Selected Topics in PDE part 3.

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Chapter 1

An interlude on the Path we will take - why go abstract?

So far we have been able to prove that for any $f \in C_c^\alpha(\mathbb{R}^n)$ we can solve $\Delta u(x) = f(x)$ in $\mathbb{R}^n$. Also, by using very similar ideas, we where able to solve the simple Dirichlet problems

$$\begin{array}{ll} \Delta u(x) = f(x) & \text{in } D \\ u(x) = g(x) & \text{on } \partial D \end{array} \tag{1.1}$$

for the simple domains $D = \mathbb{R}^n_+$ and $D = B_r(0)$. With a little bit of work we could also, Exercise 6 from last installment of the lecture notes, solve the Dirichlet problem for the simple domain $D = B_1^+(0)$.

However, in many applications we would like to solve a PDE on a very complicated domain. For instance, if we want to solve a problem involving turbulence we might want to solve a PDE describing the motion of air in a domain D that consists of $\mathbb{R}^3$ minus the shape of an airplane.

The method of solving a PDE by means of a Green's function involves finding the functions $\phi^x(y)$, that is solving the Dirichlet problem with boundary data $N(x-y)$, which we could only do for very simple domains. Even for fairly simple domains such as the one consisting of three overlapping circles in figure 1 we do not know how to calculate $\phi^x(y)$ - and thus not how to calculate the Green's function. We need to move into the abstract theory and give up any hope of finding explicit representation formulas.

Since the only way we know (at least from this course) to solve PDE in a domain is by means of a Green's function that is the only thing we can use in solving the Dirichlet problem for a more complicated domain, such as the domain consisting of three circles.

So let us try to hammer out an approach on how to solve (1.1) for $D =$ "*the union of three circles* $\mathbf{A}_1$, $\mathbf{A}_2$ *and* $\mathbf{A}_3$" and $f(x) = 0$. We may use the Green's

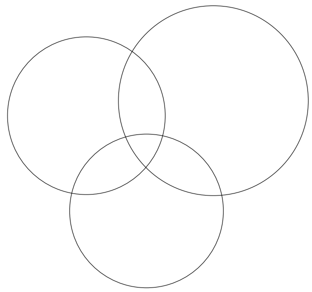

Figure 1.1: Domain consisting of Three Circles.

function to find a solution, lets call it $u^1(x)$, to the Dirichlet problem in $\mathbf{A}_1$ with boundary data $g(x)$ on the part of $\partial\mathbf{A}_1$ where $g(x)$ is defined and 0 on the other part of $\partial\mathbf{A}_1$. If we let v^1 be the function u^1 extended by 0 to the rest of D we have created a function v^1 that is harmonic in $D \setminus (\partial\mathbf{A} \cap D)$.

We may continue and use the Green's function to find a harmonic function, lets call it $u^2(x)$, in $\mathbf{A}_2$ with boundary data $g(x)$ on the part of $\partial\mathbf{A}_2$ where $g(x)$ is defined and $u^1(x)$ on the other part of $\partial\mathbf{A}_2 \cap D$ and boundary data equal to zero on the rest of $\partial\mathbf{A}_1 \cap D$. We may define

$$v^2(x) = \begin{cases} v^1(x) & \text{in } D \setminus \mathbf{A}_2 \\ u^2(x) & \text{in } \mathbf{A}_2. \end{cases}$$

Inductively we may create sequences $u^k(x)$ and $v^k(x)$ such that for any $l = 0, 1, 2, 3, ...$ and $j \in \{1, 2, 3\}$ the function u^{3l+j} solves the Dirichlet problem in $\mathbf{A}_j$:[1]

$$\begin{array}{ll} \Delta u^{3l+j}(x) = 0 & \text{in } \mathbf{A}_j \\ u^{3l+j}(x) = g(x) & \text{on } \partial\mathbf{A}_j \setminus \partial D \\ u^{3l+j}(x) = v^{3l+j-1}(x) & \text{on } \partial\mathbf{A}_j \cup D \end{array} \tag{1.2}$$

and

$$v^{3l+j} = \begin{cases} v^{3l+j-1}(x) & \text{in } D \setminus \mathbf{A}_j \\ u^{3l+j}(x) & \text{in } \mathbf{A}_j. \end{cases}$$

Notice that $v^{3l+j}(x)$ is then harmonic in $\mathbf{A}_j$ and that $v^{3l+j} = g(x)$ on ∂D for every $l \geq 1$ and $j = 1, 2, 3$. So if $\lim_{k\to\infty} v^k(x)$ converges to some function $u(x)$ then $u(x) = \lim_{l\to\infty} v^{3l+j}(x)$ in $\mathbf{A}_j$ for $j = 1, 2, 3$. That is $u(x)$ would be the limit of a sequence of harmonic functions in $\mathbf{A}_j$ for $j = 1, 2, 3$.

This leads to two questions:

1. Can we show that $\lim_{k\to\infty} v^k(x)$ exists?

2. Is harmonicity preserved under limits? That is, if a sequence of harmonic functions $v^{3l+j}(x) \to u(x)$ as $l \to \infty$ will it follow that $u(x)$ is harmonic?

[1] Since $\mathbf{A}_j$ is a ball we have no difficulties to solve this Dirichlet problem by means of a Green's function.

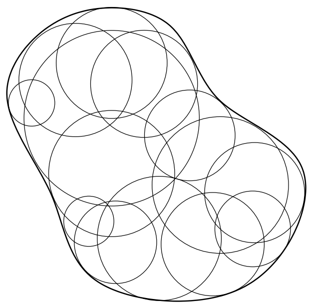

Figure 1.2: An open Domain covered by Balls.

If the answer to both these questions are affirmative then we know how to construct a solution to the Dirichlet problem, even though we don't have an explicit solution formula.

Before we try to make a brief outline of the theory that lies ahead. We will indicate that the three balls domain described above isn't as special as it looks. We could have used the same approach for a domain consisting of four, five on N balls. And if we can solve the Dirichlet problem for a domain that is the union of a finite number of balls then we should be able to use some analysis to to solve the Dirichlet problem for any domain that is the union of an infinite number of balls. Observe that any open domain is the union of all the balls in its interior.

So let us briefly indicate how we could attack the Dirichlet problem for a general domain D using the strategy used for the domain consisting of three balls. The natural way to approach this problem would be to we start with a function $v^0(x)$ defined on that domain with boundary data $g(x)$. Then we define a new function

$$v^k(x) = \begin{cases} u^k(x) & \text{in } B_r(x^0) \subset D \\ v^{k-1}(x) \text{ in } D \setminus B_r(x^0) & \end{cases} \tag{1.3}$$

for some ball $B_r(x^0) \subset D$ and u^k being a harmonic function, constructed by means of a Green's function, in $B_r(x^0)$ with boundary data $v^{k-1}(x)$. This way we can construct a sequence $v^k(x)$ that hopefully converge to a harmonic function.

The problem with this approach in a general domain is that the choice of the ball $B_r(x^0)$ was a quite arbitrary choice among infinitively many balls $B_r(x^0) \subset D$. With this arbitrariness we can not expect that $v^k(x)$ converges to a unique solution.[2] So we can not rely on an arbitrary choice of the ball $B_r(x^0)$.

Before we explain how to get rid of the problem with the arbitrary choice of the ball $B_r(x^0)$ in (1.3) let us say something brief about the convergence of

[2]If we choose the ball $B_{r/2^k}(x^0)$ in the construction of $v^k(x)$ then every function $v^k(x)$ would equal v^0 in $D \setminus B_r(0)$ so unless our starting function was harmonic in $D \setminus B_r(x^0)$ there is no chance that the limit would be harmonic in D.

$v^k(x)$. There are many ways to prove convergence of sequences of functions, but one of the simplest ways to assure convergence is to have a bounded and monotone sequence. So if, for every $k = 1, 2, 3, ...,$ $v^{k-1}(x)$ had the property that $v^k(x) \geq v^{k-1}(x)$ then the convergence of the sequence $v^k(x)$ would be easy.

