Selected Topics in PDE part 4.

John Andersson johnan@kth.se

October 13, 2014

Contents

Chapter 1

Compactness Properties of Harmonic Functions.

One of the stated reasons for the importance to develop a regularity theory for harmonic functions is that estimates implies compactness for harmonic functions. With the Arzela-Ascoli Theorem at hand (see the appendix) we can to prove the following version of Weierstrass theorem.

Theorem 1. *Let $\{u^j\}_{j=1}^\infty$ be a uniformly bounded sequence of harmonic functions in the domain Ω. That is, $u^j \in C^2(\Omega)$, $\Delta u^j(x) = 0$ in Ω and there exist a constant C_0 (independent of j) such that $\sup_{x\in\Omega} |u^j(x)| \le C_0$.*

Then there exists a sub-sequence $\{u^{j_k}\}_{k=1}^\infty$ of $\{u^j\}_{j=1}^\infty$ that is uniformly convergent on compact sets in Ω and the limit $u^0(x) = \lim_{k\to\infty} u^{j_k}$ is harmonic in Ω.

Proof: We want to show that the sequence $\{u^j\}_{j=1}^\infty$ is equicontinuous in Ω. Then the Arzela-Ascoli Theorem assures that there is a sub-sequence converging uniformly on compact sets of Ω.

To show that the sequence is equicontinuous we notice that for every point $x \in \Omega_{2r} = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > 2r\}$ we have $B_{2r}(x) \subset \Omega$. In particular for $y \in B_r(x)$ we have the estimate

$$\left|\nabla u^j(y)\right| \le \sqrt{n}\frac{n^2 2^{n+1}}{\omega_n r^{n+1}} \|u^j\|_{L^1(B_r(y))}.$$

Using that $|u^j| \le C_0$ we see that

$$\|u^j\|_{L^1(B_r(y))} = \int_{B_r(y)} |u(z)| dz \le \frac{\omega_n r^n}{n} C_0.$$

So for any $x \in \Omega_{2r}$ we have

$$\left|\nabla u^j(y)\right| \le \frac{n^{3/2} 2^{n+1}}{r} C_0, \tag{1.1}$$

for every $y \in B_r(x)$.

To show that $\{u^j\}_{j=1}^\infty$ is equicontinuos at x we need, for every $\epsilon > 0$, to find a $\delta_\epsilon > 0$ such that

$$|u^j(x) - u^j(y)| < \epsilon \quad \text{for all } y \in B_{\delta_\epsilon}(x),$$

where δ_ϵ is independent of j. There is no loss of generality to assume that $\delta_\epsilon < r$.

By the mean value Theorem (from analysis, not the mean value Theorem for harmonic functions) we get for some $t \in (0,1)$

$$|u^j(x) - u^j(y)| = \left|(y-x)\cdot\nabla u^j\big(x + t(y-x)\big)\right| \le \frac{n^{3/2}2^{n+1}}{r}C_0|x-y| \tag{1.2}$$

if $|x-y| < r$ where we also used the estimate (1.1). If we set

$$\delta_\epsilon = \inf\left(\frac{r}{C_0 n^{3/2}2^{n+1}}\epsilon, r\right),$$

then (1.2) implies that

$$|u^j(x) - u^j(y)| < \epsilon \tag{1.3}$$

for $|x-y| < \delta_\epsilon$. Since (1.3) is independent of j it follows that $\{u^j\}_{j=1}^\infty$ is equicontinious in Ω.

By the Arzela-Ascoli Theorem it follows that we can find a sub-sequence $\{u^{j_k}\}_{k=1}^\infty$ of $\{u^j\}_{j=1}^\infty$ that converges uniformly on compact sets of Ω to some $u^0 \in C(\Omega)$.

We still need to show that u^0 is harmonic. We could do that by applying the Arzela-Ascoli Theorem to the second derivatives (using estimates on the third derivatives to show that the second derivatives of $\{u^j\}_{j=1}^\infty$ forms an equicontinuous sequence). But we will use another argument based on the mean value Theorem.

Let $x^0 \in \Omega$ and $\overline{B_r(x^0)} \subset \Omega$. Then since $\overline{B_r(x^0)}$ is a compact set we know that $u^{j_k} \to u^0$ uniformly $\overline{B_r(x^0)}$. In particular for every $\epsilon > 0$ there exists an N_ϵ such that $|u^0(x) - u^{j_k}(x)| < \epsilon$ for all $k > N_\epsilon$ and $x \in \overline{B_r(x^0)}$.

Using this and the mean value property for u^{j_k} we see that when $k > N_\epsilon$

$$\epsilon > \left|u^0(x^0) - u^{j_k}(x^0)\right| = \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^{j_k}(y)dy\right| =$$

$$= \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^0(y)dy - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} \left(u^{j_k}(y) - u^0(y)\right)dy\right| \ge$$

$$\ge \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^0(y)dy\right| - \left|\frac{n}{\omega_n r^n}\int_{B_r(x^0)} \left(u^{j_k}(y) - u^0(y)\right)dy\right| \ge$$

$$\ge \left|u^0(x^0) - \frac{n}{\omega_n r^n}\int_{B_r(x^0)} u^0(y)dy\right| - \epsilon$$

where we used that $|u^0(x) - u^{j_k}(x)| < \epsilon$ in $\overline{B_r(x^0)}$ in the last inequality.

In particular

$$\left| u^0(x^0) - \frac{n}{\omega_n r^n} \int_{B_r(x^0)} u^0(y)dy \right| < 2\epsilon,$$

for any $\epsilon > 0$. That is u^0 satisfies the mean value property and is therefore harmonic. □

1.1 Appendix: The Arzela-Ascoli Theorem.

One of the main reasons that we are interested in estimating the derivatives of a harmonic function is that it gives good compactness properties of solutions, that is we can show that bounded sequences of solutions converge in C^k. One of the main compactness theorems for functions is the Arzela-Ascoli Theorem which we will prove presently. We begin with a definition.

Definition 1. *Let $\mathcal{F}$ be a set of functions defined in Ω. We say that $\mathcal{F}$ is equicontinuous at $x \in \Omega$ if for every $\epsilon > 0$ there exist an $\delta_{x,\epsilon} > 0$ such that*

$$|f(x) - f(y)| \leq \epsilon$$

for all $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon}$ and all $f \in \mathcal{F}$.

We also say that $\mathcal{F}$ is equicontinuous in Ω if $\mathcal{F}$ is equicontinuous at every $x \in \Omega$.

