Selected Topics in PDE part 2.

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Chapter 1

Green's Functions.

In this section we will begin to understand how to solve the Dirichlet problem in a domain Ω. The Dirichlet problem consists of finding a $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solving

$$\begin{aligned} \Delta u(x) = f(x) \quad & \text{in } \Omega \\ u(x) = g(x) \quad & \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

where $f \in C^\alpha(\Omega)$ and $g \in C(\partial\Omega)$ are given functions.

In this chapter we will investigate what the theory from the previous chapter would imply for solutions to (1.1). This will lead to the concept of a Green's function which is similar to the fundamental solution - but for a given domain. However, we can not, in general, calculate the Green's function. But for certain simple domains, with much symmetry, it is possible to explicitly calculate the Green's function. We will calculate the Green's function for the upper half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_n > 0\}$ and for a ball $B_r(0)$.

1.1 An informal motivation for the concept of Green's functions.

To motivate the introduction of Green's functions we have to look at the theory we have developed so far - that is all we have. In particular we have shown that we may define a solution to $\Delta u(x) = f(x)$ for any $f \in C^\alpha_c(\mathbb{R}^n)$ by

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) f(\xi) d\xi. \tag{1.2}$$

Using that $\Delta u(x) = f(x)$ we arrive at

$$u(x) = \int_{\mathbb{R}^n} N(x-\xi) \Delta_\xi u(\xi) d\xi.$$

Let us try to see if the same argument applies to solutions to (1.1). To that end we assume that $u \in C^{2,\alpha}(\Omega)$ and define

$$\tilde{u}(x) = \int_\Omega N(x-\xi)\Delta u(x)dx.$$

If $u(x)$ where defined according to (1.2) for some $f \in C_c^\alpha(\mathbb{R}^n)$ and $\Omega = \mathbb{R}^n$ then $\tilde{u}(x) = u(x)$. But we don't expect, in general, that $\tilde{u}(x) = u(x)$ for any $u \in C^{2,\alpha}(\Omega)$ for an arbitrary Ω. The point is that, using the explicit expression for $\tilde{u}(x)$, we can calculate the difference $\tilde{u}(x)$ and see what mathematics gives us back and hopefully it will give us some information about u.

Therefore we estimate

$$\int_\Omega N(x-\xi)\Delta u(x)dx = \int_{B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi + \int_{\Omega\setminus B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi = I_1 + I_2. \tag{1.3}$$

$$+\int_{\Omega\setminus B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi = I_1 + I_2.$$

We expect I_1 to be small, as a matter of fact:

$$|I_1| \leq \sup_{B_\epsilon(x)} |\Delta u| \int_{B_\epsilon(x)} \frac{C}{|x-\xi|^{n-2}} d\xi \leq C \sup_{B_\epsilon(x)} |\Delta u| \epsilon^2,$$

where we assumed, for definiteness, that $n \geq 3$. To calculate I_2 we use the second Green formula and conclude

$$I_2 = \int_{\Omega\setminus B_\epsilon(x)} N(x-\xi)\Delta u(\xi)d\xi =$$

$$= \int_{\Omega\setminus B_\epsilon(x)} u(\xi)\Delta_\xi N(x-\xi)d\xi + \int_{\partial\Omega} \left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)} \right) dA_{\partial\Omega}(\xi) +$$

$$+ \int_{\partial B_\epsilon(x)} \left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)} \right) dA_{\partial B_\epsilon(x)}(\xi) = I_3 + I_4 + I_5.$$

Notice that $I_3 = 0$ since $\Delta_\xi N(x-\xi) = 0$ in $\Omega \setminus B_\epsilon(x)$. Next we look at I_5 and estimate

$$I_5 = \int_{\partial B_\epsilon(x)} -\frac{1}{(n-2)\omega_n}\frac{1}{\epsilon^{n-2}}\frac{\partial u(\xi)}{\partial\nu}d\xi + \int_{\partial B_\epsilon(x)} -\frac{1}{\omega_n}\frac{1}{\epsilon^{n-1}}u(\xi)d\xi.$$

That is

$$|I_5 - u(x)| \leq \left| \int_{\partial B_\epsilon(x)} -\frac{1}{(n-2)\omega_n}\frac{1}{\epsilon^{n-2}}\frac{\partial u(\xi)}{\partial\nu}d\xi \right| + \tag{1.4}$$

$$+ \left| \int_{\partial B_\epsilon(x)} \frac{1}{\omega_n}\frac{1}{\epsilon^{n-1}}\left(u(\xi) - u(x)\right)d\xi \right| \leq \frac{\sup_\Omega |\nabla u|}{n-2}\epsilon + \sup_{\partial B_\epsilon(x)} |u(\xi) - u(x)|,$$

where the first term goes to zero as $\epsilon \to 0$ since $|\nabla u|$ is bounded (we assume that $u \in C^2(\Omega)$) and the second term goes to zero as $\epsilon \to 0$ since u is continuous. We can thus conclude that $I_5 \to u(x)$ as $\epsilon \to 0$.

To summarize, we have proven that

$$\int_\Omega N(x-\xi)\Delta u(\xi)d\xi = I_1 + I_2 = 0 + I_3 + I_4 + I_5 =$$

$$= 0 + 0 + \int_{\partial\Omega}\left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}\right)dA_{\partial\Omega}(\xi) + u(x),$$

as $\epsilon \to 0$.