If we could identify some functions $\mathcal{S}$ that has the property that if $v^{k-1}(x) \in \mathcal{S}$ then $v^k(x)$ defined as in (1.3) would satisfy $v^k(x) \geq v^{k-1}(x)$ and $v^k(x) \in \mathcal{S}$ for any ball $B_r(x^0) \subset D$ then it would follow that $v^0(x) \leq v^1(x) \leq ... \leq v^k(x) \leq ....$ So if $v^k(x)$ would be bounded then it would be pointwise convergent to some function $u(x)$.

But if every function in $\mathcal{S}$ is bounded then we could define

$$u(x) = \sup_{v \in \mathcal{S}} v(x) = \sup_{v \in \mathcal{S}} \tilde{v}(x) \tag{1.4}$$

where

$$\tilde{v}(x) = \begin{cases} w(x) & \text{in } B_r(x^0) \subset D \\ v(x) \text{ in } D \setminus B_r(x^0) & \end{cases} \tag{1.5}$$

where w is harmonic in $B_r(x^0)$ and equal to v on $\partial B_r(x^0)$. That (1.4) holds, for any ball $B_r(x^0) \subset D$, would follow from $v \leq \tilde{v} \in \mathcal{S}$ if $v \in \mathcal{S}$.

Notice that by considering the supremum over $\mathcal{S}$ we no longer make any choice of $B_r(x^0)$. The supremum assures that we take all balls $B_r(x^0) \subset D$ into consideration simultaneously.

So the strategy to show existence of solutions in a general domain would involve:

1. To identify a class $\mathcal{S}$ such that if $v \in \mathcal{S}$ then $v(x) \leq \tilde{v}(x)$ and $\tilde{v} \in \mathcal{S}$ where $\tilde{v}(x)$ is defined by (1.5). The class $\mathcal{S}$ will be all the sub-harmonic functions.

2. Since we will be taking a supremum over $\mathcal{S}$ we will have to understand the limit properties[3] of harmonic functions. In particular, we have to prove that if $\tilde{v}^k$ is harmonic in $B_r(x^0)$ and $\tilde{v}^k \to u$ in $B_r(x^0)$ will u be harmonic?

Step 1 of the strategy: We would like to define $\mathcal{S}$ so that $v \in \mathcal{S}$ implies $v \leq \tilde{v}$ for any ball $B_r(x^0)$. It is easy to find such a condition on $\mathcal{S}$. In particular, if $v(x) > \tilde{v}(x)$ for some point in $x \in B_r(x^0)$ then, since $v = \tilde{v}$ on $\partial B_r(x^0)$, the function $v(x) - \tilde{v}(x)$ has a strictly positive maximum at some point $\hat{x} \in B_r(x^0)$. At $\hat{x}$ we have, by first year calculus, that $\frac{\partial^2 v(\hat{x}) - \tilde{v}(\hat{x})}{\partial x_i^2} \leq 0$. Summing from $i = 1, ..., n$ we deduce that $0 \geq \Delta(v(\hat{x}) - \tilde{v}(\hat{x})) = \Delta v(\hat{x})$. This implies that if $\Delta v(x) \geq 0$ then there are no $x \in B_r(x^0)$ such that $v(x) > \tilde{v}(x)$. So we are tempted to define the class $\mathcal{S}$ to be the class of all functions $v(x)$ such that $\Delta v(x) \geq 0$.

But the other condition we impose on $\mathcal{S}$ is that $\tilde{v} \in \mathcal{S}$. For us to state that $\Delta \tilde{v} \geq 0$ we need to know that $\tilde{v} \in C^2(D)$. But even if $v(x) \in C^2(D)$ it will

[3] In taking the supremum we may consider a sequence $v^k \in \mathcal{S}$ such that $v^k(x) \to \sup_{v \in \mathcal{S}} v(x)$. So taking the supremum and a limit are more or less equivalent.

not follow that $\tilde{v}$ is C^2, or even differentiable, on $\partial B_r(x^0)$.[4] We will therefore have to find another way to define the class $\mathcal{S}$ without using derivatives. As a matter of fact we will find a way to define harmonic functions without referring to derivatives.

Step 2 of the strategy: Secondly we need to understand the convergence properties of harmonic functions. To that end we can not use monotonicity but we have to rely on compactness. We want to show that if $\tilde{v}^k(x)$ is a sequence of harmonic, and thus C^2, functions in $B_r(x^0)$ that converges to $u(x)$ then $u(x)$ is harmonic. It is enough to show that the second derivatives of $\tilde{v}^k$ converges.

In general, by the Arzela-Ascoli Theorem, it is enough for a bounded sequence of continuous functions to be equicontinuous in order for a subsequence to converge to a continuous function. Therefore we need to show that the second derivatives of $\tilde{v}^k$ are equicontinuous. This leads us to one of the more complicated aspects of the theory of partial differential equations: the regularity theory. Regularity theory involves proving that the solutions to partial differential equations are regular, that is have a certain number of derivatives defined - preferably also being able to say that the derivatives are bounded in terms of the given data.[5]

In this case we will prove that the third derivatives of $\tilde{v}^k$ are bounded uniformly which implies that the second derivatives are equicontinuous and thus convergent.

Once we have understood sub-harmonic functions and the convergence properties of harmonic functions we will be able to prove existence of solutions for general domains using the strategy outlined above - a method called **Perron's method**. That proof will be quite long and complicated.

When we consider the $\sup_{v\in\mathcal{S}} v(x)$ we do not address the issue of the boundary values. So we have to prove that our solution satisfy the boundary values[6] in a separate Theorem.

When we solve the Dirichlet problem in a general domain D we can not hope to find an explicit solution. Imagine how complicated such an explicit solution would have to be, it would have to be a function from the set of domains D, functions f and g and points $x \in D$ to the value $u(x)$ where $\Delta u(x) = f(x)$ in D and $u(x) = g(x)$ on ∂D. Just to find a reasonable way to define the space of all domains D would be rather complicated. We have to move into an abstract theory because the Dirichlet problem is very complicated and we have very few tools.

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[4] Take for instance $v(x) = |x|^2 - 1$ and $D = B_1(0)$, then $\Delta v(x) = 2n \geq 0$ but if we use $x^0 = 0$ and $r = \frac{1}{2}$ in the definition of $\tilde{v}(x)$ we will get

$$\tilde{v}(x) = \begin{cases} -\frac{3}{4} & \text{in } B_{1/2}(0) \\ |x|^2 - 1 & \text{in } B_1(0) \setminus B_{1/2}(0) \end{cases}$$

which is clearly not differentiable on $\partial B_{1/2}(0)$. However, as a distribution $\Delta\tilde{v}$ is well defined and $\Delta\tilde{v}(x) \geq 0$. But we will not discuss the theory of distributions in this course.

[5] By data I mean the domain D, the right hand side f and the boundary data g.

[6] Or does it? Under what assumptions on the domain?

Chapter 2

The Mean value Property.

If $u \in C^2(B_r(0)) \cap C(\overline{B_r(0)})$ is given by the Poisson integral then

$$u(0) = \int_{\partial B_r(0)} K(x,y)u(y)dA_{\partial B_r(0)}(y) = \int_{\partial B_r(0)} \frac{r^2}{\omega_n r}\frac{1}{|y|^n}u(y)dA_{\partial B_r(0)}(y) = \tag{2.1}$$

$$= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(0)} u(y)dA_{\partial B_r(0)}(y),$$

where we have used that $|y| = r$ on $\partial B_r(0)$. In particular, it follows that $u(0)$ equals the mean value of $u(y)$ on the boundary of $\partial B_r(0)$. This is a very powerful property and it is true for all harmonic functions.

