon}}$ and χ is the characteristic function defined in (1.8).

$$= \left| \frac{\partial}{\partial x_i} \int_{\mathbb{R}^n} N(x-\xi) f(\xi) \eta_\epsilon(|x-\xi|) d\xi - \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \right| \leq$$

$$= \left\{ \begin{array}{l} \text{diff. under} \\ \text{integral} \end{array} \right\} =$$

$$= \left| \int_{\mathbb{R}^n} \left(N(x-\xi) f(\xi) \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i} + \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) \eta_\epsilon(|x-\xi|) \right) d\xi - \right.$$

$$\left. - \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \right| \leq$$

$$\leq \left| \int_{\mathbb{R}^n} N(x-\xi) f(\xi) \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i} d\xi \right| + \tag{2.8}$$

$$+ \left| \int_{\mathbb{R}^n} \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) \left(\eta_\epsilon(|x-\xi|) - 1 \right) d\xi \right|$$

where we used the triangle inequality in the last step and differentiation under the integral is justified by Theorem 3.

Notice that $|\eta_\epsilon(|x-\xi|)-1| \leq 1$ for $\xi \in B_{2\epsilon}(x)$ and $|\eta_\epsilon(|x-\xi|)-1| = 0$ for $\xi \notin B_{2\epsilon}(x)$ and that $\left|\frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i}\right| \leq \frac{C}{\epsilon}$ for $\xi \in B_{2\epsilon}(x) \setminus B_\epsilon(x)$ and $\left|\frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_i}\right| = 0$ else.

We may thus estimate (2.8) from above by

$$\frac{C}{\epsilon} \int_{B_{2\epsilon}(x) \setminus B_\epsilon(x)} |N(x-\xi) f(\xi)| \, d\xi + \int_{B_{2\epsilon}} \left| \frac{\partial N(x-\xi)}{\partial x_i} \right| |f(\xi)| \, d\xi \leq$$

$$\leq \sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)| \left(\frac{C}{\epsilon} \int_{B_{2\epsilon}} \frac{1}{|x-\xi|^{n-2}} d\xi + C \int_{B_{2\epsilon}} \frac{1}{|x-\xi|^{n-1}} d\xi \right) \leq$$

$$\leq C \sup_{\xi \in \mathbb{R}^n} |f(\xi)| \epsilon.^4 \tag{2.9}$$

We may thus conclude that $\frac{\partial u_\epsilon}{\partial x_i} \to w_i$ uniformly and that $u_\epsilon \to u$ uniformly. It follows, from Theorem 2, that $\frac{\partial u(x)}{\partial x_i} = w_i \in C^0_{\text{loc}}(\mathbb{R}^n)$. Step 2 is thereby proved.

Step 3: *The function $u \in C^2_{loc}(\mathbb{R}^n)$ and*

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x) \right) d\xi - f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi)$$

where $B_R(x)$ is any ball such that $spt(f) \subset B_R(x)$. Here $\nu_j(\xi)$ is the j :th component of the exterior normal of $B_R(x)$ at the point $\xi \in \partial B_R(x)$ and $A(\xi)$ is the area measure with respect to ξ.

[4] Here, as is very common in PDE theory, we do not really distinguish between constants that only depend on the dimension. We will often denote them by C - but C will often mean different things within the same experssion. We may for instance write $2C \leq C$. By this we mean that for any constant C_0 there is another constant C_1 such that $2C_0 \leq C_1$. But we usually don't the indicate that we intend different constants C_0 and C_1 with an index.

Proof of step 3: Before we prove step 3, let us try to explain the idea. We will use the same method of proof as in step 2. However, since $\left|\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\right|$ grows like $\frac{1}{|x-\xi|^n}$ as $|x-\xi| \to 0$ which is not integrable in $\mathbb{R}^n$. We can not say that

$$\int_{R^n} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} f(\xi) d\xi \tag{2.10}$$

exists.

Since $f(\xi) \in C_c^\alpha(\mathbb{R}^n)$ we know that $|f(\xi)-f(x)| \leq C|x-\xi|^\alpha$ for some constant C and $\alpha > 0$. This implies that

$$\left|\frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\left(f(\xi)-f(x)\right)\right| \leq C\frac{1}{|x-\xi|^n}|x-\xi|^\alpha \leq \frac{C}{|x-\xi|^{n-\alpha}}$$

which is integrable close to the point $x = \xi$. We may thus integrate

$$\int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\left(f(\xi)-f(x)\right) d\xi \tag{2.11}$$

for any ball $B_R(x)$. The difference between what we want to integrate (2.10) and what we can integrate (2.11) is the term

$$-\int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} f(x) d\xi = -\int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial \xi_i \partial \xi_j} f(x) d\xi =$$

$$= \left\{ \begin{array}{l} \text{a very formal} \\ \text{integration by parts} \end{array} \right\} =$$

$$= \int_{B_R(x)} \frac{\partial N(x-\xi)}{\partial \xi_i} \frac{\partial f(x)}{\partial \xi_j} d\xi - \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial \xi_i} \nu_j(\xi) f(x) d\xi.$$

So at least formally

$$\int_{R^n} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} f(\xi) d\xi = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j}\left(f(\xi)-f(x)\right) d\xi - \tag{2.12}$$

$$-f(x)\int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi)$$

where all the terms on the right hand side are well defined. Of course, equation (2.12) is utter non-sense since we aren't really sure how to define the left hand side. We therefore use the right hand side, which is defined, in the expression in the statement of step 3. However, we need to be very careful when we prove step 3 to make sure that all our integrals are well defined.