Naturally, we may consider a sequence of functions $\{f_j\}_{j=1}^\infty$ defined on Ω as a set $\mathcal{F} = \{f_j;\ j \in \mathbb{N}\}$ and we may therefore say that a sequence $\{f_j\}_{j=1}^\infty$ is equicontinuous at x or in Ω.

Theorem 2. *Let $\{f_j\}_{j=1}^\infty$ be a uniformly bounded sequence of functions defined on Ω, that is $\sup_{x\in\Omega} |f_j(x)| \leq C$ for some C independent of j. Assume furthermore that $\{f_j\}_{j=1}^\infty$ is equicontinuous in Ω. Then there exist a sub-sequence $\{f_{j_k}\}_{k=1}^\infty$ such that $f_{j_k}(x)$ converges pointwise.*

If we define $f_0(x) = \lim_{k\to\infty} f_{j_k}(x)$ then $f_{j_k} \to f_0$ uniformly on compact subsets and $f_0 \in C(\Omega)$.

Proof: The proof is rather long so we will divide it into several steps.

Step 1: *There is a sub-sequence $\{f_{j_k}\}_{k=1}^\infty$ that converges pointwise on a countable dense set of Ω.*

Consider the intersection of Ω and the points with rational coordinates $\Omega_{\mathbb{Q}} \equiv \mathbb{Q}^n \cap \Omega$. Since $\mathbb{Q}^n$ is countable it follows that $\Omega_{\mathbb{Q}}$ is countable. Say $\Omega_{\mathbb{Q}} = \{y^j;\ j \in \mathbb{N},\ y^j \in \mathbb{Q}^n\}$.

We will inductively define the sub-sequence $\{f_{j_k}\}_{k=1}^\infty$ so that it converges pointwise on $\Omega_{\mathbb{Q}}$.

Consider the sequence $\{f_j(y^1)\}_{j=1}^\infty$. Since $|f_j| \leq C$ in Ω it follows that $|f_j(y^1)| \leq C$. In particular $\{f_j(y^1)\}_{j=1}^\infty$ is a bounded sequence of real numbers.

We may thus extract a convergent sub-sequence which we will denote $\{f_{1,j}\}_{j=1}^{\infty}$ where the sub-script 1 indicates that the sequence converges at y^1.

Next we make the induction assumption that we have extracted sub-sequences $\{f_{l,j}\}_{j=1}^{\infty}$ for each $l \in \{1,2,3,...,m\}$, such that

1. $\{f_{l,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{l-1,j}\}_{j=1}^{\infty}$ for $l = 2,3,4,...,m$
2. and $f_{m,j}(y^l)$ converges for $l = 1,2,3,...,m$.

In order to complete the induction we need to show that we can find a sub-sequence $\{f_{m+1,j}\}_{j=1}^{\infty}$ of $\{f_{m,j}\}_{j=1}^{\infty}$ such that $\{f_{m+1,j}(y^{m+1})\}_{j=1}^{\infty}$ converges.

Arguing as before, we see that $\{f_{m,j}(y^{m+1})\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ and we may thus extract a sub-sequence, that we denote $\{f_{m+1,j}\}_{j=1}^{\infty}$, that converges.

By induction it follows that for each $m \in \mathbb{N}$ there exist a sequence $\{f_{m,j}\}_{j=1}^{\infty}$ such that $\{f_{m,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m-1,j}\}_{j=1}^{\infty}$ and $\{f_{m,j}(y^m)\}_{j=1}^{\infty}$ is convergent.

Notice that since $\{f_{m,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m-1,j}\}_{j=1}^{\infty}$ and $\{f_{m-1,j}(y^l)\}_{j=1}^{\infty}$ converges for $1 \le l \le m-1$ it follows that $\{f_{m,j}(y^l)\}_{j=1}^{\infty}$ converges to the same limit for $1 \le l \le m-1$. In particular, $\{f_{m,j}(y^l)\}_{j=1}^{\infty}$ converges for all $l \le m$.

Now we define the sequence $\{f_{j_k}\}_{k=1}^{\infty}$ by a diagonalisation procedure

$$f_{j_k} = f_{k,k}.$$

Noticing that $\{f_{j_k}\}_{k=m}^{\infty} = \{f_{k,k}\}_{k=m}^{\infty}$ is a sub-sequence of $\{f_{m,j}\}_{j=1}^{\infty}$. This follows from the fact that $f_{k,k}$ is an element of the sequence $\{f_{k,j}\}_{j=1}^{\infty}$. But $\{f_{k,j}\}_{j=1}^{\infty}$ is a sub-sequence of $\{f_{m,j}\}_{j=1}^{\infty}$ for $k \ge m$.

We may conclude that $\{f_{j_k}\}_{k=m}^{\infty}$ converges at y^l for all $l \le k$. But k is arbitrary so $f_{j_k}(y^l)$ converges for every $l \in \mathbb{N}$. This proves step 1.

Step 2: *The sequence $\{f_{j_k}\}_{k=1}^{\infty}$ converges pointwise in Ω.*

It is enough to show that $\{f_{j_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence for every $x \in \Omega$. To that end we fix an $\epsilon > 0$. We need to show that there exist an $N_\epsilon \in \mathbb{N}$ such that $|f_{j_k}(x) - f_{j_l}(x)| < \epsilon$ for all $k, l > N_\epsilon$.

Since $\{f_{j_k}\}_{k=1}^{\infty}$ is equicontinuous at $x \in \Omega$ there exist a $\delta_{x,\epsilon/3}$ such that

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \quad \text{for all } k \in \mathbb{N}, \tag{1.4}$$

and $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon/3}$.

Moreover since $\Omega_{\mathbb{Q}}$ is dense in Ω there exist an $y^x \in \Omega_{\mathbb{Q}}$ such that $|x - y^x| < \delta_{x,\epsilon/3}$. In step 1 we showed that $f_{j_k}(y)$ was convergent for all $y \in \Omega_{\mathbb{Q}}$ in particular it follows that $\{f_{j_k}(y^x)\}_{k=1}^{\infty}$ is a Cauchy sequence. That is, there exist an $N_{y^x,\epsilon/3} \in \mathbb{N}$ such that

$$|f_{j_k}(y^x) - f_{j_l}(y^x)| < \frac{\epsilon}{3} \quad \text{for all } k, l > N_{y^x,\epsilon/3}. \tag{1.5}$$

From (1.4) and (1.5) we can deduce that

$$|f_{j_k}(x)-f_{j_l}(x)| \leq |f_{j_k}(x)-f_{j_k}(y^x)|+|f_{j_l}(x)-f_{j_l}(y^x)|+|f_{j_k}(y^x)-f_{j_l}(y^x)| < \epsilon,$$

for all $k,l > N_{y^x,\epsilon/3}$. It follows that $\{f_{j_k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence and this finishes the proof of step 2.