Rearranging terms we arrive at

$$u(x) = \int_\Omega N(x-\xi)\Delta u(\xi)d\xi - \int_{\partial\Omega}\left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - u(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}\right)dA_{\partial\Omega}(\xi). \tag{1.5}$$

So if $u(x)$ was a solution to (1.1) then

$$u(x) = \int_\Omega N(x-\xi)f(\xi)d\xi - \int_{\partial\Omega}\left(N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)} - g(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}\right)dA_{\partial\Omega}(\xi), \tag{1.6}$$

this is a rather good expression but it has one serious flaw. We do not know what value $\frac{\partial u}{\partial\nu}$ has on $\partial\Omega$. If we new that we could calculate $u(x)$ by just using $f(x)$, $g(x)$ and $\frac{\partial u}{\partial\nu}$. But if $N(x-\xi)$ happened to be equal to zero on $\partial\Omega$ then the troublesome term

$$\int_{\partial\Omega} N(x-\xi)\frac{\partial u(\xi)}{\partial\nu(\xi)}dA_{\partial\Omega}(\xi)$$

in equation (1.5) would be equal to zero and (1.6) would become

$$u(x) = \int_\Omega N(x-\xi)f(\xi)d\xi + \int_{\partial\Omega} g(\xi)\frac{\partial N(x-\xi)}{\partial\nu(\xi)}dA_{\partial\Omega}(\xi) \tag{1.7}$$

and we would have a representation formula for $u(x)$ in terms of the given data $f(x)$ and $g(x)$. This motivates us to define a function $G(x,\xi)$ that has similar properties as $N(x-\xi)$ but so that $G(x,\xi) = 0$ for $\xi \in \partial\Omega$.

Definition 1. *Let Ω be a domain with C^1 boundary and assume that for every $x \in \Omega$ we have a solution $\phi^x(\xi) \in C^2(\Omega) \cap C(\overline{\Omega})$ to*

$$\begin{array}{ll} \Delta\phi^x(\xi) = 0 & \text{in } \Omega \\ \phi^x(\xi) = N(x-\xi) & \text{on } \partial\Omega. \end{array} \tag{1.8}$$

Then we say that

$$G(x,\xi) = N(x-\xi) - \phi^x(\xi)$$

is the Green's function in Ω.

Remark: Notice that until we can prove that $\phi^x(\xi)$ is a unique solution to (1.8) we have on right to say that $G(x,\xi)$ is *the* Green's function since there might be many Green's functions satisfying the definition. Later on we will prove that ϕ^x is indeed the unique solution and that we are therefore justified in calling G *the* Green's function.

Since the Green's function $G(x,\xi)$ has the same type of singularity as $N(x-\xi)$ at $x=\xi$ so there is some hope that the above calculations should work in the same way for $G(x,\xi)$ as they did for $N(x-\xi)$. Moreover, $G(x,\xi)=0$ for $\xi\in\partial\Omega$ which makes it reasonable to hope that the representation formula (1.7) would work for $G(x,\xi)$ in place of $N(x-\xi)$. However, we need to prove this.

1.2 The Green's function.

The main reason to introduce Green's functions is the following Theorem.

Theorem 1. *Assume that $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ and that $u(x)$ solves (1.1), where Ω is a bounded domain with C^1 boundary. Assume furthermore that $G(x,\xi)$ is the Green's function for Ω. Then*

$$u(x)=\int_\Omega G(x,\xi)f(\xi)d\xi+\int_{\partial\Omega}\left(g(\xi)\frac{\partial G(x,\xi)}{\partial\nu}\right)dA_{\partial\Omega}(\xi). \tag{1.9}$$

Proof: The proof is very similar to the calculations we did in the previous section. We will use use Green's second identity on $G(x,y)\equiv N(x-y)-\phi^x(y)$ and $u(y)$.

$$\int_\Omega G(x,\xi)\Delta u(\xi)d\xi=$$

$$=\int_{B_\epsilon(x)}G(x,\xi)\Delta u(\xi)d\xi+\int_{\Omega\setminus B_\epsilon(x)}\left(-u(y)\underbrace{\Delta_y G(x,y)}_{=0}+G(x,y)\Delta_y u(y)\right)dy=$$

$$=\left\{\begin{array}{l}\text{Green's Second}\\ \text{formula on the}\\ \text{second integal}\end{array}\right\}$$

$$\begin{aligned}=\int_{B_\epsilon(x)}G(x,\xi)\Delta u(\xi)d\xi+\int_{\partial\Omega}\left(u(\xi)-\frac{\partial G(x,\xi)}{\partial\nu}+\underbrace{G(x,\xi)}_{=0}\frac{\partial u(\xi)}{\partial\nu}\right)dA_{\partial\Omega}(\xi)+\\ +\int_{\partial B_\epsilon(x)}\left(-u(\xi)\left(\frac{\partial N(x-\xi)}{\partial\nu}+\frac{\partial\phi^x(\xi)}{\partial\nu}\right)-\left(N(x-\xi)-\phi^x(\xi)\right)\frac{\partial u(\xi)}{\partial\nu}\right)dA_{\partial B_\epsilon(x)}(\xi)=\end{aligned} \tag{1.10}$$

$$=\int_{B_\epsilon(x)}G(x,\xi)\Delta u(\xi)d\xi-\int_{\partial\Omega}u(\xi)\frac{\partial G(x,\xi)}{\partial\nu}dA_{\partial\Omega}(\xi)+$$

$$+\int_{\partial B_\epsilon(x)}\left(u(\xi)\frac{\partial\phi^x(\xi)}{\partial\nu}-\phi^x(\xi)\frac{\partial u(\xi)}{\partial\nu}\right)dA_{\partial B_\epsilon(x)}(\xi)+I_5,$$

where $I_5\to u(x)$ as $\epsilon\to 0$ is the same as the expression in the previous section, see formula (1.4).

It is easy to estimate the remaining terms, in particular

$$\left|\int_{B_\epsilon(x)} G(x,\xi)\Delta u(\xi)d\xi\right| \leq \sup_\Omega |\Delta u(x)| \left(\int_{B_\epsilon(x)} |N(x-\xi)|\, d\xi + \int_{B_\epsilon(x)} |\phi(\xi)|\, d\xi\right) \leq$$

$$\leq C \sup_\Omega |\Delta u(x)| \left(\epsilon^2 + \sup_\Omega |\phi|\epsilon^n\right) \to 0$$

as $\epsilon \to 0$. And similarly

$$\left|\int_{\partial B_\epsilon(x)} \left(u(\xi)\frac{\partial \phi^x(\xi)}{\partial \nu} - \phi^x(\xi)\frac{\partial u(\xi)}{\partial \nu}\right) dA_{\partial B_\epsilon(x)}(\xi)\right| \leq$$

$$\leq C\left(\sup_\Omega |u| \sup_\Omega |\nabla \phi^x| + \sup_\Omega |\nabla u| \sup_\Omega |\phi^x|\right)\epsilon^{n-1} \to 0.$$

Using these estimates together with (1.10) we may conclude, after sending $\epsilon \to 0$, that

$$u(x) = \int_\Omega G(x,\xi)f(\xi)d\xi + \int_{\partial\Omega}\left(g(\xi)\frac{\partial G(x,\xi)}{\partial\nu}\right) dA_{\partial\Omega}(\xi),$$

this finishes the proof of the Theorem. □

Remarks: 1. Notice that (1.9) makes perfectly good sense even if we do not know that we have a solution. A good guess (which is indeed true) is that if we define u according to (1.9) then u solves the Dirichlet problem (1.1). To actually prove this will require some extra work.