As we already remarked we will use the same method as in step 2 and consider approximate functions u_ϵ and prove that the approximate functions converge uniformly in C^2 to u.

Let us start with the real proof. Following step 2 we define the function

$$w_{ij}(x) = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) d\xi - \tag{2.13}$$

$$-f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi).$$

Since the integrand in the first integrand satisfy the estimate

$$\left| \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} \left(f(\xi) - f(x)\right) \right| \leq \frac{C}{|x-\xi|^{n-\alpha}} \tag{2.14}$$

it follows that the first integrand is absolutely integrable on $B_R(x)$ for every $R > 0$ and the first integrand is therefore well defined. The second integral in (2.13) is also well defined since we integrate a continuous function over a compact set. It follows that w_{ij} is well defined.

We also define

$$v_\epsilon(x) = \int_{B_R(x)} \eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi. \tag{2.15}$$

Clearly $v_\epsilon(x) \to \frac{\partial u(x)}{\partial x_i}$ uniformly since

$$\left| v_\epsilon(x) - \frac{\partial u(x)}{\partial x_i} \right| = \left| \int_{B_R} (\eta_\epsilon(|x-\xi|) - 1) \frac{\partial N(x-\xi)}{\partial x_i} f(\xi) d\xi \right| \leq$$

$$\leq \sup_{\xi \in B_{2\epsilon}(x)} |f(\xi)| \int_{B_{2\epsilon}(x)} \left| \frac{\partial N(x-\xi)}{\partial x_i} \right| d\xi \leq C\epsilon \sup_{\xi \in \mathbb{R}^n} |f(\xi)|,$$

where we used that $\eta_\epsilon(|x-\xi|) = 1$ for $|x-\xi| \geq 2\epsilon$.

Since the integrand in (2.15) is C^1 in x we may differentiate differentiate under the integral, Theorem 3 in the appendix, and deduce that

$$\frac{\partial v_\epsilon(x)}{\partial x_j} = \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) f(\xi) d\xi$$

which is continuous, since the integrand is and the set of integration is compact. As in step 2 we want to show that $\frac{\partial v_\epsilon(x)}{\partial x_j}$ converges uniformly as $\epsilon \to 0$.

To prove that we write $\frac{\partial v_\epsilon(x)}{\partial x_j}$ in a form similar to the form of w_{ij}:

$$\frac{\partial v_\epsilon(x)}{\partial x_j} = \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) f(\xi) d\xi -$$

$$-f(x) \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) d\xi +$$

$$+f(x) \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) d\xi =$$

$$= \left\{ \begin{array}{l} \text{integration} \\ \text{by parts} \end{array} \right\} =$$

$$= \int_{B_R(x)} \frac{\partial}{\partial x_j} \left(\eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \right) (f(\xi) - f(x))\, d\xi -$$

$$-f(x) \int_{\partial B_R(x)} \eta_\epsilon(|x-\xi|) \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi) =$$

$$= \int_{B_R(x)} \frac{\partial^2 N(x-\xi) \eta_\epsilon(|x-\xi|)}{\partial x_i \partial x_j} (f(\xi) - f(x))\, d\xi -$$

$$-f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi),$$

where we used that $\eta_\epsilon(|x-\xi|) = 1$ on $\partial B_R(x)$ if $\epsilon < R$ (which we may assume) in the last equality. Notice that now we have no problem to integrate by parts since we have "cut out" the singularity by multiplying by η_ϵ.

To prove that $\frac{\partial v_\epsilon}{\partial x_j}$ converges uniformly to w_{ij} we calculate

$$\left| \frac{\partial v_\epsilon(x)}{\partial x_j} - w_{ij}(x) \right| =$$

$$= \Bigg| \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_j \partial x_i} (f(\xi) - f(x)) (\eta_\epsilon(|x-\xi| - 1))\, d\xi +$$

$$+ \int_{B_R(x)} \frac{\partial \eta_\epsilon(|x-\xi|)}{\partial x_j} \frac{\partial N(x-\xi)}{\partial x_i} (f(\xi) - f(x))\, d\xi \Bigg| \leq$$

$$\leq \int_{B_{2\epsilon}(x)} \left| \frac{\partial^2 N(x-\xi)}{\partial x_j \partial x_i} (f(\xi) - f(x)) \right| d\xi +$$

$$+ \frac{C}{\epsilon} \int_{B_{2\epsilon}(x) \setminus B_\epsilon(x)} \left| \frac{\partial N(x-\xi)}{\partial x_i} (f(\xi) - f(x)) \right| d\xi \leq$$

$$\leq C \int_{B_{2\epsilon}(x)} \frac{1}{|x-\xi|^{n-\alpha}} d\xi + \frac{C}{\epsilon} \int_{B_{2\epsilon}(x) \setminus B_\epsilon(x)} \frac{1}{|x-\xi|^{n-1-\alpha}} d\xi \leq$$

$$\leq C\epsilon^\alpha.$$

Thus $\frac{\partial v_\epsilon}{\partial x_j} \to w_{ij}$ uniformly. Since $v_\epsilon \to \frac{\partial u}{\partial x_i}$ uniformly we may use Theorem 2 conclude that

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i \partial x_j} (f(\xi) - f(x))\, d\xi - f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_j(\xi) dA(\xi)$$

which finishes the proof of step 3.