Step 3: *Define* $f_0(x) = \lim_{k\to\infty} f_{j_k}(x)$, *then* $f_0 \in C(\Omega)$.

Since $f_{j_k}(x)$ is convergent for every $x \in \Omega$ by step 2 it follows that f_0 is well defined in Ω. To show continuity we need to show that for every $x \in \Omega$ and $\epsilon > 0$ there exist a $\delta_\epsilon > 0$ such that

$$|f_0(x) - f_0(y)| < \epsilon$$

for every $y \in \Omega$ such that $|x - y| < \delta_\epsilon$. By equicontinuity there exist a $\delta_{x,\epsilon/3}$ such that

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{1.6}$$

for every $y \in \Omega$ such that $|x - y| < \delta_{x,\epsilon/3}$ and all $j \in \mathbb{N}$.

Also by step 2 there exist an $N_{x,\epsilon/3}$ such that

$$|f_0(x) - f_{j_k}(x)| < \frac{\epsilon}{3} \tag{1.7}$$

for all $k \geq N_{x,\epsilon/3}$. And an $N_{y,\epsilon/3}$ such that

$$|f_0(y) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{1.8}$$

for all $k \geq N_{y,\epsilon/3}$.

From (1.6), (1.7) and (1.8) we can deduce that for $y \in \Omega$ such that $|x-y| < \delta_{x,\epsilon/3}$

$$|f_0(x) - f_0(y)| \leq |f_0(x) - f_{j_k}(x)| + |f_0(y) - f_{j_k}(y)| + |f_{j_k}(x) - f_{j_k}(y)| < \epsilon$$

if $k > \max(N_{x,\epsilon/3}, N_{y,\epsilon/3})$.

This proves step 3.

Step 4: $\{f_{j_k}\}_{k=1}^{\infty}$ *converges uniformly on compact sets.*

We fix a compact set $K \subset \Omega$. We need to show that for every $\epsilon > 0$ there exist an N_ϵ such that when $k > N_\epsilon$ then $|f_0(x) - f_{j_k}(x)| < \epsilon$ for all $x \in K$.

Notice that by equicontinuity there exist a $\delta_{x,\epsilon/3}$ for each $x \in K$ such that for all $k \in \mathbb{N}$

$$|f_{j_k}(x) - f_{j_k}(y)| < \frac{\epsilon}{3} \tag{1.9}$$

for all $y \in B_{\delta_{x,\epsilon/3}}(x) \cap \Omega$.

Notice that the balls $B_{\delta_{x,\epsilon/3}}(x)$ forms an open cover of K: $K \subset \cup_{x\in K} B_{\delta_{x,\epsilon/3}}(x)$. Since K is compact there exist a finite sub-cover $B_{\delta_{x^l,\epsilon/3}}(x^l)$, for $l = 1, 2, 3, ..., l_0$ for some $l_0 \in \mathbb{N}$. That is $K \subset \cup_{l=1}^{l_0} B_{\delta_{x^l,\epsilon/3}}(x^l)$.

Also, using that $\lim_{k\to\infty} f_{j_k}(x^l) = f_0(x^l)$, we see that there exist an $N_{x^l,\epsilon/3}$ such that

$$|f_{j_i}(x^l) - f_{j_k}(x^l)| < \frac{\epsilon}{3} \tag{1.10}$$

for all $i, k > N_{x^l,\epsilon/3}$. We choose $N_\epsilon = \max\left(N_{x^1,\epsilon/3}, N_{x^2,\epsilon/3}, ..., N_{x^{l_0},\epsilon/3}\right)$.

Since $K \subset \cup_{l=1}^{l_0} B_{\delta_{x^l,\epsilon/3}}(x^l)$ it follows that for every $x \in K$ that $x \in B_{\delta_{x^l,\epsilon/3}}(x^l)$ for some l. Using this and (1.9) and (1.10) we see that

$$\begin{aligned} |f_{j_i}(x) - f_{j_k}(x)| \le |f_{j_i}(x) - f_{j_i}(x^l)| + |f_{j_k}(x) - f_{j_k}(x^l)| + |f_{j_i}(x^l) - f_{j_k}(x^l)| < \\ < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned} \tag{1.11}$$

for all $k \ge N_\epsilon$. Taking the limit $i \to \infty$ in (1.11) we see that

$$|f_0(x) - f_{j_k}(x)| < \epsilon$$

for all $k > N_\epsilon$. This finishes the proof of the Theorem. □

Chapter 2

Existence of Solutions.

2.1 The Perron Method.

We are now ready to prove the existence of solutions to the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array} \tag{2.1}$$

The idea of the proof is to consider the largest subharmonic function that is smaller than g on $\partial\Omega$. If a solution to (2.1) exists then, by the maximum principle, that solution has to be the largest subharmonic function that is less than or equal to g on $\partial\Omega$. This gives some hope that the largest sub-harmonic function should be the solution to (2.1).

Before we prove that the largest sub-harmonic function is harmonic we need to prove a Lemma that shows us that we can change a sub-harmonic function into a harmonic function in part of the domain without destroying the sub-harmonicity.

Lemma 1. *Suppose that $v \in C(\Omega)$ is sub-harmonic in Ω. Moreover, we assume that $\overline{B_{r_0}(x^0)} \subset \Omega$. If we define $\tilde{v}$ to by the harmonic replacement of v in $B_{r_0}(x^0)$:*

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \setminus B_{r_0}(x^0) \\ \int_{\partial B_{r_0}(x^0)} \frac{r^2 - |x-x^0|^2}{\omega_n r} \frac{1}{|x-x^0-y|^n} v(y) dA_{\partial B_{r_0}(x^0)}(y) & \text{for } x \in B_{r_0}(x^0). \end{cases}$$

Then $\tilde{v}$ is sub-harmonic in Ω.

Remark: We say that $\tilde{v}$ is defined by the harmonic replacement in $B_{r_0}(x^0)$. This language usage is natural since $\tilde{v}$ equals v outside of $B_{r_0}(x^0)$ and is defined by Poisson's formula in $B_{r_0}(x^0)$. We know that functions defined by Poisson's formula are harmonic so $\tilde{v}$ is defined by replacing the values of v by the harmonic function with boundary data v in $B_{r_0}(x^0)$.