Theorem 1. [The Mean Value Theorem.] *Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in the domain Ω and that $B_r(x^0) \subset \Omega$. Then*

1.
$$u(x^0) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x^0)} u(y)dA_{\partial B_r(x^0)}(y)$$

2. and
$$u(x^0) = \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u(y)dy.$$

Remark: *The calculation in (2.1) constitutes a proof of the first statement. We will however provide a different proof that directly uses that $\Delta u = 0$. First of all this proof is classical and should be included in the course. Secondly, we will have reason to investigate the mean value property for solutions to $\Delta u(x) \geq 0$ and for those solutions the proof given here will be easier to utilize.*

Proof: By translation invariance of the Laplace equation we may assume that $x^0 = 0$. That is the function $\tilde{u}(x) = u(x + x^0)$ is harmonic if u is. It is therefore enough to prove the theorem for $\tilde{u}(x)$ with $x^0 = 0$. By this, there is not loss of generality to assume that $x^0 = 0$ from the start.

Assuming that $x^0 = 0$ and making a change of variables in the mean value formula $rz = y$ we see that, defining the function $\Psi(r)$,

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} u(y) dA_{\partial B_r(x)}(y) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(rz) dA_{\partial B_1(0)}(z).$$

Taking the derivative with respect to r we see that

$$\Psi'(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} z \cdot \nabla u(rz) dA_{\partial B_1(0)}(z) = \tag{2.2}$$

$$= \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{\partial u(rz)}{\partial \nu} dA_{\partial B_1(0)}(z) = \frac{1}{\omega_n} \int_{B_1(0)} \Delta u(rz) dz = 0$$

since u is harmonic. We also used the divergence theorem in the second to last equality. In particular $\Psi(r)$ =constant= $\lim_{r\to 0} \Psi(r)$. Since $u \in C(\Omega)$ we have

$$\Psi(r) = \lim_{r\to 0} \Psi(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} \lim_{r\to 0} u(rz) dA_{\partial B_1(0)}(z) =$$

$$= \frac{1}{\omega_n} \int_{\partial B_1(0)} u(0) dA_{\partial B_1(0)}(z) = u(0).$$

This proves the first version of the mean value Theorem.

To prove the second part of the mean value Theorem we use polar coordinates.

$$\frac{n}{\omega_n r^n} \int_{B_r(x^0)} u(y) dy = \frac{1}{n\omega_n r^n} \int_0^r \left(\int_{\partial B_s(0)} u(y) dA_{\partial B_s(0)}(y) \right) ds.$$

Using the mean value Theorem on spheres we see that the integral in the brackets can be evaluated

$$\int_{\partial B_s(0)} u(y) dA_{\partial B_s(0)}(y) = \omega_n s^{n-1} u(0).$$

This implies that

$$\frac{n}{\omega_n r^n} \int_{B_r(x^0)} u(y) dy = \frac{n}{\omega_n r^n} \int_0^r \omega_n s^{n-1} u(0) ds = u(0).$$

This concludes the proof. □

As a matter of fact the mean value property characterises harmonic functions as the following Corollary shows.

Corollary 1. *Assume that $u \in C^2(\Omega)$ and that u satisfies the mean value property in Ω. That is, for every ball $\overline{B_r(x^0)} \subset \Omega$ the following equality holds*

$$u(x^0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y). \tag{2.3}$$

Then u is harmonic in Ω.

Proof: We will argue by contradiction and assume that there exist an $x^0 \in \Omega$ such that $\Delta u(x^0) \neq 0$ and derive a contradiction. For definiteness we assume that $\Delta u(x^0) = \delta > 0$.

Since $u \in C^2(\Omega)$ there exist an $r_\delta < \text{dist}(x^0, \partial\Omega)$ such that

$$\left|\Delta u(x^0) - \Delta u(y)\right| < \frac{\delta}{2}$$

for all y such that $|x^0 - y| < r_\delta$. In particular $\Delta u(y) > \delta/2$ in $B_{r_\delta}(x^0)$.

Define $\Psi(r)$ according to

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y).$$

That is, using the mean value property (2.3), $\Psi(r) = u(x^0)$. It follows, for $r < r_\delta$ and using the calculation in the proof of the mean value property,

$$0 = \Psi'(r) = \frac{1}{\omega_n} \int_{B_1(x^9)} \Delta u(rz) dz > \frac{1}{\omega_n} \int_{B_1(x^0)} r^2 \frac{\delta}{2} dz > 0.$$

This is a contradiction. It follows that $\Delta u(x) = 0$ in Ω. $\square$

Remark: *Something important, but subtle, happens in this section. We show that there is a property that is equivalent to $\Delta u(x) = 0$ for C^2 functions - the mean value property. But the mean value property is in itself independent of the function being C^2. So could we define any function, regardless of whether it is C^2 or not, to be harmonic if it satisfies the mean value property? Indeed we can, it even turns out as we will see later that the mean value property for a function u implies that $u \in C^2$. It is also through the mean value property that we will be able to define something like $\Delta v(x) \geq 0$ without assuming that $v \in C^2$ which will be a crucial step in defining the class $\mathcal{S}$ of sub-harmonic functions.*

Chapter 3

The maximum Principle.

From the mean value Theorem it follows that if $u(x)$ is harmonic in a domain Ω and if $u(x)$ equals its supremum at a point $x^0 \in \Omega$ then u must equal its supremum in every ball contained in Ω with center at x^0. It is a direct consequence that only constant harmonic functions achieve their maximum in their domain of harmonicity (if the domain is bounded and connected). The next Theorem proves this.

Theorem 2. [The Strong Maximum Principle.] *Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in the bounded domain Ω. Then*

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x).$$

Furthermore if Ω is also connected and there exist a point $x^0 \in \Omega$ such that $u(x^0) = \sup_{x\in\overline{\Omega}} u(x)$ then $u(x)$ is a constant.

Proof: Lets denote $M = \sup_{x\in\overline{\Omega}} u(x)$. Since $u \in C(\overline{\Omega})$ it follows that the set

$$\Omega_M = \{x \in \Omega;\ u(x) = M\}$$

is a relatively closed set in Ω. Now assume that there is a point $x^0 \in \Omega$ such that $u(x^0) = M$ then for any r such that $0 < r < \mathrm{dist}(x^0, \partial\Omega)$ we have by the mean value property

$$M = u(x^0) = \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u(y)dy \leq \frac{n}{\omega_n r^n}\int_{B_r(x^0)} M dy = M, \tag{3.1}$$

where the inequality is an equality (which it obviously is) if and only if $u(y) = M$ for all $y \in B_r(x^0)$. It follows that for any $x \in \Omega_M$ there is a ball $B_r(x) \subset \Omega_M$, that is Ω_M is an open set in Ω. Since Ω_M is both open and relatively closed in Ω it follows that Ω_M is either empty or a component of Ω.

If Ω_M is the empty set it follows that the supremum of u is attained on the boundary of Ω. If Ω_M is a component of Ω it still follows that $u(x) = M$ on the boundary of that component of Ω.

Finally, if Ω is connected and there exist an $x^0 \in \Omega$ such that $u(x^0) = M$ then it follows that $\emptyset \neq \Omega_M$ and therefore $\Omega_M = \Omega$, that is $u(x) = M$ in Ω. $\square$

Remark: If $u(x)$ is harmonic so is $-u(x)$. It is therefore an immediate consequence of this theorem that if Ω is bounded and u is harmonic in Ω then

$$\inf_{x\in\Omega} u(x) = \inf_{x\in\partial\Omega} u(x).$$

If Ω is also connected and if u attains its infimum at a point $x^0 \in \Omega$ then u is constant.

The maximum principle has many consequences, one of the most important consequences is that it implies that solutions to the Dirichlet problem are unique.

Theorem 3. *Let Ω be a bounded domain and suppose that $u^1, u^2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be two solutions to*

$$\begin{array}{ll} \Delta u(x) = f(x) & \textit{in } \Omega \\ u(x) = g(x) & \textit{on } \partial\Omega. \end{array}$$

Then $u^1 = u^2$ in Ω.