Step 4. *The function $u(x)$ satisfies $\Delta u(x) = f(x)$.*

Proof of step 4: By step 3 we know that $u(x) \in C^2_{\text{loc}}(\mathbb{R}^n)$ and that

$$\frac{\partial^2 u(x)}{\partial x_i^2} = \int_{B_R(x)} \frac{\partial^2 N(x-\xi)}{\partial x_i^2} \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_R(x)} \frac{\partial N(x-\xi)}{\partial x_i} \nu_i(\xi) dA(\xi).$$

This implies in particular that

$$\Delta u(x) = \int_{B_R(x)} \sum_{i=1}^n \frac{\partial^2 N(x-\xi)}{\partial x_i^2} \left(f(\xi) - f(x)\right) d\xi -$$

$$-f(x) \int_{\partial B_R(x)} \sum_{i=1}^n \frac{\partial N(x-\xi)}{\partial x_i} \nu_i(\xi) dA(\xi) =$$

$$= \int_{B_R(x)} \Delta N(x-\xi) \left(f(\xi) - f(x)\right) d\xi - f(x) \int_{\partial B_R(x)} \nabla N(x-\xi) \cdot \nu(\xi) dA(\xi). \tag{2.16}$$

But $\Delta N(x-\xi) = 0$ at almost every point which implies that the first integral in (2.16) is zero. To calculate the second integral in (2.16) we notice that

$$\nu(\xi) = \frac{x-\xi}{|x-\xi|}$$

and

$$\nabla N(x-\xi) = \frac{1}{\omega_n} \frac{x-\xi}{|x-\xi|^n} = \frac{1}{\omega_n} \frac{\nu(\xi)}{R^{n-1}},$$

on $\partial B_R(x)$. Thus

$$\Delta u(x) = f(x) \int_{\partial B_R(x)} \nabla N(x-\xi) \cdot \nu(\xi) dA(\xi) =$$

$$= \frac{f(x)}{R^{n-1}\omega_n} \int_{\partial B_R(x)} |\nu(\xi)|^2 dA(\xi) = f(x)$$

since $|\nu(\xi)|^2 = 1$ and $\int_{\partial B_R(x)} dA(\xi) = \omega_n R^n$ by the definition of ω_n. This finishes the proof. $\square$

Remark: *The proof is rather long and difficult to overview. But the bulk of the proof consists in using the cut off function η_ϵ to make sure that the integrals involved are well defined. The real important step in the proof is in step 3 where we use the Hölder continuity of $f(x)$ to assure that the second derivatives of $u(x)$ are well defined. It is in the very last equation of the proof where we see why we choose the rather strange constant $\frac{1}{(n-2)\omega_n}$ in the definition of $N(x)$.*

2.2 Appendix: Some Integral Formulas and Facts from Analysis.

In this appendix we repeat some results form analysis.

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a domain and assume that u_ϵ is a family of continuous functions on D such that*

1. *$u_\epsilon \to u$ locally uniformly as $\epsilon \to 0$,*
2. *$\frac{\partial u_\epsilon}{\partial x_i}$ is locally continuous on D.*
3. *$\frac{\partial u_\epsilon}{\partial x_i} \to w_i$ locally uniformly as $\epsilon \to 0$.*

Then $\frac{\partial u}{\partial x_i}$ exists, is locally continuous and $\frac{\partial u}{\partial x_i} = w_i$.

Proof:

Step 1: *The function w_i is locally continuous.*

Proof of step 1: We will argue by contradiction and assume that w_i has a discontinuity point $x^0 \in D$. That means that there exists two sequences $x^j \to x^0$ and $y^j \to x^0$ such that

$$\lim_{j\to\infty} |w_i(x^j) - w_i(y^j)| = \delta > 0. \tag{2.17}$$

Since D is open there exists some $r > 0$ such that $\overline{B_r(x^0)} \subset D$. Also, since $\frac{\partial u_\epsilon}{\partial x_i} \to w_i$ locally uniformly there exists an $\epsilon_0 > 0$ such that

$$\left| \frac{\partial u_\epsilon(x)}{\partial x_i} - w_i(x) \right| < \frac{\delta}{4} \tag{2.18}$$

for all $x \in B_r(x^0)$ and $\epsilon < 2\epsilon_0$.