At times the harmonic replacement is referred to as the harmonic lifting in $B_{r_0}(x^0)$. The reason for that terminology is that, by the maximum principle,

$\tilde{v} \geq v$ in $B_{r_0}(x^0)$. So $\tilde{v}$ is defined by increasing, or lifting, the values of v in $B_{r_0}(x^0)$.

Proof: Since $\tilde{v}$ is defined by Poisson's formula in $B_r(x^0)$ it follows that $\tilde{v}$ is harmonic in $B_{r_0}(x^0)$. Since $\tilde{v}$ is harmonic in $B_{r_0}(x^0)$ it follows that $\tilde{v}$ is sub-harmonic in $B_{r_0}(x^0)$.

Also $\tilde{v} = v$ in $\Omega \setminus B_{r_0}(x^0)$ so $\tilde{v}$ is sub-harmonic in $\Omega \setminus B_{r_0}(x^0)$.

This does not imply that $\tilde{v}$ is sub-harmonic in Ω. We need to show that $\tilde{v}$ satisfies the sub-meanvalue property for every ball $\overline{B_{r_0}(y)} \subset \Omega$.

To that end we fix an arbitrary ball $\overline{B_r(y)} \subset \Omega$. If $B_r(y) \subset \Omega \setminus B_{r_0}(x^0)$ then $\tilde{v}$ satisfies the sub-meanvalue property for the ball $B_r(y)$ since $\tilde{v} = v$ in $B_r(y)$. Similarly, if $B_r(y) \subset B_{r_0}(x^0)$ then $\tilde{v}$ satisfies the sub-meanvalue property (and even the mean value property) for the ball $B_r(y)$ since $\tilde{v}$ is harmonic in $B_r(y) \subset B_{r_0}(x^0)$.

We therefore only need to prove that $\tilde{v}$ satisfies the sub-meanvalue property for balls $\overline{B_r(y)} \subset \Omega$ such that $B_r(y) \cap B_{r_0}(x^0) \neq \emptyset$ and $B_r(y) \cap \big(\Omega \setminus B_{r_0}(x^0)\big) \neq \emptyset$. Fix such a ball $B_r(y)$. We continue the proof in several steps.

Step 1: *Let $\tilde{h}$ be the harmonic function in $B_r(y)$ with $\tilde{h}(x) = \tilde{v}(x)$ on $\partial B_r(y)$. We claim that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \setminus B_{r_0}(x^0)$.*

Notice that $v - \tilde{v}$ is sub-harmonic in $B_{r_0}(x^0)$ and that $v - \tilde{v} = 0$ on $\partial B_{r_0}(x^0)$. So by the maximum principle for sub-harmonic functions $v \leq \tilde{v}$ in $B_{r_0}(x^0)$.

Also, if we let h solve

$$\begin{array}{ll} \Delta h = 0 & \text{in } B_r(y) \\ h = v & \text{on } \partial B_r(y), \end{array}$$

then again, by the sub-harmonicity of v and the maximum principle $v \leq h$ in $B_r(y)$.

Since $v \leq \tilde{v}$ we have that $h \leq \tilde{h}$ on $\partial B_r(y)$ and since both h and $\tilde{h}$ are harmonic it follows that $\tilde{h} \geq h$ in $B_r(y)$. That is $v \leq h \leq \tilde{h}$ in $B_r(y)$.

Using that $\tilde{v} = v$ in $B_r(y) \setminus B_{r_0}(x^0)$ the claim in step 1 follows.

Step 2: *Let, as in step 1, $\tilde{h}$ be the harmonic function in $B_r(y)$ with $\tilde{h}(x) = \tilde{v}(x)$ on $\partial B_r(y)$. We claim that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \cap B_{r_0}(x^0)$.*

By step 1 we know that $\tilde{h} \geq \tilde{v}$ in $B_r(y) \setminus B_{r_0}(x^0)$. Since $\tilde{v}$ and $\tilde{h}$ are continuous functions it follows that $\tilde{h} \geq \tilde{v}$ on $\big(\partial B_{r_0}(x^0)\big) \cap B_r(y)$. On $\big(\partial B_r(y)\big) \cap B_{r_0}(x^0)$ we have that $\tilde{h} = \tilde{v}$ by the definition of $\tilde{h}$.

In particular, $\Delta \tilde{v} = \Delta \tilde{h} = 0$ in $B_{r_0}(x^0) \cap B_r(y)$ and $\tilde{v} \leq \tilde{h}$ on $\partial\big(B_{r_0}(x^0) \cap B_r(y)\big)$. It follows that $w = \tilde{v} - \tilde{h}$ solves

$$\begin{array}{ll} \Delta w = 0 & \text{in } B_{r_0}(x^0) \cap B_r(y) \\ w \leq 0 & \text{on } \partial\big(B_{r_0}(x^0) \cap B_r(y)\big). \end{array}$$

By the maximum principle $w \leq 0$ in $B_{r_0}(x^0) \cap B_r(y)$, that is $\tilde{v} \leq \tilde{h}$ in $B_{r_0}(x^0) \cap B_r(y)$.

Step 3: *$\tilde{v}$ satisfies the sub-meanvalue property in Ω.*

Pick any ball $\overline{B_r(y)} \subset \Omega$. If $B_r(y) \cap B_{r_0}(x^0) = \emptyset$ we have already shown that $\tilde{v}$ satisfies the sub-meanvalue property in $B_r(y)$. So we may assume that $B_r(y) \cap B_{r_0}(x^0) \neq \emptyset$.

Define $\tilde{h}$ as in step 1 and 2, then

$$\tilde{v}(y) \leq h(y) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} \tilde{h}(z) dA_{\partial B_r(y)}(z) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} \tilde{v}(z) dA_{\partial B_r(y)}(z), \tag{2.2}$$

$$= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(y)} \tilde{v}(z) dA_{\partial B_r(y)}(z),$$

where we have used step 1 if $y \in B_r(y) \setminus B_{r_0}(x^0)$ and step 2 if $y \in B_r(y) \cap B_{r_0}(x^0)$ in the first inequality, the meanvalue property for the harmonic function h in and finally that $\tilde{h} = \tilde{v}$ on $\partial B_r(y)$.

Notice that (2.2) is nothing by the sub-meanvalue property for $\tilde{v}$.