2) In some sense we hide the difficulties in this Theorem. In particular we assume that we can solve (1.8) in order to define the Green's function. But to solve the Dirichlet problem is exactly what we are aiming to do. So we assume that we have a solution to one Dirichlet problem, namely (1.8), in order to find a representation for the solution to another Dirichlet problem (1.1).

The Theorem is however useful since the Dirichlet problem (1.8) has $f = 0$ *and very special boundary data. So Theorem 1 states that if we can calculate a solution to the Dirichlet problem with simple boundary data (1.8) in* Ω *then we can find a representation for the solution to the Dirichlet problem in* Ω *with any boundary data* $g \in C(\partial\Omega)$.

Our next goal will be to actually solve the Dirichlet problem (1.8) in some simple domains Ω.

1.3 The Dirichlet Problem in $\mathbb{R}^n_+$.

As pointed out in the last section, if we know that we have a solution u to the Dirichlet problem in Ω, it is enough to solve the Dirichlet problem

$$\begin{array}{ll} \Delta\phi^x(\xi) = 0 & \text{in } \Omega \\ \phi^x(\xi) = -N(\xi - x) & \text{on } \partial\Omega. \end{array} \tag{1.11}$$

for every x in order to find a representation formula for u.

If Ω is very complicated it will be very hard to find a solution to (1.11). But if Ω has some simple symmetries it is indeed possible to explicitly write down the solutions to (1.11). In this section we will consider $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_n > 0\}$.

We need to find a $\phi^x(\xi)$ solving

$$\begin{array}{ll} \Delta\phi^x(\xi) = 0 & \text{in } \mathbb{R}^n_+ \\ \phi^x(\xi) = N(\xi - x) & \text{on } \partial\mathbb{R}^n_+ = \{\xi \in \mathbb{R}^n;\ \xi_n = 0\}. \end{array}$$

Notice that

$$\begin{array}{ll} \Delta N(\xi - x) = 0 & \text{in } \mathbb{R}^n_- = \{\xi \in \mathbb{R}^n;\ \xi_n < 0\} \\ N(\xi - x) = N(\xi - x) & \text{on } \partial\mathbb{R}^n_- = \partial\mathbb{R}^n_+ = \{\xi \in \mathbb{R}^n;\ \xi_n = 0\}. \end{array}$$

So $N(y\xi - x)$ is a solution, but in the wrong half space! This is however very easy to fix by a simple reflection. We define

$$\phi^x(\xi) = N(\xi - \tilde{x}) \tag{1.12}$$

where $\tilde{x} = (x_1, x_2, ..., x_{n-1}, -x_n)$. Then we have, for $\xi \in \partial\mathbb{R}^n_+$ that is $\xi_n = 0$,

$$\phi^x(\xi) = -\frac{1}{(n-2)}\frac{1}{|\xi - x|^{n-2}} = -\frac{1}{(n-2)}\frac{1}{(|\xi' - x'|^2 + |x_n|^2)^{\frac{n-2}{2}}} = N(\xi - x),$$

where we have used the notation $x' = (x_1, x_2, ..., x_{n-1}, 0)$ and $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$. We also assumed that $n \geq 3$ for simplicity. The calculations for $n = 2$ is very similar.

We have thus proved the following Lemma.

Lemma 1. *The Green's function in* $\mathbb{R}^n_+$ *is*

$$G(x, \xi) = N(\xi - x) - N(\xi - \tilde{x})$$

where $\tilde{x} = (x_1, x_2, ..., x_{n-1}, -x_n)$.

1.3.1 The Poisson Kernel in $\mathbb{R}^n_+$.

We know that if we have a solution to the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

then we can represent the solution by the formula

$$u(x) = \int_\Omega G(x, \xi)\Delta u(\xi)d\xi + \int_{\partial\Omega}\left(u(\xi)\frac{\partial G(x, \xi)}{\partial\nu}\right)dA_{\partial\Omega}(\xi),$$

if Ω is C^1 and bounded and the Green's function $G(x, \xi)$ exists and is $C^1(\overline{\Omega}) \cap C^2(\Omega)$.

If $f = 0$ this reduces to

$$u(x) = \int_{\partial\Omega} \left(u(\xi) \frac{\partial G(x,\xi)}{\partial \nu} \right) dA_{\partial\Omega}(\xi).$$

This representation formula indicates that $\frac{\partial G(x,\xi)}{\partial \nu}$ is of special importance. We make the following definition.

Definition 2. *Let Ω be a domain and $G(x,\xi)$ be the corresponding Green's function. Call assume that the normal derivative of $G(x,\xi)$ exists on $\partial\Omega$ and callit*

$$K(x,\xi) = \frac{\partial G(x,\xi)}{\partial \nu}$$

the Poisson Kernel for Ω and the representation formula

$$u(x) = \int_{\partial\Omega} \left(u(\xi) \frac{\partial G(x,\xi)}{\partial \nu} \right) dA_{\partial\Omega}(\xi)$$

we call the Poisson formula.

Since we know the Green's function in $\mathbb{R}^n_+$ we are able to calculate the Poisson kernel for $\mathbb{R}^n_+$.