Using that $\frac{\partial u_{\epsilon_0}}{\partial x_i}$ is locally continuous on D and that $\overline{B_r(x^0)}$ is compact we may conclude that there exists a $\mu > 0$ (for simplicity of notation we may assume that $\mu < r$) such that

$$\left| \frac{\partial u_{\epsilon_0}(x)}{\partial x_i} - \frac{\partial u_{\epsilon_0}(y)}{\partial x_i} \right| < \frac{\delta}{4} \tag{2.19}$$

for all $x, y \in B_\mu(x^0)$.

In particular we may conclude that for any j large enough such that $x^j, y^j \in B_\mu(x^0)$

$$\left| w_i(x^j) - w_i(x^j) \right| =$$

$$= \left| w_i(x^j) - \frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} - \left(w_i(y^j) - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right) + \left(\frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right) \right| \le$$

$$\le \left| w_i(x^j) - \frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} \right| + \left| w_i(y^j) - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right| + \left| \frac{\partial u_{\epsilon_0}(x^j)}{\partial x_i} - \frac{\partial u_{\epsilon_0}(y^j)}{\partial x_i} \right| < \frac{3\delta}{4},$$

where we have used (2.18) and (2.19). This clearly contradicts (2.17) which finishes the proof of step 1.

Step 2: *Assume that $B_r(x) \subset D$. Then*

$$u(x + e_i h) = u(x) + \int_0^h w_i(x + s e_i) ds$$

for any $|h| < r$*. Here* e_i *is the* i *:th unit vector* $e_i = (0, 0, ..., 1, ..., 0)$ *where the* 1 *is in the* i *:th coordinate place.*

Proof of step 2: Since $u_\epsilon \to u$ locally uniformly it follows that

$$u(x + he_i) - u(x) = \lim_{\epsilon \to 0} (u_\epsilon(x + he_i) - u_\epsilon(x)) =$$

$$= \left\{ \begin{array}{l} \text{fundamental} \\ \text{Theorem} \\ \text{of calculus} \end{array} \right\} = \lim_{\epsilon \to 0} \int_0^h \frac{\partial u_\epsilon(x + se_i)}{\partial x_i} ds = \int_0^h w_i(x + se_i) ds,$$

where the last step follows form the uniform convergence $\frac{\partial u_\epsilon}{\partial x_i} \to w_i$. Step 2 follows.

Step 3: *The end of the proof.*

Form the fundamental Theorem of calculus and step 2 it follows that

$$\frac{\partial u(x)}{\partial x_i} = w_i(x)$$

which is continuous by step 1. □

Theorem 3. *Let* D_0 *and* D_1 *be domains and assume that* $f(x, \xi)$ *is locally continuous on* $D_0 \times D_1 = \{(x, \xi);\ x \in D_0 \text{ and } \xi \in D_1\}$*. Assume furthermore that* $\frac{\partial f(x,\xi)}{\partial x_i}$ *is locally continuous on* $D_0 \times D_1$*.*

Then for any compact set $K \subset D_1$

$$f_i(x, \xi) \equiv \frac{\partial}{\partial x_i} \int_K f(x, \xi) d\xi = \int_K \frac{\partial f(x, \xi)}{\partial x_i} d\xi \tag{2.20}$$

and f_i *is locally continuous on* D_0*.*

Proof: Since D_0 is open and $x \in D_0$ there exists a ball $\overline{B_\mu(x)} \subset D_0$. Notice that $\overline{B_\mu(x)} \times K \subset D_0 \times D_1$ is compact since $\overline{B_\mu(x)}$ and K are[5].

Since $\overline{B_\mu(x)} \times K \subset D_0 \times D_1$ is compact it follows that $\frac{f(x,\xi)}{\partial x_i}$ is uniformly continuous on $\overline{B_\mu(x)} \times K \subset D_0 \times D_1$.[6]

By definition

$$\frac{\partial}{\partial x_i} \int_K f(x, \xi) d\xi = \lim_{h \to 0} \int_K \frac{f(x + he_i, \xi) - f(x, \xi)}{h} d\xi. \tag{2.21}$$

Next, using the mean value property for the derivative we see that there exists a $\gamma(x, \xi)$ such that $\gamma(x, \xi) \in [0, h]$ and

$$f(x + he_i, \xi) - f(x, \xi) = \frac{\partial f(x + \gamma(x, \xi) e_i)}{\partial x_i} h.$$

[5]Remember that the product of two compact sets are compact.

[6]Remember that continuous functions are uniformly continuous of compact sets.

We may conclude that

$$\left|\int_K \frac{f(x+he_i,\xi)-f(x,\xi)}{h}d\xi - \int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi\right| =$$
$$= \left|\int_K \frac{\partial f(x+\gamma(x,\xi)e_i,\xi)}{\partial x_i}d\xi - \int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi\right| \leq$$
$$\leq \int_K \left|\frac{\partial f(x+\gamma(x,\xi)e_i,\xi)}{\partial x_i}d\xi - \frac{\partial f(x,\xi)}{\partial x_i}\right| d\xi.$$