We have thus shown that $\tilde{v}$ satisfies the sub-meanvalue property in Ω and is thus sub-harmonic. □

Definition 2. *Let $g \in C(\partial\Omega)$ where Ω is a bounded domain. We define $S_g(\Omega)$ to be the class of sub-harmonic functions $v \in C(\overline{\Omega})$ such that $v(x) \leq g(x)$ on $\partial\Omega$. That is*

$$S_g(\Omega) = \left\{ v \in C(\overline{\Omega});\ v \text{ is sub-harmonic in } \Omega \text{ and } v(x) \leq g(x) \text{ on } \partial\Omega \right\}.$$

Since $g \in C(\partial\Omega)$ it follows that the constant $\inf_{x \in \partial\Omega} g(x) \in S_g(\Omega)$. That is $S_g(\Omega) \neq \emptyset$.

The first part of the existence theorem is:

Theorem 3. [Perron's Method.] *Suppose that Ω is a bounded domain and $g(x) \in C(\partial\Omega)$. Define*

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

Then $u(x)$ is harmonic in Ω.

Proof: As we remarked before $S_g(\partial\Omega) \neq \emptyset$. Also by the maximum principle for sub-harmonic functions we have

$$\sup_{x \in \Omega} \big(v(x) - \sup_{y \in \partial\Omega} g(y)\big) \leq \sup_{x \in \partial\Omega} \big(v(x) - \sup_{y \in \partial\Omega} (g(y))\big) = \sup_{x \in \partial\Omega} \big(v(x)\big) - \sup_{y \in \partial\Omega} g(y) \leq 0$$

for every $v \in S_g(\Omega)$ since if $v \in S_g(\Omega)$ then v is sub-harmonic and $v \leq g$ on $\partial\Omega$.

It follows that

$$\sup_{v \in S_g(\Omega)} v(x) \leq \sup_{y \in \partial\Omega} g(y).$$

Using that a non-empty set of real numbers that is bounded from above has a supremum (the completeness property of $\mathbb{R}$) we may conclude that $u(x) =$

$\sup_{v\in S_g(\Omega)} v(x)$ is well defined. Moreover, for every $x \in \Omega$ we can find a sequence $\{v^k\}_{k=1}^\infty$ in $S_g(\Omega)$ so $u(x) = \lim_{k\to\infty} v^k(x)$. We fix an arbitrary $x^0 \in \Omega$ and sequence $\{v^k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} v^k(x^0) = u(x^0)$. Since Ω is a domain, in particular Ω is open, there exist an $r > 0$ such that $B_r(x^0) \subset \Omega$.

We may assume that

$$v^k \geq \inf_{x\in\partial\Omega} g(x). \tag{2.3}$$

If (2.3) where not true then we could consider the sequence $\max\left(v^k(x), \inf_{x\in\partial\Omega} g(x)\right) \in S_g(\Omega)$ instead.

In order to proceed we define the harmonic replacement of v^k in $B_r(x^0)$ according to

$$\tilde{v}^k(x) = \begin{cases} v^k(x) & \text{if } x \notin B_r(x^0) \\ \int_{\partial B_r(x^0)} \frac{r^2-|x-x^0|^2}{\omega_n r} \frac{1}{|x-x^0-y|^n} v^k(y) dA_{\partial B_r(x^0)}(y) & \text{for } x \in B_r(x^0). \end{cases}$$

Notice that $\tilde{v}^k$ is defined by the Poisson integral in $B_r(x^0)$. It follows that

$$\begin{aligned} \Delta\tilde{v}^k(x) = 0 \qquad & \text{in } B_r(x^0) \\ \tilde{v}^k(x) = v^k(x) \qquad & \text{on } \partial B_r(x^0). \end{aligned}$$

By Lemma 1 it follows that $\tilde{v}^k$ is sub-harmonic in Ω and since $\tilde{v}^k = v^k \leq g$ on $\partial\Omega$ it follows that $\tilde{v}^k \in S_g(\Omega)$.

Moreover, since v^k is sub-harmonic and $\tilde{v}^k$ is harmonic in $B_r(x^0)$ and $\tilde{v}^k = v^k$ on $\partial B_r(x^0)$ we can conclude that $\tilde{v}^k \geq v^k$ in $B_r(x^0)$. Also $u(x^0) \geq \tilde{v}^k(x^0)$ since $\tilde{v}^k \in S_g(\Omega)$.

It follows that

$$u(x^0) = \lim_{k\to\infty} v^k(x^0) \leq \lim_{k\to\infty} \tilde{v}^k(x^0) \leq u(x^0),$$

so $\tilde{v}^k(x^0) \to u(x^0)$.

From the compactness Lemma 1 we know that there exist a sub-sequence $\{\tilde{v}^{k_j}\}_{j=1}^\infty$ of $\{\tilde{v}^k\}_{k=1}^\infty$ such that $\tilde{v}^{k_j} \to \tilde{v}^0$ uniformly on compact sets in $B_r(x^0)$ and that $\tilde{v}^0$ is harmonic in $B_r(x^0)$.We claim that $\tilde{v}^{k_j}(x) \to u(x)$ uniformly on compact sets in $B_r(x^0)$.

By the definition of u it follows that $u \geq \tilde{v}^0$ in $B_r(x^0)$.

Claim: *We claim that $u(x) = \tilde{v}^0(x)$ for all $x \in B_r(x^0)$. Since x^0 i arbitrary this implies that $\Delta u(x) = 0$ in any ball $B_s(y) \subset \Omega$ and finishes the proof of the theorem.*

We prove this claim by an argument of contradiction. Aiming for a contradiction we assume that there exist a $z \in B_r(x^0)$ such that $\tilde{v}^0(z) < u(z)$. Since $u(z) = \sup_{w\in S_g(\Omega)} w(z)$ there exist a $w \in S_g(\Omega)$ such that $\tilde{v}^0(z) < w(z)$.

We define $w^j = \sup(w, \tilde{v}^{k_j})$. Then w^j is sub-harmonic since w and $\tilde{v}^{k_j}$ are sub-harmonic.