Lemma 2. *The Poisson kernel for $\mathbb{R}^n_+$ is*

$$K(x,\xi) = \frac{x_n}{\omega_n} \frac{1}{|x-\xi|^n}.$$

Proof: The Poisson kernel is by definition

$$\frac{\partial G(x,\xi)}{\partial \nu}.$$

The normal of $\mathbb{R}^n_+$ is $-e_n$ so the Poisson kernel is

$$K(x,\xi) = -\frac{\partial G(x,\xi)}{\partial \xi_n}$$

and

$$G(x,\xi) = N(\xi - x) - N(\xi - \tilde{x}).$$

The Lemma follows by a simple calculation. □

Lemma 3. *For every $x_n > 0$ we have*

$$\int_{\mathbb{R}^{n-1}} K(x,\xi')d\xi' = 1,$$

where $y' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$ and $d\xi' = d\xi_1 d\xi_2 ... d\xi_{n-1}$.

Proof: In order to evaluate the integral of the Poisson kernel we will resort to a trick. I am not particularly fond of tricks in mathematics but in this case it will save us some calculation (which I do not like any more than tricks).

First we notice that by translating $\xi' \to z + x'$ we get

$$\int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x|^n} d\xi' = \int_{\mathbb{R}^n} \frac{x_n}{|z' - x_n e_n|^n} dz' = \int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x_n e_n|^n} d\xi'$$

so we might assume that $x' = (0, 0, ..., 0)$ without changing the value of the integral.

Secondly, we notice that if we change variables $\xi' \to sz'$ for any $s > 0$ then we get

$$\int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x|^n} dy' = \int_{\mathbb{R}^{n-1}} \frac{s^{n-1} x_n}{|sz' - x|^n} dz' \int_{\mathbb{R}^n} \frac{\frac{x_n}{s}}{\left|z' - \frac{x}{s}\right|^n} dz'$$

which implies that the value of the integral of the Poisson kernel is independent of $x_n > 0$. So there is a constant c_n such that

$$c_n = \int_{\mathbb{R}^n} \frac{x_n}{|\xi' - x|^n} d\xi',$$

where c_n is independent of x as long as $x_n > 0$.

The difficult part is to evaluate

$$\int_{\mathbb{R}^{n-1}} \frac{x_n}{|\xi' - x|^n} d\xi' = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|\xi'|^2 + |x_n|^2\right)^{n/2}} d\xi' = c_n. \tag{1.13}$$

In order to evaluate (1.13) we notice that

$$\int_0^\infty \frac{1}{1 + x_n^2} \left(\int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|\xi'|^2 + x_n^2\right)^{n/2}} d\xi' \right) dx_n \tag{1.14}$$

$$= c_n \int_0^\infty \frac{1}{1 + x_n^2} dx_n = c_n \left(\arctan(\infty) - \arctan(0)\right) = \frac{c_n \pi}{2}.$$

We may also evaluate (1.14)

$$\int_0^\infty \frac{1}{1 + x_n^2} \left(\int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|\xi'|^2 + x_n^2\right)^{n/2}} d\xi' \right) dx_n =$$

$$= \int_0^\infty \int_{\mathbb{R}^{n-1}_+} \frac{1}{1 + x_n^2} \frac{x_n}{\left(|\xi'|^2 + x_n^2\right)^{n/2}} d\xi' dx_n,$$

changing to polar coordinates $x_n = r\cos(\psi) r$, $r^2 = |\xi'|^2 + x_n^2$ we may continue the equality,

$$= \int_0^\infty \int_{\partial B_1^+(0)} \frac{r\cos(\psi)}{1 + r^2\cos^2(\psi)} \frac{1}{r^n} r^{n-1} dA_{\partial B_1^+(0)} dr =$$

$$= \int_{\partial B_1^+(0)} \int_0^\infty \frac{\cos(\psi)}{1 + r^2\cos^2(\psi)} dA_{\partial B_r(0)} dr = \int_{\partial B_1^+(0)} \frac{\pi}{2} dA_{\partial B_1^+} = \frac{\pi \Omega_n}{2}, \quad (1.15)$$

where we again used that $\int \frac{a}{1+a^2x^2} = \arctan(ax)$ and that $r\cos(\psi) \to \infty$ as $r \to \infty$.

Comparing (1.15) and (1.14) we see that $c_n = \omega_n$ this implies that

$$\int_{\mathbb{R}^{n-1}} K(x, \xi')dy' = \frac{c_n}{\omega_n} = 1.$$

□

The next Theorem establishes that we may indeed use the Poisson kernel to calculate a solution to the Dirichlet problem in $\mathbb{R}^n_+$.

Theorem 2. *Let $g \in C_c(\partial\mathbb{R}^n_+)$ and define*

$$u(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi')}{|x - \xi'|^n} d\xi' \quad (1.16)$$

where $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$ and $d\xi' = d\xi_1 d\xi_2 ... d\xi_{n-1}$. Then

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^n_+ \\ \lim_{x_n \to 0^+} u(x', x_n) = g(x') & \text{uniformly on compact sets } x' \in K \subset\subset \mathbb{R}^{n-1}. \end{array} \quad (1.17)$$

Remark: There is a slight abuse of notation in this Theorem. We use the notation $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}, 0)$ as a vector in $\mathbb{R}^n$ with zero as its last component. But we also use $\xi' = (\xi_1, \xi_2, ..., \xi_{n-1}) \in \mathbb{R}^{n-1}$ without the zero in the n :th component when we write $g(\xi')$. It should be clear from context which convention we are using.

Proof: We will do the proof into two steps.

Step 1: *The function u defined in (1.16) is well defined and is harmonic in $\mathbb{R}^n_+$.*

That the function is well defined and that we may differentiate under the integral sign is clear since the integrand has compact support in ξ' and is C^∞ in x for each $x \in \mathbb{R}^n_+$. To show that $\Delta u(x) = 0$ follows from a simple calculation.

Step 2: *Showing that $\lim_{x_n \to 0^+} u(x', x_n) = g(x')$ uniformly on compact sets.*

To show that $\lim_{x_n \to 0^+} u(x', x_n) = g(x')$ uniformly on compact sets we notice that since $g \in C(\mathbb{R}^{n-1})$ it follows that g is uniformly continuous on compact sets $K \subset\subset \mathbb{R}^{n-1}$. Fix a compact set $K \subset\subset \mathbb{R}^{n-1}$. For technical reasons that will become clear later we will define

$$K^1 = \cup_{x \in K} \overline{B_1(x)},$$

that is K^1 is the closed set containing all points that are at a distance at most one from K. Since K is compact it is closed and bounded which implies that K^1

is closed and bounded and thus compact. Therefore g is uniformly continuous on K^1.