But since $\frac{\partial f(x,\xi)}{\partial x_i}$ is uniformly continuous there exists an $h_\epsilon > 0$ for each $\epsilon > 0$ such that

$$\left|\frac{\partial f(x+se_i,\xi)}{\partial x_i}d\xi - \frac{\partial f(x,\xi)}{\partial x_i}\right| < \epsilon$$

for each $|s| < h_\epsilon$. Since $|\gamma(x,\xi)| < h$ it follows, for $|h| < h_0$, that

$$\int_K \left|\frac{\partial f(x+\gamma(x,\xi)e_i,\xi)}{\partial x_i}d\xi - \frac{\partial f(x,\xi)}{\partial x_i}\right| d\xi < |K|\epsilon,$$

where $|K|$ denotes the volume of the set K.[7]

In particular we may conclude that for each $\epsilon > 0$

$$\lim_{h\to 0}\left|\int_K \frac{f(x+he_i,\xi)-f(x,\xi)}{h}d\xi - \int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi\right| < |K|\epsilon.$$

It follows that (2.20).

To see that $f_i(x,t)$ is locally continuous in x we again notice that $\frac{\partial f(x,\xi)}{\partial x_i}$ is uniformly continuous on $\overline{B_\mu(x)} \times K$ which implies that for every $\epsilon > 0$ there exists a $h_\epsilon > 0$ such that

$$\left|\frac{\partial f(x,\xi)}{\partial x_i} - \frac{\partial f(x,\xi)}{\partial x_i}\right| =$$
$$= \left|\int_K \frac{\partial f(x,\xi)}{\partial x_i}d\xi - \int_K \frac{\partial f(y,\xi)}{\partial x_i}d\xi\right| \leq$$
$$\int_K \epsilon d\xi \leq |K|\epsilon$$

for every $y \in B_\mu(x)$ such that $|x-y| < h_\epsilon$. Continuity follows. □

Let us remind ourselves of the following results from calculus.

Theorem 4. [The Divergence Theorem.] *Let Ω be a C^1 domain (that is the boundary $\partial\Omega$ is locally the graph of a C^1 function) in $\mathbb{R}^n$ and $v = (v^1, v^2, ..., v^n) \in C^1(\Omega;\mathbb{R}^n)$. Then*

$$\int_\Omega div(v)dx = \int_{\partial\Omega} v(x)\cdot\nu(x)dA(x)$$

where $div(v) = \sum_{j-1}^n \frac{\partial v^j}{\partial x_j}$ is the divergence of v and $\nu(x)$ is the outward pointing unit normal of $\partial\Omega$ the point x.

[7]Rather the measure of K (in case you have studied measure theory)

We will not prove this theorem.

Corollary 1. [Integration by parts.] *Let Ω be a C^1 domain (that is the boundary $\partial\Omega$ is locally the graph of a C^1 function) in $\mathbb{R}^n$ and $v, w \in C^1(\Omega)$. Then*

$$\int_\Omega \frac{\partial v(x)}{\partial x_i} dx = -\int_\Omega v(x)\frac{\partial w(x)}{\partial x_i} dx + \int_{\partial\Omega} w(x)v(x)\nu_i(x)dA(x)$$

where $\nu_i(x)$ is the i :th component of the outward pointing unit normal of $\partial\Omega$ the point x.

Proof: If we apply the divergence theorem to the vector function $v(x)w(x)e_i$ we see that

$$\int_\Omega \mathrm{div}\,(v(x)w(x)e_i)\,dx = \int_{\partial\Omega} v(x)w(x)e_i \cdot \nu(x)dA(x).$$

The left hand side in the last expression is

$$\int_\Omega \frac{\partial v(x)}{\partial x_i} dx + \int_\Omega v(x)\frac{\partial w(x)}{\partial x_i} dx.$$

Putting these two expression together gives the Corollary. $\square$

Theorem 5. [Green's Formulas.] *Let Ω be a C^1 domain and $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ then*

1. $$\int_\Omega v(x)\Delta u(x)dx + \int_\Omega \nabla v(x)\cdot\nabla u(x)dx = \int_{\partial\Omega} v(x)\frac{\partial u(x)}{\partial\nu} dA(x)$$

   where ν is the outward pointing unit normal of Ω and $\frac{\partial u(x)}{\partial\nu} = \nu\cdot\nabla u(x)$ and $dA(x)$ is an area element of $\partial\Omega$.

2. $$\int_\Omega (v(x)\Delta u(x) - u(x)\Delta v(x))\,dx = \int_{\partial\Omega}\left(v(x)\frac{\partial u(x)}{\partial\nu} - u(x)\frac{\partial v(x)}{\partial\nu}\right) dA(x).$$

Proof: For the first Green identity we apply the divergence theorem to $v(x)\nabla u(x)$. The second identity follows from interchanging u and v in the first identity and subtract the result. $\square$

2.3 Appendix: An Excursion into the subject of Regularization.

In this appendix we remind ourselves of a fact from regularization theory. The goal of this section is to show that we may approximate any continuous function uniformly by a function in C^∞. This is an important tool in analysis to approximate irregular functions by infinitely differentiable functions. We start by introducing the standard mollifier.

Definition 5. *Let*

$$\phi(x) = \begin{cases} c_0 e^{-\frac{1}{1-|x|^2}} & \textit{for } |x| < 1 \\ 0 & \textit{for } |x| \geq 1, \end{cases}$$

where c_0 is chosen so that $\int_{\mathbb{R}^n} \phi(x)dx = 1$.