Proof: Define $v = u^1 - u^2$ then

$$\begin{array}{ll} \Delta v(x) = 0 & \text{in } \Omega \\ v(x) = 0 & \text{on } \partial\Omega. \end{array}$$

So by the maximum principle it follows that $\sup_{x\in\Omega} v(x) \leq \sup_{x\in\partial\Omega} v(x) = 0$. Applying the maximum principle on $-v(x)$ we see that

$$-\inf_{x\in\Omega} v(x) = \sup_{x\in\Omega}(-v(x)) \leq \sup_{x\in\partial\Omega}(-v(x)) = 0.$$

It follows that $0 \leq v(x) \leq 0$, that is $v(x) = 0$ or $u^1(x) = u^2(x)$ in Ω. $\square$

Chapter 4

Sub-harmonic functions.

If we assume that $\Delta u(x) \geq 0$ in Ω and define

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} u(y) dA_{\partial B_r(0)}(y)$$

for all r such that $\overline{B_r(0)} \subset \Omega$. Then we see, following the calculation in (2.2), that

$$\Psi'(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} z \cdot \nabla u(rz) dA_{\partial B_1(0)}(z) =$$

$$= \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{\partial u(rz)}{\partial \nu} dA_{\partial B_1(0)}(z) = \frac{1}{\omega_n} \int_{B_1(0)} \Delta u(rz) dz \geq 0.$$

In particular $\Psi(r)$ is a non-decreasing function and since u is continuous we have

$$u(0) = \lim_{r \to 0^+} \Psi(r) \leq \Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(0)} u(y) dA_{\partial B_r(0)}(y). \tag{4.1}$$

We will say that u satisfies the sub-mean value property if it satisfy (4.1).

Definition 1. *We say that $u \in C(\Omega)$ is sub-harmonic if it satisfies the sub-mean value property:*

$$u(x^0) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y)$$

for all $x^0 \in \Omega$ and $r \geq 0$ such that $\overline{B_r(x^0)} \subset \Omega$.

We say that $u \in C(\Omega)$ is super-harmonic if $-u(x)$ is sub-harmonic. Equivalently, $u \in C(\Omega)$ is super-harmonic if it satisfies the super-meanvalue property:

$$u(x^0) \geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y)$$

for all $x^0 \in \Omega$ and $r \geq 0$ such that $\overline{B_r(x^0)} \subset \Omega$.

Since u being sub-harmonic implies that $-u$ is super-harmonic it follows that every theorem for subharmonic functions have a corresponding theorem for super-harmonic functions.

Many of the theorems in the previous two sections have versions for sub and super-harmonic functions with very similar proofs. In particular we have the following, corresponding to Corollary 1.

Lemma 1. *Assume that $u \in C^2(\Omega)$ and that u is sub-harmonic in Ω. Then $\Delta u(x) \geq 0$ in Ω.*

Conversely if $u \in C^2(\Omega)$ and $\Delta u(x) \geq 0$ in Ω then $u(x)$ is subharmonic in Ω.

Proof: The proof is very similar to the proof of Corollary 1.

We will argue by contradiction and assume that there exist an $x^0 \in \Omega$ such that $\Delta u(x^0) < 0$ and derive a contradiction. For definiteness we assume that $\Delta u(x^0) = -\delta < 0$.

Since $u \in C^2(\Omega)$ there exist an $r_\delta < \text{dist}(x^0, \partial\Omega)$ such that

$$\left|\Delta u(x^0) - \Delta u(y)\right| < \frac{\delta}{2}$$

for all y such that $|x^0 - y| < r_\delta$. In particular $\Delta u(y) < -\delta/2$ in $B_{r_\delta}(x^0)$.

Define $\Psi(r)$ according to

$$\Psi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y).$$

It follows, for $r < r_\delta$, that

$$\Psi'(r) = \frac{1}{\omega_n} \int_{B_1(0)} \Delta u(rz + x^0) dz < -\frac{1}{\omega_n} \int_{B_1(0)} r^2 \frac{\delta}{2} dz < 0. \tag{4.2}$$

Since $u \in C(\Omega)$ it also follows that $\lim_{r \to 0^+} \Psi(r) = u(x^0)$. Using (4.2) we see that $\psi(r) < u(x^0)$ for $r \in (0, r_\delta)$. This contradicts the sub-mean value property.

The second part follows by the calculation in the beginning of this section. $\square$

Remark: Here we use a wonderful technique of mathematics. In principle we could define $u(x)$ to be sub-harmonic if $u \in C^2(\Omega)$ and $\Delta u(x) \geq 0$. Instead we use the sub-mean value property and are able to define sub-harmonicity for functions that are only in $C(\Omega)$ which is a much more flexible class of functions. In particular, which we will show and use later, if u and v are subharmonic so is $\max(u(x), v(x))$ (this would not be true if we demanded that subharmonic functions had to be in C^2).

The Lemma shows that we are not giving up anything in our definition based on the sub-mean value property. If a sub-harmonic function happens $u(x)$ to be in $C^2(\Omega)$ then it satisfies the equation $\Delta u(x) \geq 0$.

Since our proof of the maximum principle was based on the mean value property it is not surprising that the same result holds for sub-harmonic functions

Theorem 4. THE STRONG MAXIMUM PRINCIPLE FOR SUB-HARMONIC FUNCTIONS. *Suppose that $u \in C(\overline{\Omega})$ is sub-harmonic in the bounded domain Ω. Then*

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\partial\Omega} u(x).$$

Furthermore if Ω is also connected and there exist a point $x^0 \in \Omega$ such that $u(x^0) = \sup_{x\in\overline{\Omega}} u(x)$ then $u(x)$ is a constant.

Proof: The proof is exactly the same as for the strong maximum principle. The only difference is that the second equality in (3.1) should be an inequality and every time we referred to the mean value property we now have to refer to the sub-mean value property. □

Next we state a theorem that will be very important in our proof of existence of solutions for the Dirichlet problem in a general domain. We will state it for super-harmonic functions, but a similar statement is also true for sub-harmonic functions.

Theorem 5. *Let $u, v \in C(\Omega)$ be super-harmonic functions. Define*

$$w(x) = min(u(x), v(x)).$$

Then $w(x)$ is super-harmonic.

Proof: It is clear that $w(x)$ is continuous so we only need to show that w satisfies the super-mean value property. Notice that by definition $w(x) \leq u(x)$ and $w(x) \leq v(x)$, with one of the inequalities being an equality. We will fix an arbitrary point $x^0 \in \Omega$ and for definiteness assume that $w(x^0) = u(x^0)$. Then since $u(x)$ is super-harmonic we have, for any ball $\overline{B_r(x^0)} \subset \Omega$, that

$$w(x^0) = u(x^0) \geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} u(y) dA_{\partial B_r(x^0)}(y) \geq$$

$$\geq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x^0)} w(y) dA_{\partial B_r(x^0)}(y),$$

where we used that $w(x) \leq u(x)$ for all x. But this shows that w satisfies the super-mean value property. □

4.1 Sub and Super-Solutions.

It is possible to extend the concept of sub and super-harmonic functions to general solutions to the Dirichlet problem.

Definition 2. *We say that $w(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a sub-solution to*

$$\begin{aligned} \Delta u(x) &= f(x) \quad && \text{in } \Omega \\ u(x) &= g(x) && \text{on } \partial\Omega \end{aligned} \tag{4.3}$$

for $f \in C(\Omega)$ *and* $g \in C(\partial\Omega)$ *if*

$$\begin{aligned} \Delta u(x) &\geq f(x) \quad && \textit{in } \Omega \\ u(x) &= g(x) && \textit{on } \partial\Omega. \end{aligned}$$

Similarly we say that $w(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ *is a super-solution to (4.3) if*

$$\begin{aligned} \Delta u(x) &\leq f(x) \quad && \textit{in } \Omega \\ u(x) &= g(x) && \textit{on } \partial\Omega. \end{aligned}$$

Remark: Notice that if $u \in C^2(\Omega)$ *is sub-harmonic then by Lemma 1* u *is a sub-solution to* $\Delta u(x) = 0$.