In particular for every $\epsilon > 0$ we have a $\delta_{\epsilon/2} > 0$, which we may assume to satisfy $\delta_{\epsilon/2} < 1$, such that

$$|g(x') - g(\xi')| < \frac{\epsilon}{2} \tag{1.18}$$

for every $x' \in K$ such that $|x' - \xi'| < \delta_{\epsilon/2}$. Here we use that g is uniformly continuous on K^1, notice that if $x' \in K$ and $|x' - \xi'| < \delta_{\epsilon/2} < 1$ then $x', \xi' \in K^1$ and we may use the same $\delta_{\epsilon/2}$ for all $x' \in K$.

Using that $\int_{\mathbb{R}^{n-1}} K(x,\xi')dy = 1$ (Lemma 3) we see that for any $x' \in K$

$$|u(x',x_n) - g(x')| = \left| \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi') - g(x')}{|x-\xi'|^n} dy' \right| \leq$$

$$\leq \left| \int_{\mathbb{R}^{n-1} \setminus B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n} \frac{g(\xi') - g(x')}{|x-\xi'|^n} dy' \right| + \left| \int_{B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n} \frac{g(\xi') - g(x')}{|x-\xi'|^n} d\xi' \right| = \tag{1.19}$$

$$= I_{\epsilon/2} + J_{\epsilon/2}$$

It is easy to see that

$$J_{\epsilon/2} \leq \int_{B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n} \frac{|g(\xi') - g(x')|}{|x-\xi'|^n} d\xi' < \frac{\epsilon}{2} \int_{B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n} \frac{1}{|x-\xi'|^n} d\xi' < \frac{\epsilon}{2} \tag{1.20}$$

since $\int_{\mathbb{R}^{n-1}} K(x,\xi')dy = 1$, $|g(\xi') - g(x')| < \epsilon$ for all $\xi' \in B_{\delta_\epsilon}(x')$ and $K(x,\xi) > 0$. Noticew that the estimate (1.20) is independent of $x_n > 0$.

Also, if we chose R so large that $g(\xi') = 0$ outside of $B_R(x')$ for all $x' \in K$,

$$\left|I_{\epsilon/2}\right| = \left| \int_{\mathbb{R}^{n-1} \setminus B_{\delta_{\epsilon/2}}} \frac{x_n}{\omega_n} \frac{g(\xi') - g(x')}{|x-\xi'|^n} d\xi' \right| \leq$$

$$\leq \sup_{\xi' \in \mathbb{R}^{n-1},\, x' \in K} |g(\xi') - g(x')| \left| \int_{B_R(0) \setminus B_{\delta_{\epsilon/2}}(x')} \frac{x_n}{\omega_n} \frac{1}{|x-\xi'|^n} d\xi' \right| \leq \tag{1.21}$$

$$\leq \frac{2x_n \sup_{y \in \mathbb{R}^{n-1}} |g(y)|}{\omega_n} \left| \int_{B_R(0) \setminus B_{\delta_{\epsilon/2}}(x')} \frac{1}{\delta_{\epsilon/2}^n} d\xi' \right| \leq$$

$$\leq \left(\frac{2 \sup_{y \in \mathbb{R}^{n-1}} |g(y)| R^n}{n \delta_{\epsilon/2}^n} \right) x_n.$$

From (1.21) it follows that $\left|I_{\epsilon/2}\right| < \frac{\epsilon}{2}$ if $x_n < \tilde{\delta}_\epsilon$ where

$$\tilde{\delta}_\epsilon = \frac{n \delta_{\epsilon/2}^n}{4 \sup_{y \in \mathbb{R}^{n-1}} |g(y)| R^n}$$

only depend on g and the dimension.[1]

Putting (1.19), (1.20) and (1.21) together we have shown that for each compact set K and each $\epsilon > 0$ there is a $\tilde{\delta}_\epsilon$ such that for each $x' \in K$

$$|u(x', x_n) - g(x')| < \epsilon \quad \text{for all } x_n < \tilde{\delta}_\epsilon.$$

It follows that

$$\lim_{x_n \to 0^+} u(x', x_n) = g(x')$$

uniformly on compact sets. □

Corollary 1. *Theorem 2 is still true under the assumption that $g(\xi')$ is continuous and bounded.*

Sketch of the Proof: The proof of the corollary is almost the same as the proof of the Theorem. We only need to make sure that the integrals are convergent. We will show that the integral in (1.16) is convergent and leave the rest of the details to the reader.

Notice that $u(x)$ is still well defined if $g(\xi)$ is bounded and integrable. In particular, under those assumptions there exists a constant C, depending only on x_n, $\sup_{\mathbb{R}^{n-1}} |g(\xi')|$ and the dimension such that

$$\left| \frac{x_n}{\omega_n} \frac{g(\xi')}{|x - \xi'|^n} \right| \leq C \text{ in } B_1(x'),$$

and

$$\left| \frac{x_n}{\omega_n} \frac{g(\xi')}{|x - \xi'|^n} \right| \leq \frac{C}{|x' - \xi'|^n} \text{ in } \mathbb{R}^{n-1} \setminus B_1(x'),$$

it follows that the integral (1.16) is convergent under the assumptions in the Corollary, see exercise 2C in the previous set of lecture notes. □

Notice that we can now solve the Dirichlet problem in $\mathbb{R}^+_+$. In particular if $f \in C^\alpha_c(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^{n-1})$ then

$$u^1(x) = \int_{\mathbb{R}^n} N(x - \xi) f(\xi) d\xi$$

solves

$$\Delta u^1(x) = f(x) \text{ in } \mathbb{R}^n$$

and

$$u^2(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n}{\omega_n} \frac{g(\xi') - u^1(\xi')}{|x - \xi'|^n} dy'$$

solves

$$\begin{array}{ll} \Delta u^2(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u^2(x', 0) = g(x') - u^1(x', 0) & \text{on } \partial\mathbb{R}^{n-1}, \end{array}$$

[1] In particular, $\tilde{\delta}_\epsilon$ only depend on g through $\sup |g|$, the support of g and the continuity properties of g, that is on $\delta_{\epsilon/2}$.

where the second identity is interpreted in the sense of limits as in Theorem 2. In particular $u(x) = u^1(x) + u^2(x)$ will solve

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } \mathbb{R}^n_+ \\ u(x', 0) = g(x') & \text{on } \partial\mathbb{R}^{n-1}, \end{array}$$

that is we now know how to solve the Dirichlet problem in $\mathbb{R}^n_+$.