We will, for $\epsilon > 0$, call $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon)$ the standard mollifier.

Slightly abusing notation we will at times write $\phi_\epsilon(x) = \phi_\epsilon(|x|)$.

Before we state the main theorem for mollifiers we need to introduce some notation.

Definition 6. *We say that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$ (where $\mathbb{N}_0 = \{0, 1, 2, ...\}$) is a multiindex. We will say that the length of α is $|\alpha| = \sum_{j=1}^n \alpha_j$. And for $u(x) \in C^k(\Omega)$ and $|\alpha| = l \leq k$ we will write*

$$\frac{\partial^{|\alpha|}u(x)}{\partial x^\alpha} \equiv \frac{\partial^l u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

Note that a multiindex is just a shorthand way of writing derivatives.

The standard mollifier is important because of the following Lemma.

Lemma 3. *Let $\epsilon > 0$ and $\phi_\epsilon(x)$ be the standard mollifier then*

1. *spt$(\phi_\epsilon) = \overline{B_\epsilon(0)}$ where spt$(\phi_\epsilon) = \overline{\{x \in \mathbb{R}^n;\ \phi_\epsilon(x) \neq 0\}}$ is the support of ϕ_ϵ,*
2. *$\phi_\epsilon \in C^\infty(\mathbb{R}^n)$,*
3. *$\int_{\mathbb{R}^n} \phi_\epsilon(x)dx = 1$,*
4. *if $u \in C(\Omega)$ (or if u is locally integrable) and we define*

$$u_\epsilon(x) = \int_\Omega u(y)\phi_\epsilon(x-y)dy$$

   for $x \in \Omega_\epsilon = \{x \in \Omega;\ dist(x, \partial\Omega) > \epsilon\}$ then $u_\epsilon \in C^\infty(\Omega_\epsilon)$

5. *and if $u \in C(\Omega)$, where Ω is open, then $\lim_{\epsilon\to 0} u_\epsilon(x) \to u(x)$ uniformly on compact sets of Ω.*

Proof: We will prove each part individually.

Part 1: To show that the support of ϕ_ϵ is $\overline{B_\epsilon(0)}$ we notice that for $|x| \geq \epsilon$ we have

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon) = 0$$

since $|x/\epsilon| \geq 1$ and $\phi(x) = 0$ for $|x| \geq 1$. For $|x| < \epsilon$ we have $\phi_\epsilon(x) = \frac{c_0}{\epsilon^n}e^{-\frac{\epsilon^2}{\epsilon^2-|x|^2}} > 0$. That is $\phi_\epsilon > 0$ in $B_\epsilon(0)$ and $\phi_\epsilon = 0$ in $\mathbb{R}^n \setminus B_\epsilon(0)$. By definition the support of ϕ_ϵ is the closure of the set where $\phi_\epsilon \neq 0$.

Part 2: In order to see that $\phi_\epsilon \in C^\infty(\mathbb{R}^n)$ we notice that it is enough to show that $\phi(x) \in C^\infty(\mathbb{R}^n)$. In particular we have

$$\frac{\partial^{|\alpha|}\phi_\epsilon(x)}{\partial x^\alpha} = \frac{1}{\epsilon^{n+|\alpha|}} \frac{\partial^{|\alpha|}\phi(x/\epsilon)}{\partial x^\alpha},$$

so if $\phi \in C^\infty$ then $\phi_\epsilon \in C^\infty$.

We will show, by induction, that, for any multiindex α,

$$\frac{\partial^{|\alpha|}\phi(x)}{\partial x^\alpha} = \frac{p_\alpha(x)}{q_\alpha(x)}\phi(x), \tag{2.22}$$

where $p_\alpha(x)$ and $q_\alpha(x)$ are polynomials, $q_\alpha(x) > 0$ in $B_1(0)$. When $|\alpha| = 0$ the representation is obviously true with $p_0(x) = q_0(x) = 1$. If we assume that (2.22) is true for all multiindexes α of length $k-1$ then for any multiindex β of length k we have some $j \in \{0, 1, 2, ..., n\}$ and multiindex α of length $k-1$ such that in $B_1(0)$

$$\frac{\partial^{|\beta|}\phi(x)}{\partial x^\beta} = \frac{\partial}{\partial x_j}\frac{\partial^{|\alpha|}\phi(x)}{\partial x^\alpha} = \frac{\partial}{\partial x_j}\left(\frac{p_\alpha(x)}{q_\alpha(x)}\phi(x)\right) =$$

$$= \frac{q_\alpha(x)\frac{\partial p_\alpha(x)}{\partial x_j} - p_\alpha(x)\frac{\partial q(x)}{\partial x_j}}{q_\alpha^2(x)}\phi(x) + \frac{p_\alpha(x)}{q_\alpha(x)}\frac{\partial\phi(x)}{\partial x_j} =$$

$$= \left(\frac{q_\alpha(x)\frac{\partial p_\alpha(x)}{\partial x_j} - p_\alpha(x)\frac{\partial q(x)}{\partial x_j}}{q_\alpha^2(x)} + \frac{x_j}{(1-|x|^2)^2}\right)\phi(x),$$

we may define the quantity in the brackets on the right hand side as $\frac{p_\beta(x)}{q_\beta(x)}$. Since p_α, q_α and $(1-|x|^2)^2$ are all polynomials it follows that p_β and q_β are polynomials. Moreover we see, by a simple induction, that q_β is a power of $1-|x|^2$ so $q_\beta(x) > 0$ in $B_1(0)$.