When we defined sub-harmonicity, we only needed to assume that $u \in C(\Omega)$ *(see Definition 1) whereas we demand general sub-solutions to be* C^2. *It is noteworthy that there are other definitions of sub-solutions that require less stringent assumptions*[1] *- and most of the Theorems we show for sub-solutions would still be true. For simplicity we will assume that sub and super-solutions are* $C^2(\Omega)$ *for this course.*

The following Theorem will be important in our proof of existence of solutions to the Dirichlet problem.

Theorem 6. [THE COMPARISON PRINCIPLE.] *Let* Ω *be a bounded domain and suppose that* $u^1(x) \in C^2(\Omega) \cap C(\Omega)$ *be a sub-solution and* $u^2(x) \in C^2(\Omega) \cap C(\Omega)$ *be a super-solution to*

$$\begin{aligned} \Delta u(x) &= f(x) \quad && \textit{in } \Omega \\ u(x) &= g(x) && \textit{on } \partial\Omega. \end{aligned} \tag{4.4}$$

Then $u^1(x) \leq u^2(x)$ *in* Ω.

Proof: Notice that $w(x) = u^1(x) - u^2(x)$ solves

$$\begin{aligned} \Delta w(x) &\geq 0 \quad && \text{in } \Omega \\ w(x) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

That is $w(x)$ is sub-harmonic. By the maximum principle for sub-harmonic functions it follows that $w(x) \leq 0$ which implies that $u^1(x) \leq u^2(x)$. $\square$

Notice that if $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to (4.4) then u is both a sub and a super-solution so this Theorem directly implies Theorem 3.

[1] One that comes readily to mind would be that $u(x)$ is a sub-solution to (4.3) if $u(x) - v(x)$ is sub-harmonic for any solution $\Delta v(x) = f(x)$ in Ω. By this definition we would only need $u \in C(\Omega)$ to verify the definition.

Chapter 5

Interior Regularity of Harmonic Functions.

A major part of the study of partial differential equations (PDEs), a part that can be a little difficult to grasp, is the regularity theory. Regularity theory is the branch of PDE studies that investigates how regular a solution is, basically how many derivatives the solution has and if one can bound those derivatives.

We have already seen that the mean value property is equivalent to harmonicity for C^2 functions. But we only need the function to be continuous in order to define the mean value property. So if we would define a function to be harmonic if it is continuous and satisfies the mean value property could we still make sense of the equation $\Delta u(x) = 0$?

There are many different definitions of a function being a solution to a PDE; classical solutions (solutions that are continuously differentiable), weak solutions (defined by means of integration by parts), variational solutions (functions that minimise a certain energy), viscosity solutions (solutions defined by the comparison principle) etc. The only solutions that a priori have enough derivatives to satisfy the equation in the classical sense are classical solutions. These are the solutions that we have been working with so far, we assume that a solution to $\Delta u(x) = 0$ are in C^2 which makes it unproblematic to interpret whether a given function is a solution or not.

There are several reasons that regularity theory is so important for the study of partial differential equations. One reason is that it is often easier to prove the existence of a, say, weak solution than it would be to show the existence of a classical solution. But it is of obvious interest to know if the weak solution, once we have it, is in fact a classical solution. Other reasons for doing regularity theory is that one can use regularity theory to show properties of solutions, something that we will exemplify by by the Liouiville Theorem below. Regularity is also strongly related to existence theory, often it is only possible to show that a solution exists by approximating the PDE and by a limit procedure for which we need compactness. As a final motivation we should mention that only in very special cases are we able to write down a solution to a PDE. Instead we rely on numerical analysis to calculate approximate solutions with computers. In order to verify that we actually get a good approximation, and to say how good our approximation is, we need to know something about the regularity of the solutions.

In this section we will start to do some easy regularity theory. Our first theorem states that if $u(x)$ satisfies the mean value property in Ω then $u \in C^\infty(\Omega)$.

Theorem 7. *Let Ω be a domain. Suppose that $u \in C(\Omega)$ and satisfies the mean value property in Ω. Then $u \in C^\infty(\Omega)$ and $\Delta u(x) = 0$ in Ω.*

Proof: It is enough to show that $u \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$ where

$$\Omega_\epsilon = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \epsilon\}.$$

Fix an $\epsilon > 0$ and define u_ϵ by means of the standard mollifier

$$u_\epsilon(x) = \int_\Omega u(y)\phi_\epsilon(x-y)dy,$$

where $\phi_\epsilon(x)$ is a standard mollifier.It follows that $u_\epsilon \in C^\infty(\Omega_\epsilon)$.[1]

We will show that $u_\epsilon(x) = u(x)$ in Ω_ϵ. This is established in the following calculation

$$u_\epsilon(x) = \int_{B_\epsilon(x)} \phi_\epsilon(x-y)u(y)dy = \left\{ \begin{array}{l} \text{change to} \\ \text{polar coordinates} \end{array} \right\} =$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \phi\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B_s(x)} u(y)dA_{\partial B_s(x)}\right)ds =$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \phi\left(\frac{r}{\epsilon}\right)\omega_n s^{n-1}u(x)ds,$$

where we have used the mean value property in the last equality. Noticing that $\int_{\partial B_s(0)} dA_{\partial B_s(0)}(y) = \omega_n s^{n-1}$ we may continue the calculation

$$\frac{1}{\epsilon^n}\int_0^\epsilon \phi\left(\frac{r}{\epsilon}\right)\omega_n s^{n-1}u(x)ds =$$

$$= u(x)\int_0^r \int_{\partial B_s(0)} \frac{1}{\epsilon^n}\phi\left(\frac{r}{\epsilon}\right)dA_{\partial B_s(0)}(y)ds = u(x)\int_{B_\epsilon(0)} \phi_\epsilon(y)dy = u(x),$$

where we used part 3 of Lemma 2 in week 5's lecture notes in the final step. In particular it follows that $u(x) = u_\epsilon(x) \in C^\infty(\Omega_\epsilon)$ for every $x \in \Omega_\epsilon$. Since Ω is open it follows that $u \in C^\infty\left(\cup_{\epsilon>0}\Omega_\epsilon\right) = C^\infty(\Omega)$. □

[1]See the appendix in the first part of these lecture notes for the definition of standard mollifiers and the C^∞-proof.

Naively, one might think that the above result, that if u is harmonic then $u \in C^\infty$, is the best possible result in regularity theory, which is after all about showing that solutions to partial differential equations have derivatives.

There are two reasons why this result is not the best possible. The first reason is that one can show (but we will not) that harmonic functions are in fact analytic (can be expressed in a Taylor series). That is if $\Delta u(x) = 0$ in the domain Ω and $x^0 \in \Omega$ then there is a ball $B_r(x^0) \subset \Omega$ such that $u(x)$ equals it Taylor expansion

$$u(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha (x - x^0)^\alpha \text{ in } B_r(x^0),$$

where we have used the multiindex notation again; $\alpha = (\alpha_1, ..., \alpha_n)$ in a multi-index and $(x - x^0)^\alpha = (x_1 - x_1^0)^{\alpha_1}(x_2 - x_2^0)^{\alpha_2}...(x_n - x_n^0)^{\alpha_n}$. That analyticity is stronger than C^∞ is easy to see since the standard mollifier $\phi(x) \in C^\infty(\mathbb{R}^n)$ but the Taylor expansion at any point on $\partial B_1(0)$ must be identically zero since all derivatives vanish on $\partial B_1(0)$. Thus we can not express ϕ by means of a Taylor series.

The other reason why the above C^∞ result is not the best possible (in every respect) is more subtle. We already know that *any* continuous function f may be approximated as closely as we want by a C^∞ function, namely f_ϵ (see Lemma 2 in the Lectures from week 5). This means that a function being in C^∞ does not mean very much, in particular convergence and compactness properties of C^∞ functions are not good.