In Theorem 1 we made an assumption that Ω was bounded. Obviously $\mathbb{R}^n_+$ is not a bounded set so we can not apply Theorem 1 to $\mathbb{R}^n_+$. We can however, Theorem 2, construct a solution in $\mathbb{R}^n_+$.

The difference between Theorem 1 and Theorem 2 is that in Theorem 1 we assume that we have a solution and we find a representation formula for that solution. In Theorem 2 we do not assume that we have a solution - we prove that we have a solution.

However to state that the solution we construct in Theorem 2 is the same as the any given solution we would have to know that the solutions are unique. In bounded domains Ω it is indeed the case that solutions that are $C^2(\Omega) \cap C(\overline{\Omega})$ are unique, a fact that we will prove later. In unbounded domains, in particular in $\mathbb{R}^n_+$, the solutions are not uniquely determined by the boundary data. A simple example is that $u(x) = ax_n$ is a solution to

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u(x', 0) = 0 & \text{for every } x' \in \mathbb{R}^{n-1}. \end{array} \tag{1.22}$$

for any $a \in \mathbb{R}$. Clearly the Dirichlet problem (1.22) does not admit a unique solution.

Before we end our discussion about the Dirichlet problem in $\mathbb{R}^n_+$ we should mention something about the conclusion in Theorem 2 that $\lim_{x_n \to 0^+} u(x', x_n) = g(x')$ uniformly on compact sets. We start by an example.

Example: *There are infinitely many solutions to the following Dirichlet problem in $\mathbb{R}^n_+$:*

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^{n-1} \\ \lim_{x_n \to 0^+} u(x', x_n) = 0 & \text{for all } x' \in \mathbb{R}^{n-1} \lim_{x \to \infty, x_n > 0} u(x) = 0. \end{array} \tag{1.23}$$

Only one of these solutions, the trivial solution $u(x) = 0$, is bounded.

To see that there are infinitively many solutions we just notice that for each $a \in \mathbb{R}$ and $i = 1, 2, ..., n-1$ the function

$$u_a(x) = a\frac{x_i x_n}{|x|^{n+2}}$$

solves (1.23). In particular, u_a is just a constant multiple of the derivative of the Poisson kernel $\frac{\partial K(x,0)}{\partial x_i}$ which is harmonic in $\mathbb{R}^n_+$. That $\lim_{x \to \infty, x_n > 0} u(x) = 0$ follows easily from $|u_a(x)| \le \frac{a}{|x|^n} \to 0$ as $|x| \to \infty$. The proof that $u(x', x_n) \to 0$ as $x_n \to 0^+$ splits up into two cases. If $|x'| = \delta \neq 0$ then $|u_a(x)| \le \frac{a x_n}{|\delta|^{n+1}} \to 0$ as $x_n \to 0^+$ and if $|x'| = 0$ then $x_i = 0$ and thus $u_a(x) = 0$.

Clearly, if $a \neq 0$ then the limit $\lim_{x_n \to 0^+} u_a(x', x_n) = 0$ is not uniform since u_a is not bounded close to $x = 0$.

We have not developed enough theory, yet, to show that $u(x) = 0$ is the only bounded solution. But we will in the next few weeks.

This example shows that the solution defined in Theorem 2 is a particularly good solution. And that we have to be very careful when we investigate uniqueness properties of the solutions. In general, it is not enough that the boundary values are obtained in a limit sense for the solution to be unique. That is we need the solution to the Dirichlet problem in Ω is continuous up to the boundary for the solution to be unique.

1.4 The Green's function in B_r.

In this section we will repeat the analysis in the previous section for the domain $\Omega = B_r(0)$. We will leave some calculations for the reader.

For every $x \in B_r(0)$ we need to find a solution to

$$\begin{array}{ll} \Delta\phi^x(\xi) = 0 & \text{in } B_r(0) \\ \phi^x(\xi) = N(\xi - x) & \text{on } \partial B_r(0). \end{array}$$

As before we want to use the particular symmetry of the domain to explicitly calculate ϕ^x. To do that we need the following definition and Lemma.

Definition 3. *For any $x \in \mathbb{R}^n$ we denote*

$$x^* = \frac{r^2 x}{|x|^2} \qquad \text{if } |x| \neq 0.$$

We say that x^* is the reflection of x in $\partial B_r(0)$. And if u a function defined in Ω then we say that

$$u^*(x) = \frac{r^{n-2}}{|x|^{n-2}} u(x^*) \text{ for } x^* \in \Omega$$

is the Kelvin transform of u.

Lemma 4. *The * operator maps the ball $B_r(0)$ onto $\mathbb{R}^n \setminus \overline{B_r(0)}$:*

$$\{x^*;\ x \in B_r(0) \setminus \{0\}\} = \mathbb{R}^n \setminus \overline{B_r(0)}.$$

Furthermore $(x^*)^* = x$ for all $x \neq 0$.