For $x \in \mathbb{R}^n \setminus \overline{B_1(0)}$ it follows that

$$\frac{\partial^{|\alpha|}\phi(x)}{\partial x^\alpha} = 0$$

since $\phi(x) = 0$ for $x \in \mathbb{R}^n \setminus \overline{B_1(0)}$.

Finally we notice that since $p_\alpha(x)$ is a polynomial we have for every multiindex α a constant C_α such that $\sup_{B_1(0)} |p_\alpha(x)| \le C_\alpha$. And for each multiindex α there is some k such that $q_\alpha(x) = (1-|x|^2)^k$. We may therefore estimate

$$\lim_{|x|\to 1}\left|\frac{p_\alpha(x)}{q_\alpha(x)}\phi(x)\right| \le \lim_{t\to 1}\left|c_n\frac{C_\alpha}{(1-t^2)^k}e^{-\frac{1}{1-t^2}}\right| = 0$$

since $e^{-\frac{1}{1-t^2}} \to 0$ with exponential speed as $t \to 1$ whereas $\frac{C_\alpha}{(1-t^2)^k} \to \infty$ with polynomial speed.

We have therefore shown that $\phi(x)$ is continuously differentiable for any α.

Part 3: This follows by a change of variables

$$\int_{\mathbb{R}^n} \phi_\epsilon(x)dx = \int_{\mathbb{R}^n} \frac{1}{\epsilon^n}\phi\left(\frac{x}{\epsilon}\right)dx = \left\{ \begin{array}{l} \text{set } x = \epsilon y \\ \text{then } dx = \epsilon^n dy \end{array} \right\} = \int_{\mathbb{R}^n} \phi(y)dy = 1.$$

Part 4: Follows from Theorem 3.

Part 5: Let $K \subset \Omega$ be compact. Since $u \in C(\Omega)$ it follows that for any $x \in \Omega$ and $\delta > 0$ there exists a $\frac{1}{2}\inf(1, \mathrm{dist}(K, \partial\Omega)) > \kappa_\delta > 0$ such that

$$|u(x) - u(y)| < \delta$$

for all $x \in K$ and y such that $|x - y| < \kappa_\delta$. In particular if $\epsilon < \kappa_\delta$ then

$$\left|\int_\Omega \phi_\epsilon(x-y)u(y)dy - u(x)\right| = \left|\int_{B_\epsilon(x)} \phi_\epsilon(x-y)u(y)dy - u(x)\right| \leq$$

$$\left|\int_{B_\epsilon(x)} \phi_\epsilon(x-y)\big(u(y) - u(x)\big)dy\right| \leq \int_{B_\epsilon(x)} \phi_\epsilon(x-y)|u(y) - u(x)|dy <$$

$$< \delta \int_{B_\epsilon(x)} \phi_\epsilon(x-y)dy = \delta$$

where we used that $\phi_\epsilon(x-y) = 0$ in $\mathbb{R}^n \setminus B_\epsilon(x)$ in the first inequality, that $\int_{B_\epsilon(x)} \phi(x-y)dy = 1$ in the second and that $|u(y) - u(x)| < \delta$ in the forth and and that $\int_{B_\epsilon(x)} \phi(x-y)dy = 1$ again in the last equality. □

2.4 Exercises Chapter 3.

Exercise 1.

1. Show that all affine functions $u(x) = a + \mathbf{b} \cdot x$ are harmonic.

2. Let A be an $n \times n-$matrix and show that if $u(x) = \langle x, A\rangle \cdot x = \sum_{i,j=1}^n a_{ij}x_i x_j$ is harmonic if and only if $\mathrm{trace}(A) = 0$.

3. Find all harmonic third order polynomials in $\mathbb{R}^2$.

4. Let $u(z)$ be an analytic function in a domain $D \subset \mathbb{C}$. Define the function $v(x, y) = \mathcal{RE}(u(x+iy))$ (the real part of the complex valued u) and prove that $\Delta v(x, y) = 0$ in the set $\{(x, y);\ x + iy \in D\}$. In particular there are polynomial harmonic functions of any order in $\mathbb{R}^2$.

   Hint: *The Cauchy-Riemann equations.*

Exercise 2:

A: Let $K \subset \mathbb{R}^n$ be a compact set and $x \in K^\circ$ (the interior of K). Prove the following

1. $\int_K \frac{1}{|x-y|^q} dy$ converges if and only if $q < n$.
2. If $|f(y)| \leq \frac{C}{|x-y|^q}$ and $q < n$ then $\int_K f(y)dy$ is well defined.