We need *estimates* in order to deduce desirable compactness properties of solutions. By estimates we mean some inequality where we control higher derivatives by means of lower derivatives. A typical, and important, estimate is presented in the following theorem where we show that derivatives on any order of a harmonic function can be controlled by the integral of the function (that is higher derivatives are controlled by the zeroth order derivatives). Before we state the theorem we need a definition.

Definition 3. *If u is a function whose absolute value is integrable in Ω we write*

$$\|u\|_{L^1(\Omega)} = \int_\Omega |u(x)| dx.$$

More generally, if $|u|^p$ is integrable in Ω we write

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)| dx \right)^{1/p}.$$

Remark: We can consider the space of all integrable functions v such that $\|v\|_{L^p(\Omega)} < \infty$, call this space $L^p(\Omega)$. If $1 \le p < \infty$ then $\| \cdot \|_{L^p(\Omega)}$ is a norm on $L^p(\Omega)$. The most important result in integration theory is that $L^p(\Omega)$ is a complete space with the norm $\| \cdot \|_{L^p(\Omega)}$ if we interpret the integral in the Lebesgue sense. These considerations are not important for us in this course.

Theorem 8. *Suppose that $u \in C^2(\Omega)$ is harmonic in Ω. Then for each ball $B_r(x^0) \subset \Omega$ and each multiindex α of length $|\alpha| = k \geq 1$ we have the following estimate*

$$\left|\frac{\partial^{|\alpha|} u(x^0)}{\partial x^\alpha}\right| \leq \frac{n(2^{n+1}nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_r(x^0))}.$$

Proof: Since u is harmonic in Ω we know that $u \in C^\infty(\Omega)$. Writing $\frac{\partial u}{\partial x_i} = u_i$ we see by changing the order of differentiation that

$$\Delta u_i(x) = \frac{\partial}{\partial x_i}\big(\Delta u(x)\big) = \frac{\partial}{\partial x_i}\big(0\big) = 0.$$

So u_i is harmonic and satisfies therefore the mean value property. In particular for $B_r(x^0) \subset \Omega$ we may apply the mean value property to the ball $B_{r/2}(x^0)$:

$$u_i(x^0) = \frac{n2^n}{\omega_n r^n} \int_{B_{r/2}(x^0)} u_i(y) dy.$$

Taking the absolute values and integrating by parts we get

$$|u_i(x^0)| = \left| \frac{n2^n}{\omega_n r^n} \int_{B_{r/2}(x^0)} \frac{\partial u(y)}{\partial x_i} dy \right| =$$

$$= \left| \frac{n2^n}{\omega_n r^n} \int_{\partial B_{r/2}(x^0)} u(y) \nu_i dA_{\partial B_{r/2(x^0)}}(y) \right| \leq \frac{2n}{r} \sup_{\partial B_{r/2}(x^0)} \big(|u|\big) \tag{5.1}$$

where we used the notation $\nu_i = \nu \cdot e_i$ where ν is the unit normal of $\partial B_{r/2}(x^0)$ and that

$$\left| \int_{\partial B_{r/2}(x^0)} u(y) dA_{\partial B_{r/2}(x^0)}(y) \right| \leq \sup_{\partial B_{r/2}(x^0)} |u| \int_{\partial B_{r/2}(x^0)} dA_{\partial B_{r/2}(x^0)}(y) =$$

$$\frac{1}{\omega (r/2)^{n-1}} \sup_{\partial B_{r/2}(x^0)} |u|$$

in the last inequality. To estimate $\sup_{y \in \partial B_{r/2}(x^0)} \big(|u(y)|\big)$ we use the mean value formula again. Since $B_r(x^0) \subset \Omega$ we have that $B_{r/2}(y) \subset \Omega$ for each $y \in \partial B_{r/2}(x^0)$. We can therefore apply the mean value formula to the ball $B_{r/2}(y) \subset \Omega$:

$$|u(y)| \leq \frac{n2^n}{\omega_n r^n} \left| \int_{B_{r/2}(y)} u(z) dz \right| \leq \frac{n2^n}{\omega_n r^n} \int_{B_{r/2}(y)} |u(z)| dz \leq \tag{5.2}$$

$$\leq \frac{n2^n}{\omega_n r^n} \int_{B_r(x^0)} |u(z)| dz = \frac{n2^n}{\omega_n r^n} \|u\|_{L^1(B_r(x^0))},$$

where we used that $\int_{B_{r/2}(y)} |u(z)| dz \leq \int_{B_r(x^0)} |u(z)| dz$ since $B_{r/2}(y) \subset B_r(x^0)$ and that the integrand is non negative.

Taking the supremum over $\partial B_{r/2}(x^0)$ on both sides in (5.2) and inserting this in (5.1) we get

$$|u_i(x^0)| \leq \frac{n^2 2^n}{\omega_n r^{n+1}} \|u\|_{L^1(B_r(x^0))}$$

which proves the theorem for $|\alpha| = 1$.

In order to prove the Theorem for general α we will use induction on the length of $|\alpha|$. We will assume that we have proved the theorem for all multiindexes α of length $k-1$. Now fix a multiindex β of length k and assume that $\frac{\partial^{|\beta|}}{\partial x^\beta} = \frac{\partial}{\partial x_i}\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ where α is a multiindex of length $k-1$. Writing $u_\gamma(x) = \frac{\partial^{|\gamma|}u(x)}{\partial x^\gamma}$ for any multiindex γ we have for any $B_r(x^0) \subset \Omega$ that

$$|u_\beta(x^0)| = \left| \frac{nk^n}{\omega_n r^n} \int_{B_{r/k}(x^0)} \frac{\partial u_\alpha(y)}{\partial x_i} dy \right| =$$

$$= \left| \frac{nk^n}{\omega_n r^n} \int_{\partial B_{r/k}(x^0)} u_\alpha(y)\nu_i dA_{\partial B_{r/k(x^0)}}(y) \right| \leq \frac{kn}{r} \sup_{\partial B_{r/k}(x^0)} \big(|u_\alpha|\big). \tag{5.3}$$

Using the induction hypothesis we see that, for $y \in \partial B_{r/k}(x^0)$

$$|u_\alpha(y)| \leq \frac{n\big(2^{n+1}n(k-1)\big)^{k-1}k^{n+k-1}}{\omega_n\big((k-1)r\big)^{n+k-1}} \left| \int_{B_{(k-1)r/k}(y)} u(z)dz \right| \leq$$

$$\leq \frac{n\big(2^{n+1}n\big)^{k-1}k^{n+k-1}}{\omega_n(k-1)^n r^{n+k-1}} \int_{B_{(k-1)r/k}(y)} |u(z)|dz \leq \tag{5.4}$$

$$\leq \frac{n\big(2^{n+1}n\big)^{k-1}k^{n+k-1}}{\omega_n(k-1)^n r^{n+k-1}} \|u\|_{L^1(B_r(x^0))},$$

Putting (5.3) and (5.4) together we see that

$$|u_\beta(x^0)| \leq \frac{n^2\big(2^{n+1}n\big)^{k-1}k^{n+k}}{\omega_n(k-1)^n r^{n+k}} \|u\|_{L^1(B_r(x^0))} =$$

$$= \left(\frac{k^n}{2^{n+1}(k-1)^n}\right)\left(\frac{n(2^{n+1}nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_r(x^0))}\right),$$

noticing that the first bracket to the right in the last equation is less than one gives the desired estimate. □

As a direct consequence of Theorem 8 we state the following theorem.