We also define the harmonic lifting of w^j according to

$$\tilde{w}^j(x) = \begin{cases} w^j(x) & \text{if } x \notin B_r(x^0) \\ \int_{\partial B_r(x^0)} \frac{r^2-|x-x^0|^2}{\omega_n r} \frac{1}{|x-x^0-y|^n} w^j(y) dA_{\partial B_r(x^0)}(y) & \text{for } x \in B_r(x^0) \end{cases}$$

Arguing as before, using the maximum principle, we see that $\tilde{w}^j \geq w^j$. Notice that

$$\tilde{w}^j \geq w^j = \sup(w, \tilde{v}^{k_j}) \geq \tilde{v}^{k_j}. \tag{2.4}$$

Using Lemma 1 again we can extract a sub-sequence $\{\tilde{w}^{j_l}\}_{l=1}^{\infty}$ of $\{\tilde{w}^j\}_{j=1}^{\infty}$ such that $\{\tilde{w}^{j_l}\}_{l=1}^{\infty}$ converges uniformly on compact sets to some harmonic function $\tilde{w}^0$. Notice that

$$\tilde{w}^0(z) = \lim_{l\to\infty} \tilde{w}^{j_l}(z) \geq \lim_{l\to\infty} w^{j_l}(z) = \lim_{l\to\infty} \sup(w(z), \tilde{v}^{k_{j_l}}(z)) = w(z). \tag{2.5}$$

In particular this imples that

$$\tilde{w}^0(z) > \tilde{v}^0(z). \tag{2.6}$$

Also, since

$$u(x^0) \geq \tilde{w}^{j_l}(x^0) \geq \tilde{v}^{k_{j_l}}(x^0) \to u(x^0)$$

we get that $\tilde{w}^0(x^0) = u(x^0)$. Using (2.4), (2.5) and that $\tilde{w}^0(x^0) = \tilde{v}^0(x^0) = u(x^0)$ we get

$$\begin{array}{ll} \Delta(\tilde{v}^0(x) - \tilde{w}^0(x)) = 0 & \text{in } B_r(x^0) \\ \tilde{v}^0(x) - \tilde{w}^0(x) \leq 0 \text{ on } \partial B_r(x^0) & \\ \tilde{v}^0(x^0) - \tilde{w}^0(x^0) = 0. & \end{array}$$

From the last two lines and the strong maximum principle we can conclude that $\tilde{v}^0(x) - \tilde{w}^0 = 0$ in $B_r(x^0)$. This contradicts (2.6).

We have thus finished our contradiction argument and proved that $\tilde{v}^{k_j} \to u^0$ uniformly on compact sets in $B_r(x^0)$. But $\lim_{j\to\infty} \tilde{v}^{k_j} = \tilde{v}^0$ where $\Delta\tilde{v}^0 = 0$ in $B_r(x^0)$. It follows that $\Delta u(x) = 0$ in $B_r(x^0)$. But $x^0 \in \Omega$ was arbitrary so we may conclude that $\Delta u(x) = 0$ in Ω. □

2.2 Attaining the Boundary Data.

In the previous section we showed that

$$u(x) = \sup_{v \in S_g(\Omega)} v(x)$$

is harmonic in Ω. This is not enough in order to show existence of solutions to the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega. \end{array}$$

Perron's method gives a harmonic function but it does not prove that the harmonic function actually attain the boundary values $u(x) = g(x)$. In order to solve the Dirichlet problem we need to show that the solution attained from the Perron process actually satisfy the boundary data.

We will show that, at least in some cases, by the method of using barriers.

Definition 3. *Let Ω be a domain and $\xi \in \partial\Omega$. We say that w is a barrier at ξ relative to Ω if*

1. *$w \in C(\overline{\Omega})$,*
2. *$w > 0$ in $\overline{\Omega} \setminus \{\xi\}$, $w(\xi) = 0$ and*
3. *w is super-harmonic in Ω.*

If Ω is a domain and there exist a barrier at ξ relatively to Ω then we say that ξ is a regular point of $\partial\Omega$.

Theorem 4. *Let Ω be a bounded domain and $g \in C(\partial\Omega)$. Furthermore let*

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

If ξ is a regular point of $\partial\Omega$ then

$$\lim_{x \to \xi} u(x) = g(\xi).$$

Proof: We need to find, for each $\epsilon > 0$, a $\delta_\epsilon > 0$ such that

$$\sup_{x \in B_{\delta_\epsilon}(\xi) \cap \Omega} |u(x) - g(\xi)| < \epsilon.$$

Since $g \in C(\partial\Omega)$ there exist an $\delta_{g,\epsilon/2}$ such that

$$\sup_{x \in \partial\Omega \cap B_{\delta_{g,\epsilon/2}}(\xi)} |g(x) - g(\xi)| < \frac{\epsilon}{2}.$$

Let w be a barrier at ξ and define

$$\kappa = \inf_{x \in \partial\Omega \setminus B_{\delta_{g,\epsilon/2}}(\xi)} w(x).$$

Using that $w > 0$ in $\overline{\Omega} \setminus \{\xi\}$, $\partial\Omega$ is compact and $w \in C(\overline{\Omega})$ we see that $\kappa > 0$.

If we define

$$k = \frac{1}{\kappa} \sup_{x \in \partial\Omega} |g(x) - g(\xi)|$$

then it follows that

$$-\frac{\epsilon}{2} - kw(x) \leq g(x) - g(\xi) \leq \frac{\epsilon}{2} + kw(x). \tag{2.7}$$

We know that $w \in C(\overline{\Omega})$ and $w(\xi) = 0$ so there is a $\delta_{w,\epsilon/(2k)}$ such that

$$\sup_{x \in B_{\delta_{w,\epsilon/(2k)}}(\xi)} |w(x)| < \frac{\epsilon}{2k}. \tag{2.8}$$

Since w is super-harmonic it follows from the comparison principle and (2.7) that

$$v(x) \leq g(\xi) + \frac{\epsilon}{2} + kw(x) \tag{2.9}$$

for every $v \in S_g(\Omega)$.

From (2.9) it follows that

$$u(x) \leq g(\xi) + \frac{\epsilon}{2} + kw(x). \tag{2.10}$$

Since w is super-harmonic it follows that $g(\xi) - \frac{\epsilon}{2} - kw(x)$ is sub-harmonic so by (2.7) it follows that $-\frac{\epsilon}{2} - kw(x) \in S_g(\Omega)$. In particular

$$u(x) \geq g(\xi) - \frac{\epsilon}{2} - kw(x). \tag{2.11}$$

From (2.10) and (2.11) we deduce that

$$|u(x) - g(\xi)| \leq \frac{\epsilon}{2} + kw(x). \tag{2.12}$$

Finally we see that if $\delta < \delta_{w,\epsilon/2k}$ and $x \in B_\delta(\xi) \cap \Omega$ then, from (2.12) we may estimate

$$|u(x) - g(\xi)| \leq |\frac{\epsilon}{2} + kw(x)| < \frac{\epsilon}{2} + \frac{k\epsilon}{2k} = \epsilon$$

where we used (2.8) in the strict inequality. This proves the Theorem. □

We are now in the position to create harmonic functions by the Perron method and we also have a criteria to assure that the function so created satisfies the boundary values.