B: Let $f(x)$ be a continuous function defined on $\mathbb{R}^n$. Prove that if there exists a constant C such that $|f(x)| \leq C|x|^{-p}$ and $p > n$ then the integral $\int_{\mathbb{R}^n} f(x)dx$ is well defined.

C: Assume that $f(x)$ is a function defined on $\mathbb{R}^n$ and that $f(x)$ is continuous on $\mathbb{R}^n \setminus \{x^0\}$. Assume furthermore that there exists constants C_p, C_q, $0 \leq q < n$ and $p > n$ such that $|f(x)| \leq C_p|x|^{-p}$ for all $x \notin B_1(x^0)$ and $|f(x)| \leq C_q|x - x^0|^{-q}$ for all $x \in B_1(x^0)$. Prove that $\int_{\mathbb{R}^n} f(x)dx$ is well defined.

D: Prove that (2.7), (2.14) and (2.4) are well defined under the weaker assumption that $f(x)$ is continuous on $\mathbb{R}^n$ and satisfies $|f(x)| \leq C|x|^{-2+\epsilon}$ for some $\epsilon > 0$.

Exercise 3: In the informal discussion leading up to Theorem 1 we indicated that we needed to assume that $f \in C^\alpha_{\text{loc}}$ in order to make sense of the second derivatives of $u(x)$ defined as in (2.4). It is always good in mathematics to make sure that our assumptions are necessary. In this exercise we will prove that the expression in Step 3 in the proof of Theorem 1 is not well defined under the assumption that $f(x)$ is continuous with compact support. We will also slightly weaken the assumption that $f \in C^\alpha$ in Theorem 1.

A: Define $f(x)$ in $B_{1/2}(0)$ according to

$$f(x) = \frac{x_1 x_2}{|x|^2 |\ln(|x|)|}.$$

Show that $f(x)$ is continuous.

B: Prove that we may extend $f(x)$ to a continuous function on $\mathbb{R}^2$ with support in $B_1(0)$.

HINT: *Can we find a function $g(x) \in C_c^\infty(B_1(0))$ such that $g(x) = 1$ in $B_{1/2}$? Then $f(x)g(x)$ would be a good candidate for a solution.*

C: Show that the right hand side in the expression

$$\frac{\partial^2 u(x)}{\partial x_1 \partial x_2} = \int_{B_2(x)} \frac{\partial^2 N(x-\xi)}{\partial x_1 \partial x_2} \left(f(\xi) - f(x)\right) d\xi -$$

$$-f(x) \int_{\partial B_2(x)} \frac{\partial N(x-\xi)}{\partial x_1} \nu_2(\xi) dA(\xi)$$

from step 3 in the proof of Theorem 1 is not well defined at $x = 0$ with the $f(x)$ defined in step **B**. Conclude that it is not enough to assume that $f(x)$ is continuous with compact support in order for our current proof of Theorem 1 to work.

D: In the theory of PDE one often uses Dini continuity. We say that a function $f(x)$ is Dini continuous if there exists a continuous function $\sigma \geq 0$ defined on $[0,1)$ such that $\sigma(0)=0$ and

$$\int_0^1 \frac{\sigma(t)}{t} dt < \infty$$

such that

$$|f(x)-f(y)| \leq \sigma(|x-y|) \quad \text{for all } x, y \text{ s.t. } |x-y|<1.$$

Prove that Theorem 1 holds under the weaker assumption that $f(x)$ is Dini continuous with compact support in $\mathbb{R}^n$.

Remark on Exercise 3: *Notice that we have only proved that the expression in Step 3 in not well defined for this particular f. One might ask if there is another way to define a solution so that $\Delta u = f$. We will se later in the course that that is not the case. With our particular function f the only possible solutions to the Laplace equation are not C^2. But before we reach the point where we can understand how to define solutions that are not C^2 we need to develop more understanding of the Laplace equation.*

Exercise 4: Very often in PDE books one proves the weaker statement that if $f \in C_c^2(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi$$

is a $C^2_{\text{loc}}(\mathbb{R}^n)$ function and $\Delta u(x) = f(x)$. It might be a good exercise to prove this statement directly using the following steps.

A: Show that

$$u(x) = \int_{\mathbb{R}^n} N(\xi) f(x-\xi) d\xi.$$

B: Prove that

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} N(\xi) \frac{\partial^2 f(x-\xi)}{\partial x_i \partial x_j} d\xi$$

and that $u \in C^2_{\text{loc}}$.

C: Notice that

$$\Delta u(x) = \int_{\mathbb{R}^n} N(\xi) \Delta_x f(x-\xi) d\xi = \int_{\mathbb{R}^n} N(\xi) \Delta_\xi f(x-\xi) d\xi =$$

$$= \int_{B_\delta(x)} N(\xi) \Delta_\xi f(x-\xi) d\xi + \int_{B_R(x) \setminus B_\delta(x)} N(\xi) \Delta_\xi f(x-\xi) d\xi, \qquad (2.23)$$

if R is chosen large enough. Then show that for any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that the first integral in (2.23) has absolute value less than ϵ and the second integral differs from $f(x)$ by at most ϵ. Conclude the theorem.

HINT: *Use Green's second formula when you estimate the second integral.*