Theorem 9. [THE LIOUIVILLE THEOREM] *Suppose that $u \in C_0(\mathbb{R}^n)$ is harmonic. Then if $|u(x)| \leq C$ for every $x \in \mathbb{R}^n$ and for some constant C_0 (C_0 is independent of x) then $u(x)$ is constant in $\mathbb{R}^n$.*

Proof: We use Theorem 8 for $k = 1$ and deduce that for any $j \in \{1, 2, ..., n\}$

$$\left| \frac{\partial u(x^0)}{\partial x_j} \right| \leq \frac{n^2 2^n}{\omega_n r^{n+1}} \|u\|_{L^1(B_r(x^0))} = \frac{n^2 2^n}{\omega_n r^{n+1}} \int_{B_r(x^0)} |u(y)| dy \leq \frac{n 2^n}{r} C_0. \quad (5.5)$$

If we let $r \to \infty$ in (5.5) we can deduce that

$$\left| \frac{\partial u(x^0)}{\partial x_j} \right| = 0$$

for every $x^0 \in \mathbb{R}^n$ and j. It follows that u is constant. □

Corollary 2. *Suppose that $u \in C_0(\mathbb{R}^n)$ is harmonic. Then if $|u(x)| \leq C(1 + |x|^{k+\alpha})$ for every $x \in \mathbb{R}^n$ and for some constant C_0, $k \in \mathbb{N}$ and $0 \leq \alpha < 1$ then $u(x)$ is a polynomial of degree at most k in $\mathbb{R}^n$.*

Proof: The argument is similar to the argument in Theorem 9. From Theorem 8 we deduce that

$$\frac{\partial^{|\beta|} u(x)}{\partial x^\beta} \leq \frac{C}{r^{1-\alpha}}$$

for any multiindex β of length $k + 1$. In particular, sending $r \to \infty$ we see that the $(k+1)$:st derivatives of $u(x)$ are zero. That is the k :th derivatives of u are constant. It follows that u is a polynomial of degree at most k. □

5.1 The Harnack Inequality.

In this section we will state a very important important theorem known as the Harnack inequality. At this point I am not sure if we are going to further explore its consequences in this course. We will certainly not talk more about it in the first part of the course.

Theorem 10. (The Harnack Inequality.) *Let Ω be a domain. Then for every connected compact set $K \subset \Omega$ there exist a constant C_K such that*

$$\sup_{x \in K} u(x) \leq C_K \inf_{x \in K} u(x)$$

for all non-negative harmonic functions u in Ω.

Proof: From the mean value property (used both in the first and in the last equality) and standard estimates we may conclude that

$$u(x) = \frac{n}{\omega_n (2r)^n} \int_{B_{2r}(x)} u(z) dz \geq \frac{n}{\omega_n (2r^n} \int_{B_r(y)} u(z) dz = \frac{1}{2^n} u(y) \quad (5.6)$$

for any $y \in B_r(x)$. Notice that we use that $u \geq 0$ in the inequality of (5.6).

We have thus shown that

$$u(x) \geq \frac{1}{2^n} u(y) \quad (5.7)$$

for any $y \in B_r(x)$.

Next we let $r_0 = \frac{1}{4}\text{dist}(K, \partial\Omega)$ and notice that for any $z \in K$ there is a path of balls (that will be chosen below), for $j = 1, 2, ..., j_0$, $B_{r_0}(y^j)$ such that $B_{r_0}(x) \cup B_{r_0}(y^j) \neq \emptyset$ and $B_{r_0}(y^j) \cup B_{r_0}(y^{j+1}) \neq \emptyset$ and $z \in B_{r_0}(y^{j_0})$. Since K is compact we see that j_0 is finite. In particular, the set $\cup_{z\in K} B_{r_0}(z)$ is an open cover of K so there is a finite sub-cover $K \subset \cup_{k=1}^N B_{r_0}(z^k)$. It follows that we may choose $y^j = z^{k_j}$ for some k_j and conclude that $j_0 \leq N$.

We may pick a sequence $\tilde{x}^0 = x$, $\tilde{x}^j \in B_{r_0}(y^j) \cap B_{r_0}(y^{j+1})$ and $\tilde{x}^{j_0+1} = z$ and apply (5.7) with $\tilde{x}^j$ in place of x and $\tilde{x}^{j+1}$ in place of y and $r = 2r_0$. Since $\tilde{x}^j, \tilde{x}^{j+1} \in B_{r_0}(y^{j+1}) \subset B_{2r_0}(\tilde{x}^j)$ we it is justified to apply (5.7).

In particular we have shown that

$$u(x) = u(\tilde{x}^0) \geq 2^{-n} u(\tilde{x}^1) \geq 2^{-n}\left(2^{-n} u(\tilde{x}^2)\right) \geq ... \geq$$

$$\geq 2^{-(j_0+1)n} u(\tilde{x}^{j_0+1}) = 2^{-(j_0+1)n} u(z).$$

But this holds for arbitrary $x, z \in K$. In particular we can choose x such that $u(x) = \inf_{y\in K}\left(u(y)\right)$ and z such that $u(z) = \sup_{y\in K}\left(u(y)\right)$. The theorem follows. $\square$

Remark: Notice that we may view the Harnack inequality as a quantitative version of the strong maximum principle. In particular if $u \geq 0$ is a harmonic function in the bounded connected domain Ω. Then by the strong maximum principle we know that $-u$ (which is also harmonic) satisfies either $-u(x) < 0$ in Ω or there exist a point $x^0 \in \Omega$ such that $-u(x^0) = 0$ in which case $-u(x) = 0$ in Ω. However the strong maximum principle says nothing if $-u(x^0) < 0$ but $|u(x^0)|$ is small.

But if we assume that $u(x^0) = \epsilon$ for some $x^0 \in \Omega$ then the Harnack inequality states that $0 \leq u(x) \leq C_K\epsilon$ for all $x \in K$, where K is some compact connected set containing x^0. If $\epsilon = 0$ then it follows that $u = 0$ on every compact set in Ω, that is $u = 0$ in Ω so we recover the strong maximum principle.

But the estimate $0 \leq u(x) \leq C_K\epsilon$ is stronger than the strong maximum principle in that it provides information even if $\epsilon > 0$.

Chapter 6

Exercises.

Exercise 1. The following Theorem is known as the weak maximum principle

Theorem: *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ where Ω is a bounded domain. Furthermore assume that $\Delta u(x) \geq 0$ in Ω. Then*

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x).$$

Prove this Theorem using the following steps:

Step 1: Assume that $x \in \Omega$ and that x is a local maximum of $u(x)$ show that $\Delta u(x) \leq 0$.
(Hint: What do we know about the second derivatives at a local maximum?)

Step 2: Prove the Theorem under the assumption that $\Delta u(x) > 0$.
(Hint: If the Theorem is false can you find a contradiction to step 1?)

Step 3: Define $u_\epsilon(x) = u(x) - \epsilon|x|^2$ and show that the Theorem is true for u_ϵ. Pass to the limit $\epsilon \to 0$ and conclude that the Theorem is true for u.

Exercise 2. Assume that $u \in C(\mathbb{R}^n)$ and that for every $\phi \in C_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u(x)\phi(x) = 0.$$

Show that $u(x) = 0$.
(Hint: Assume that $u(x^0) > 0$ and let $\phi(x) = \max(\delta - |x - x^0|, 0)$ chose δ small enough and derive a contradiction.)

Exercise 3 a) Let $u \in C^2(\Omega)$ solve $\Delta u = f(x)$ in Ω, where $\Omega \subset \mathbb{R}^n$ is some domain in $\mathbb{R}^n$ and $f \in C(\Omega)$. Show that

$$\int_\Omega \big(\nabla u(x) \cdot \nabla \phi(x) + \phi(x) f(x)\big) dx = 0 \tag{6.1}$$

for every $\phi \in C_0^1(\Omega) \equiv \{\phi \in C^1(\Omega);\ \phi = 0 \text{ on } \partial\Omega\}$.

(HINT: Use Green's formula.)

b) Let $u \in C^2(\mathbb{R}^n)$ and assume that (6.1) holds for every $\phi \in C^1(\Omega)$. Prove that $\Delta u = f$.