The criteria that assures that the Perron solution assumes the boundary data at $\xi \in \partial\Omega$ is that there exists a barrier at ξ relative to Ω. Since the definition of a barrier is rather abstract we need to develop some adequate theory for the existence of barriers.

The simplest condition that assures the existence of a barrier is the exterior ball condition.

Definition 4. *Let Ω be a domain. We say that Ω satisfies the exterior ball condition at ξ if there exist a ball $B_s(y^\xi) \subset \Omega^c$ such that $\xi \in \overline{B_s(y^\xi)} \cap \overline{\Omega}$*

We say that the domain Ω satisfies the exterior ball condition if Ω satisfies the exterior ball condition at every $\xi \in \partial\Omega$.

We say that that the domain Ω satisfies the exterior ball condition uniformly if Ω satisfies the exterior ball condition at every $\xi \in \partial\Omega$ and the radius of the touching balls have radius $s > 0$ independent of ξ.

Lemma 2. *Let Ω be a bounded domain and assume that Ω satisfies the exterior ball condition at $\xi \in \partial\Omega$.*

Then ξ is a regular point of $\partial\Omega$. That is there exist a barrier at ξ relatively to Ω.

Proof: Let the touching ball at ξ be $B_s(y)$. We define the the following function that is zero on $\partial B_{s/2}((\xi+y)/2)$

$$w(x) = \begin{cases} \ln\left|x - \frac{y+\xi}{2}\right| - \ln\left(\frac{s}{2}\right) & \text{if } n = 2 \\ \frac{2^{n-2}}{s^{n-2}} - \frac{1}{\left|x - \frac{y+\xi}{2}\right|^{n-2}} & \text{if } n \geq 3. \end{cases}$$

Since w is a multiple of the Newtonian potential plus a constant it is clear that $\Delta w(x) = 0$ in $\mathbb{R}^n \setminus \{(y-\xi)/2\}$.

Notice that $B_{s/2}((\xi+y)/2)$ touches $\partial\Omega$ at only the point $\xi \in \partial\Omega$. The original ball $B_s(y)$ might touch at a larger set.

Moreover $w(x) = 0$ on $\partial B_{s/2}((y+\xi)/2)$, $w > 0$ in $\mathbb{R}^n \setminus \overline{B_{s/2}((y+\xi)/2)}$. Finally notice that $\xi \in \partial B_{s/2}((y^\xi - \xi)/2)$ so $w(\xi) = 0$. It follows that w is a barrier. $\square$

A somewhat stronger sufficient (but not necessary) condition for a the existence of a barrier is the exterior cone condition.

Definition 5. *Let Ω be a domain. We say that Ω satisfies the exterior cone condition at ξ relative to $B_r(\xi)$ if $\Omega^c \cap B_r(\xi)$ contains a circular cone. That is if there exist a $\kappa > 0$ and a unit vector η such that*

$$\{x \in B_r(\xi);\ \eta \cdot (x-\xi) > 0 \text{ and } |x - \eta \cdot (x-\xi)| < \kappa|x-\xi|\} \subset \Omega^c.$$

We say that the domain Ω satisfies the exterior cone condition if Ω satisfies the exterior cone condition at every $\xi \in \partial\Omega$ with respect to some ball $B_{r_\xi}(\xi)$ and $r_\xi > 0$ and some $\kappa_\xi > 0$.

We say that that the domain Ω satisfies the exterior cone condition uniformly if Ω satisfies the exterior cone condition at every $\xi \in \partial\Omega$ with respect to some ball $B_r(\xi)$ and $r > 0$ and some $\kappa > 0$ where r and κ is independent of ξ.

Proposition 1. *Let Ω be a bounded domain and assume that Ω satisfies the exterior cone condition at $\xi \in \partial\Omega$.*

The ξ is a regular point of $\partial\Omega$. That is there exist a barrier at ξ relatively to Ω.

Proof (only in $\mathbb{R}^2$ and $\mathbb{R}^3$ (sort of)): There is no loss of generality to assume that $\xi = 0$. If $\xi \neq 0$ we may simply translate the coordinate system by the translation $x \to x - \xi$ to attain this situation.

By assumption there exist a unit vector η and $r, \kappa > 0$ such that

$$K_\kappa = \{x \in B_s(0);\ \eta \cdot x > 0 \text{ and } |x - \eta \cdot x| < \kappa|x|\} \subset \Omega^c.$$

By rotation the coordinate system we may assume that $\eta = e_n$.

Proof in $\mathbb{R}^2$: If we change to polar coordinates $x_1 = r\sin(\phi)$ and $x_2 = r\cos(\phi)$ then the cone becomes

$$K_\kappa = \{(r,\phi);\ |\sin(\phi)| < \kappa,\ \sin(\phi) > 0\}.$$

Recalling that Laplace's equation in polar coordinates is

$$\Delta w(r,\phi) = \frac{\partial^2 w(r,\phi)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\phi)}{\partial \phi^2} = 0$$

it is easy to verify that

$$w(r,\phi) = \sqrt{r}\sin\left(\frac{\phi}{2}\right)$$

is harmonic in $\mathbb{R}^2 \setminus K$ and $w(r,\phi) = \sqrt{r}\kappa > 0$ on ∂K. So $w(r,\phi)$ is a barrier at $\xi = 0$ relative to Ω.

Sketch of the proof in Proof in $\mathbb{R}^3$: Laplace equation in spherical coordinates, $(x_1, x_2, x_3) = r(\sin(\psi)\cos(\phi), \sin(\psi)\sin(\phi), \cos(\psi)$, is

$$\frac{\partial^2 w(r,\phi,\psi)}{\partial r^2} + \frac{2}{r}\frac{\partial w(r,\phi,\psi)}{\partial r} + \frac{1}{r^2\sin(\psi)}\frac{\partial}{\partial \psi}\left(\sin(\psi)\frac{\partial w(r,\phi,\psi)}{\partial \psi}\right) +$$

$$+\frac{1}{r^2\sin^2(\psi)}\frac{\partial^2 w(r,\phi,\psi)}{\partial \phi^2} = 0.$$

To simplify the expression somewhat we assume that we can find a solution $w(r,\phi,\psi) = r^\alpha \tilde{w}(\psi)$ that is independent of ϕ and homogeneous in r. Taking into consideration that we also want our barrier to be zero on $\partial K_{\kappa/2}$ we end up with the following ordinary differential equation

$$\begin{array}{ll} r^{\alpha-2}\left(\alpha(\alpha+1)\tilde{w}(r,\psi) + \frac{1}{\sin(\psi)}\frac{\partial}{\partial\psi}\left(\sin(\psi)\frac{\partial \tilde{w}(r,\psi)}{\partial \psi}\right)\right) = 0 & \text{in } K_{\kappa/2} \\ \tilde{w}(r,\psi) = 0 & \text{on } \partial K_{\kappa/2}. \end{array}$$

It turns out that this ordinary differential equation is solvable and that there exists a unique $\alpha_\kappa > 0$ such that the solution is positive in $\mathbb{R}^3 \setminus K_\kappa$. It follows that $w(r,\phi,\psi) = r^{\alpha_\kappa}\tilde{w}(r,\phi,\psi)$. $\square$

Knowing that we have barriers in some cases it is natural to ask the question if we always have barriers. The answer to that is: No.