Proof: Clearly if $|x| < r$ then

$$|x^*| = \left|\frac{r^2}{|x|}\right| > r.$$

Also

$$(x^*)^* = \left(\frac{r^2 x}{|x|^2}\right)^* = \frac{r^2 \frac{r^2 x}{|x|^2}}{r^4 \frac{|x|^2}{|x|^4}} = x.$$

□

Lemma 5. *Assume that u^* is harmonic on Ω. Then u^* is harmonic on*

$$\Omega^* = \{x;\ x^* \in \Omega\}.$$

Proof: The proof is a straightforward, although rather tedious, calculation

$$\Delta u^*(x) = \Delta\left(\frac{r^{n-2}}{|x|^{n-2}} u\left(\frac{r^2 x}{|x|^2}\right)\right) = 0.$$

□

Next we notice that for $x \in B_r(0)$ we have that $N(\xi - x)$ is harmonic in both x and ξ for $x \neq \xi$. We want to do the Kelvin transform of $N(y-x)$ with respect to $x \in B_r(0)$. That is

$$N^*(\xi - x) = \begin{cases} -\frac{r^{n-2}}{|x|^{n-2}} \frac{1}{(n-2)\omega_n} \frac{1}{|\xi - x^*|^{n-2}} = -\frac{r^{n-2}}{(n-2)\omega_n} \frac{1}{(|x||x^* - \xi|)^{n-2}} & \text{if } x \neq 0 \\ -\frac{1}{(n-2)\omega_n} & \text{if } x = 0, \end{cases}$$

when $n > 2$.

By Lemma 5 that $\Delta_\xi N^*(\xi - x) = 0$ whenever $\xi \neq x^*$. In particular, since $x \in B_r(0)$ so $x^* \notin B_r(0)$, we have that for every $\xi \in B_r(0)$ $x^* \neq xi$ and thus $\Delta_\xi N^*(\xi - x) = 0$.

We have that if $\xi \in \partial B_r(0)$ then

$$\begin{aligned} N^*(\xi - x) &= -\frac{1}{(n-2)\omega_n} \frac{r^{n-2}}{\left(|x|^2 |x^* - \xi|^2\right)^{(n-2)/2}} = \\ &= -\frac{1}{(n-2)\omega_n} \frac{r^{n-2}}{\left(\left|\frac{r^2 x}{|x|} - \xi|x|\right|^2\right)^{(n-2)/2}} = \\ &= -\frac{1}{(n-2)\omega_n} \frac{r^{n-2}}{\left(r^4 + 2r^2 x \cdot \xi + r^2 |x|^2\right)^{(n-2)/2}}, \end{aligned} \tag{1.24}$$

where we used that $|\xi|^2 = r^2$ since $\xi \in \partial B_r(0)$. Again using that $|\xi|^2 = r^2$ we deduce that $r^4 + 2r^2 x \cdot \xi + r^2|x|^2 = r^2|\xi - x|^2$. So we may write (1.24) as

$$N^*(\xi - x) = -\frac{1}{(n-2)\omega_n} \frac{1}{|\xi - x|^{n-2}} = N(\xi - x)$$

for all $\xi \in \partial B_r(0)$. In particular $N^*(\xi - x)$ satisfies the criteria of being the corrector $\phi^x(\xi)$ in the definition of the Green's function. All these calculations also works for $n = 2$. We have thus proved the following Lemma

Lemma 6. *The Green's function for $B_r(0)$ is*

$$G(x,\xi) = N(\xi - x) - N^*(\xi - x),$$

where

$$N^*(\xi - x) = \begin{cases} -\frac{1}{(n-2)\omega_n}\frac{r^{n-2}}{(|x||x^*-\xi|)^{n-2}} & \text{if } x \neq 0 \\ -\frac{1}{(n-2)\omega_n} & \text{if } x = 0, \end{cases}$$

when $n > 2$ and

$$N^*(\xi - x) = \begin{cases} -\frac{1}{2\pi}\left(\ln(|\xi - x^*|) - \ln(r^2)\right) = & \text{if } x \neq 0 \\ -\frac{1}{2\pi} & \text{if } x = 0, \end{cases}$$

when $n = 2$.

When we know the Green's function we can calculate the Poisson kernel for the ball.

Lemma 7. *The Poisson kernel for the ball $B_r(0)$ is*

$$K(x,\xi) = \frac{r^2 - |x|^2}{\omega_n r}\frac{1}{|x-\xi|^n}.$$

Proof: The proof is a simple calculation. We know, for $n > 2$, that

$$G(x,\xi) = N(\xi - x) - N^*(\xi - x) =$$

$$-\frac{1}{(n-2)\omega_n}\frac{1}{|x-\xi|^{n-2}} + \frac{r^{n-2}}{(n-2)\omega_n}\frac{1}{(|x|\,|\tilde{x}-\xi|)^{n-2}}.$$

The outward normal of $B_r(0)$ is $\nu = \frac{\xi}{|\xi|} = \frac{\xi}{r}$ which implies that for $|\xi| = r$ we have

$$K(x,\xi) = \frac{\xi}{r}\cdot\nabla_\xi G(x,\xi) = \sum_{j=1}^{n}\frac{\xi_j}{r}\left(\frac{1}{\omega_n}\frac{x_j - \xi_j}{|x-\xi|^n} - \frac{r^n}{\omega_n}\frac{x_j - |x|^2\xi_j}{(|x||\tilde{x}-\xi)|)^n}\right) =$$

$$= \frac{\xi}{r}\cdot\nabla_\xi G(x,\xi) = \sum_{j=1}^{n}\frac{\xi_j}{r}\left(\frac{1}{\omega_n}\frac{x_j - \xi_j}{|x-\xi|^n} - \frac{1}{\omega_n}\frac{x_j - |x|^2\xi_j}{(|x-\xi)|)^n}\right) \tag{1.25}$$

where we used the same argument as in and the lines following (1.24). Simplifying (1.25) we get

$$K(x,\xi) = \frac{r^2 - |x|^2}{\omega_n r}\frac{1}{|x-\xi|^n}.$$

□

Continuing as we did with the Poisson's equation in $\mathbb{R}^n_+$ we need the following Lemma.

Lemma 8. *Let $K(x,\xi)$ be the Poisson kernel for a $B_r(0)$ then for each $x \in B_r(0)$*

$$\int_{\partial B_r(0)} K(x,\xi) dA_{\partial B_r(0)}(\xi) = 1.$$

Proof: We know, Theorem 1, that if $\Delta u(x) = 0$ in $B_r(0)$ and $u(x) = g(x)$ on $\partial B_r(0)$ then

$$u(x) = \int_{\partial\Omega} \left(g(\xi) \frac{\partial G(x,\xi)}{\partial \nu} \right) dA_{\partial\Omega}(\xi) = \tag{1.26}$$

$$= \int_{\partial\Omega} g(\xi) K(x,\xi) dA_{\partial\Omega}(\xi),$$

where we also used the definition of $K(x,\xi)$ in the last equality.