(HINT: Look at Exercise 2.)

c) Note that the equation (6.1) makes perfectly good sense even if $f \notin C(\Omega)$ and in particular (6.1) makes sense even if $u \in C^1(\Omega)$ but $u \notin C^2(\Omega)$. We will say that u is a *weak solution* of $\Delta u = f$ if $u \in C^1(\Omega)$ and if (6.1) holds for every $\phi \in C_0^1(\Omega)$

Try to find a weak solution in $\mathbb{R}^3$ to

$$\Delta u = \begin{cases} 1 & \text{when } |x| \leq 1 \\ 0 & \text{when } |x| > 1. \end{cases}$$

(HINT: Look for a radial $u(x)$, that is $u(x) = u(|x|) = u(r)$.)

d) Let u be your weak solution form c) and define

$$v(x) = \begin{cases} u(x) & \text{when } |x| \leq 1 \\ u(x) + \frac{1}{|x|} - 1 & \text{when } |x| > 1. \end{cases}$$

Then v is continuous and $\Delta v = 1$ when $|x| < 1$ and $\Delta v = 0$ when $|x| > 1$. However, $v \notin C^1(\mathbb{R}^3)$ prove that v does not satisfy (6.1) and v is therefore not a weak solution.

Remark: Notice that what we do in this exercise is very similar to what we did when we defined sub-harmonic functions. Both solutions and sub-harmonic functions can be defined by using C^2. But we may relax the C^2 assumption when we define sub-harmonic functions by using the mean value formula. In the same way we can relax the notion of solution to weak solution where a weak solution is defined in a bigger function space (C^1 instead of C^2). This allows us to talk about solutions with discontinuous right hand sides such as the solution in part c).

Exercise 4. Let $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)})$ and

$$\Delta u = f(x) \quad \text{in } B_1(0) \subset \mathbb{R}^n \text{ and}$$

$$u(x) = g(x) \quad \text{on } \partial B_1(0).$$

Where $f, g \in C(\mathbb{R}^n)$ are some given functions. Show that

$$\sup_{B_1} u \leq \sup_{\partial B_1(0)} g + \frac{1}{2n} \sup_{B_1} f^-(x),$$

where $f^-(x) = \max(0, -f(x))$.

(HINT: What equation will $v = u + \alpha|x|^2 - \alpha$ solve when α is a constant, when is v sub-harmonic?)

Exercise 5. Hopf's Boundary Lemma. Let $u \in C^2(\overline{B_1^+})$, where $B_1^+ = \{x \in B_1;\ x_n > 0\}$, and

$$\begin{array}{ll} \Delta u = 0 & \text{in } B_1^+ \\ u = g \in C^2 & \text{on } \partial B_1 \cap \{x_n > 0\} \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{array}$$

Assume furthermore that $0 \leq g$ and that g is not identically zero.

Then the Hopf boundary lemma states that

$$\frac{\partial u(0)}{\partial x_n} > 0,$$

the important point is that the inequality is strict. The aim of this exercise is to prove this.

a.) Let u be as above. Show that the maximum principle implies that

$$\frac{\partial u(0)}{\partial x_n} \geq 0$$

b.) Define $v(x)$ to be

$$v(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, ..., x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Show that $v \in C^2(B_1)$ and that $\Delta v = 0$ in B_1.

c.) Use the mean value formula to express $\frac{\partial v(0)}{\partial x_n}$. Use this expression to show that $\frac{\partial v(0)}{\partial x_n} > 0$.

(Hint: Let $e_n = (0, 0, 0..., 0, 1)$ as usual, then $\int_{B_1} div(e_n v(x))dx = \int_{B_1} \frac{\partial v}{\partial x_n} dx$, also if ν is the normal of $\partial B_1(0)$ then $\nu \cdot e_n > 0$ at points on ∂B_1 where $x_n > 0$...)

d.) Use b.) and c.) to prove Hopf's lemma.

Exercise 6. Suppose that $u \in C^2(\overline{\mathbb{R}^n_+})$ and that $\Delta u(x) = 0$ in $\mathbb{R}^n_+$ and $u(x_1, x_2, ..., x_{n-1}, 0) = 0$. Furthermore assume that $\lim_{|x|\to\infty} \left(|x|^{-1}|u(x)|\right) = 0$ uniformly.

a) Define

$$v(x) = \begin{cases} u(x) & \text{for } x_n \geq 0 \\ -u(x_1, x_2, ..., x_{n-1}, -x_n) & \text{for } x_n < 0. \end{cases}$$

Show that $\Delta v = 0$ in $\mathbb{R}^n$.

b) Use the estimates on the first derivatives of v to prove that $\nabla v(x) = 0$. Conclude that $u(x) = 0$.

Exercise 7. Let Ω be an unbounded domain and assume that

$$\begin{aligned} \Delta u^i(x) = f(x) \quad & \text{in } \Omega \\ u^i(x) = g(x) \quad & \text{on } \partial\Omega, \end{aligned}$$

for $i = 1, 2$.

a) Show that if $\lim_{\Omega \ni x \to \infty} |u^1(x) - u^2(x)| = 0$ uniformly then $u^1 = u^2$.

b) Assume that $\Omega = \mathbb{R}^n_+$ and show that if $\lim_{|x| \to \infty} \left(|x|^{-1} |u^1(x) - u^2(x)| \right) = 0$ then $u^1 = u^2$.

(Hint: Look at Exercise 6.)

c) Assume that $\Omega \subset \mathbb{R}^2$ and that, in polar coordinates, $\Omega = \{(r, \phi);\ \phi \in (0, \phi_0)\}$ for some $\phi_0 \in (0, 2\pi)$. Show that if $\lim_{|x| \to \infty} \left(|x|^{-\pi/\phi_0} |u^1(x) - u^2(x)| \right) = 0$ then $u^1 = u^2$.

(Hint: Let $w(x) = u^1(x) - u^2(x) - \epsilon r^{\pi/\phi_0} \sin(\pi\phi/\phi_0)$. Is $w(x)$ harmonic? Does $w(x)$ have a sign on $\partial\left(\Omega \cap B_r(0)\right)$ if R is large enough?)

Exercise 8. Use the Harnack inequality to show that if $\{u^j\}_{j=1}^\infty$ is an increasing sequence of harmonic functions in the connected domain Ω then if $u^j(x^0)$ converges for some $x^0 \in \Omega$ then there exist a harmonic function u^0 such that $u^j \to u^0$ uniformly on compact sets $K \subset\subset \Omega$.

(Hint: What can you say about $u^{j+k} - u^j$ for $k \geq 1$?)

Exercise 9. Let $\phi_\epsilon(x)$ be the standard mollifier. Use the estimate

$$\sup_{\mathbb{R}^n} \left| \frac{\partial^{|\alpha|} \phi(x)}{\partial x^\alpha} \right| \leq C_\alpha$$

for any multiindex α together with

$$\frac{\partial^{|\alpha|} \phi_\epsilon(x)}{\partial x^\alpha} = \frac{1}{\epsilon^{n+|\alpha|}} \frac{\partial^{|\alpha|} \phi(x/\epsilon)}{\partial x^\alpha}$$

to directly show that if u is harmonic in Ω then for any $x^0 \in \Omega$

$$\left| \frac{\partial^{|\alpha|} u(x^0)}{\partial x^\alpha} \right| \leq C_0 C_\alpha \frac{1}{\mathrm{dist}(x^0, \partial\Omega)^{n+|\alpha|}} \|u\|_{L^1(B_{\mathrm{dist}(x^0,\partial\Omega)}(x^0))},$$

for some constant C_0.

Exercise 10. Suppose that u is harmonic in Ω. Prove that u^2 is sub-harmonic in Ω.

(Hint: Is $u^2 \in C^2(\Omega)$?)

Exercise 11. Show that the following definition is equivalent to our definition of sub-harmonicity:

We say that $u \in C(\Omega)$ is sub-harmonic if for any $D \subset \Omega$ we have $u \leq h$ for all h that are harmonic in D and $h \geq u$ on ∂D.