Example of a non-regular point: Consider the domain $\Omega = B_1(0) \setminus \{0\}$, for simplicity we assume that $n \geq 3$. We want to solve the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{array}$$

where

$$\begin{array}{ll} g(x) = 0 & \text{on } \Omega \setminus \{0\} \\ g(0) = -1. & \end{array}$$

By the Perron method we would want to construct the solution

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

Notice that for any $j \in \mathbb{N}$

$$w_j(x) = \max\left(-\frac{1}{j}\frac{1}{|x|^{n-2}}, -1\right) \in S_g(\Omega).$$

In particular, w_j is the supremum of two harmonic functions and is thus subharmonic. Also both $-\frac{1}{j}\frac{1}{|x|^{n-2}}$ and -1 are less than g on $\partial\Omega$. Since $w_j \in S_g(\Omega)$ we have

$$u(x) \geq w_j(x)$$

for all $j \in \mathbb{N}$. But $w_j(x) \to 0$ as $j \to \infty$ for every $x \in \Omega$. This implies that $u(x) \geq 0$. But the maximum principle implies that $u(x) \leq \sup_{x \in \partial\Omega} g(x) = 0$. That is $u(x) = 0$. So $\lim_{x\to 0} u(x) = 0 \neq -1 = g(0)$ and therefore there is no barrier at $\xi = 0$.

In general we have a barrier at ξ if the complement on Ω is "large" close to ξ. In the above example the complement of Ω near the origin consists of just one point and that is why we do not have a barrier at the origin.

2.3 Existence of Solutions to the Dirichlet Problem.

We can now prove our first general existence Theorem for the Dirichlet problem.

Theorem 5. *Let Ω be a bounded domain that satisfies the exterior cone condition. Moreover, assume that $f \in C_c^2(\mathbb{R}^n)$ and $g \in C(\partial\Omega)$. Then there exists a unique solution to*

$$\begin{aligned} \Delta u(x) &= f(x) && \text{in } \Omega \\ u(x) &= g(x) && \text{on } \partial\Omega. \end{aligned} \tag{2.13}$$

Proof: We know that, if N is the Newtonian potential then

$$v(x) = \int_{\mathbb{R}^n} N(x-y) f(y) dy$$

solves $\Delta v(x) = f(x)$ in $\mathbb{R}^n$. It is therefore enough to show that there exist a solution to

$$\begin{aligned} \Delta w(x) &= 0 && \text{in } \Omega \\ w(x) &= \tilde{g}(x) = g(x) - v(x) && \text{on } \partial\Omega, \end{aligned} \tag{2.14}$$

since then $u(x) = v(x) + w(x)$ would be a solution to (2.13).

By the Perron process we can find a harmonic function

$$w(x) = \sup_{h \in S_{\tilde{g}}(\Omega)} h(x) \tag{2.15}$$

and since every point in $\partial\Omega$ is regular it follows from Theorem 4 that $\lim_{x\to\xi} w(x) = \tilde{g}(\xi)$ for any $\xi \in \partial\Omega$. It follows that the function defined by (2.15) solves the boundary value problem (2.14).

Uniqueness is an easy consequence of the maximum principle. In particular if u^1 and u^2 are two solutions to (2.13) then $u^1(x)-u^2(x)$ is a harmonic function with zero boundary data in Ω. From the maximum principle we can deduce that $u^1(x) - u^2(x) = 0$, that is $u^1 = u^2$. $\square$

Chapter 3

Exercises:

Exercise 1. Let $g \in C(\partial\Omega)$ where Ω is a bounded domain. Define the Perron solution

$$u(x) = \sup_{v \in S_g(\Omega)} v(x).$$

Assume that u is sub-harmonic and prove that $\Delta u(x) = 0$ in Ω.

(Hint: Consider the harmonic replacement $\tilde{u}$ in some ball $B_r(x) \subset \Omega$. Use comparison to conclude that $\tilde{u} \geq u$, how does that relate to the definition of u?)

Remark: The above proof is much simpler than the one we gave during the lectures. The reason that we did not use that proof is that in order to show that u is sub-harmonic one need to show that u is integrable. That is to show that the supremum of an uncountable family of integrable functions is integrable. That requires measure theory (and also a slightly different definition of subharmonicity) which we do not assume for this course.

Exercise 2. Let $\Omega = B_1(0) \setminus \{x \in \mathbb{R}^3;\ x_1 = x_2 = 0\}$ be the unit ball in $\mathbb{R}^3$ minus the x_3-axis. Show that the origin is not a regular point with respect to Ω.

(Hint: How did we prove that the origin was not regular with respect to the punctured disk $B_1(0) \setminus \{0\}$ in $\mathbb{R}^2$? What is the relation between the punctured disk in $\mathbb{R}^2$ and Ω?)

Exercise 3. We say that $u \in C(\Omega)$ is a viscosity solution to $\Delta u(x) = 0$ in Ω if for any second order polynomial $p(x)$ the following holds:

1. if $u(x) - p(x)$ has a local maximum at x^0 then $\Delta p(x) \leq 0$ and
2. if $u(x) - p(x)$ has a local minimum at x^0 then $\Delta p(x) \geq 0$.

Prove that if $u \in C^2(\Omega)$ is harmonic then u is a viscosity solution to $\Delta u(x) = 0$. Prove that if $u \in C^2(\Omega)$ is a viscosity solution to $\Delta u(x) = 0$ then $u(x)$ is harmonic.

(HINT: Assume that 1 or 2 holds at a point $x^0 \in \Omega$ what is the second order Taylor expansion at x^0?)

Exercise 4. Assume that $u \in C^2(\Omega)$ is a solution to the following partial differential equation

$$\begin{array}{ll} \Delta u(x) = u(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega. \end{array}$$

Prove that $u(x) = 0$ in Ω.

Exercise 5. Prove that any convex function is subharmonic.