Clearly $u(x) = 1$ is a C^2 solution to the Dirichlet problem with $g(x) = 1$. Inserting this in (1.26) gives the lemma. □

We are now ready to state the main Theorem of this section. The proof is parallel to the proof of Theorem (2) and left to the reader (see the exercises).

Theorem 3. *Let $g \in C(\partial B_r(0))$ and define*

$$u(x) = \int_{\partial B_r(0)} \frac{r^2 - |x|^2}{\omega_n r} \frac{1}{|x-y|^n} g(y) dA_{\partial B_r(0)}(y). \tag{1.27}$$

Then $u \in C^2(B_r 0)$ and

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } B_r(0) \\ \lim_{s\to 1^-} u(sx) = g(x) & \text{uniformly for every } x \in \partial B_r(0). \end{array} \tag{1.28}$$

Notice that the second line in (1.28) only says that u satisfies the boundary conditions in some sense.

1.5 Exercises:

Exercise 1: Let $K((\xi, x)$ be the Poisson kernel for the half space $\mathbb{R}^n_+$. Prove that $\Delta_x K(\xi,x) = 0$ for all $\xi \in \partial\mathbb{R}^n_+$. Conclude that an integral of the kind $\int_{\mathbb{R}^{n-1}} K(x,\xi')g(\xi')d\xi'$ is nothing more that a summation of harmonic functions $K(x,\cdot)$.

Exercise 2: Verify that $v(x) = x_n$ is a solution to

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u(x) = 0 & \text{on } \partial\mathbb{R}^n_+. \end{array}$$

Define $u(x)$ as in Theorem 2 and verify that $u(x) + av(x)$ is a solution to (1.17) for any $a \in \mathbb{R}$.

Draw the conclusion that the solution to (1.17) are not unique.

Exercise 3: We say that the functions $K_\epsilon(x,\xi)$ defined for every $\epsilon > 0$ and $x, \xi \in \mathbb{R}^n$ is a family of "Good Kernels" if

1. For every $\epsilon > 0$ and every $x \in \mathbb{R}^n$ the function $K_\epsilon(x,\xi)$ is integrable in ξ and
$$\int_{\mathbb{R}^n} K_\epsilon(x,\xi)d\xi = 1.$$
2. For every $\epsilon > 0$ and $x \in \mathbb{R}^n$ there is a constant C that is independent of ϵ such that
$$\int_{\mathbb{R}^n} |K_\epsilon(x,\xi)|\, d\xi \leq C.$$
3. For every, $\delta > 0$, and every $x \in \mathbb{R}^n$
$$\int_{\mathbb{R}^n \setminus B_\delta(x)} |F(x,\xi)|d\xi \to 0 \text{ as } \epsilon \to 0.$$

A: Prove the Poisson kernel $K(x,\xi')$ for $\mathbb{R}^n_+$ is a family of "Good Kernels" on $\mathbb{R}^{n-1}$ if we interpret $K(x,\xi') = \tilde{K}_{x_n}(x',\xi')$ with $x_n > 0$ playing the role of ϵ.

B: Prove that if $F_\epsilon(x,\xi)$ is a family of "Good Kernels", g is continuous and bounded on $\mathbb{R}^n$ and
$$u_\epsilon(x) = \int_{\mathbb{R}^n} K_\epsilon(x,\xi)g(\xi)d\xi.$$
Then $\lim_{\epsilon\to 0^+} u(x) = g(x)$.

HINT: *Look at step 2 in the proof of Theorem 2. As a matter of fact, my main reason for putting this exercise here is to force you to think through that proof.*

C: Can you formulate what it would mean for $K_\epsilon(x,\xi)$ to be a family of "Good Kernels" on the unit sphere $\partial B_1(0)$? Use this to prove Theorem 3.

Exercise 4: Assume that $u(x) \in C^2_{\text{loc}}(D)$ and that $\Delta u(x) = 0$ in D. Assume furthermore that $B_r(x^0) \subset D$ is any ball. Prove that
$$u(x^0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(x)dA_{\partial B_r(x^0)}(x).$$
This is known as *"The mean value property for harmonic functions"* since it states that if $u(x)$ is harmonic in a domain then $u(x^0)$ is equal to the mean value of u on the boundary of any ball with center at x^0.

HINT: *Can you use Theorem 3?*

Exercise 5: Show that if $u,v \in C^2(B_r(0)) \cap C^1(\overline{B_1(0)})$ both solve the Dirichlet problem
$$\begin{aligned} \Delta w(x) &= 0 && \text{in } B_1(0)\\ w(x) &= g(x) && \text{on } \partial B_1(0)\end{aligned}$$
then $u(x) = v(x)$ for all $x \in B_1(0)$. This shows that $C^2(B_r(0)) \cap C^1(\overline{B_1(0)})$ solutions to the Dirichlet problem in $B_1(0)$ are unique.

HINT: *Representation formulas are great! They tell us exactly what the solutions to the problem are.*

Exercise 6: Derive a representation formula for the solutions to the following Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } B_1^+(0) = \{x \in B_1(0);\ x_n > 0\} \\ u(x) = f(x) & \text{for } x \in (\partial B_1(0))^+ = \{x \in \partial B_1(0);\ x_n > 0\} \\ u(x) = g(x) & \text{for } x \in B_1(0) \cap \{x;\ x_n = 0\} \end{array}$$

Where $f(x)$ and $g(x)$ are given functions.

HINT: *First prove that if we define the function $\hat{f}$ on $\partial B_1(0)$ according to*

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } x \in (\partial B_1(0))^+ \\ -f(x', -x_n) & \text{for } x \in (\partial B_1(0))^- . \end{cases}$$

Then the solution, $v(x)$, to the Dirichlet problem in $B_1(0)$ that satisfies $u(x) = \hat{f}(x)$ on $\partial B_1(0)$ also satisfies $v(x', 0) = 0$. Use this together with Theorem 2 to get your representation formula - it doesn't have to be pretty.